

Iterative Delegations in Liquid Democracy with Restricted Preferences

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Abstract

Liquid democracy is a collective decision making paradigm which lies between direct and representative democracy. One main feature of liquid democracy is that voters can delegate their votes in a transitive manner so that: A delegates to B and B delegates to C leads to A delegates to C. Unfortunately, because voters' preferences over delegates may be conflicting, this process may not converge. There may not even exist a stable state (also called equilibrium). In this paper, we investigate the stability of the delegation process in liquid democracy when voters have restricted types of preference on the agent representing them (e.g., single-peaked preferences). We show that various natural structures of preference guarantee the existence of an equilibrium and we obtain both tractability and hardness results for the problem of computing several equilibria with some desirable properties.

On the other hand, a concern about LD is the stability of the induced delegation process (Bloembergen, Grossi, and Lackner 2019; Escoffier, Gilbert, and Pass-Lanneau 2019a). Indeed, as the preferences of voters over their possible gurus can be conflicting, this process may end up in an unstable situation, i.e., a situation in which some voters would change their delegations. Unfortunately, it was shown in (Escoffier, Gilbert, and Pass-Lanneau 2019a) that an equilibrium of LD's delegation process may not exist, and that the existence of such an equilibrium is even NP-hard to decide. In the present work, we obtain more positive results by considering structures of preference. We show that various natural structures of preference guarantee the existence of an equilibrium and we obtain both tractability and hardness results for the problem of computing several equilibria with some desirable properties.

1 Introduction

Interactive democracy aims at using modern information technology, as Social Networks (SN), in order to make democracy more flexible, interactive and accurate (Brill 2018). One of its most well-known implementation is *Liquid Democracy* (LD) (Green-Armytage 2015). In a nutshell, LD allows voters to delegate transitively along an SN. More precisely, each voter may decide to vote directly or to delegate her vote to one of her neighbors, i.e., to use a *representative*. In LD this representative can in turn delegate her vote and the votes that have been delegated to her to someone else. As a result, the delegations of the voters will flow along the SN until they reach a voter who decides to vote. This voter is called the *guru* of the people she represents and has a voting weight equal to the number of people who directly or indirectly delegated to her. This approach is implemented in several online tools (Behrens et al. 2014; Hardt and Lopes 2015) and has been used by several political parties (e.g., the German Pirate party). One main advantage of this framework is its flexibility, as it enables voters to vote directly for issues on which they feel both concerned and expert and to delegate for others.

2 Related Work

We review here several algorithmic issues of liquid democracy recently studied in the AI literature.

One promise of LD is that its flexibility should make the resulting collective decision more accurate. Indeed, it should be possible for each voter to make an informed vote either by voting or through delegation. This claim has been investigated by several works. On the positive side, Green-Armytage (2015) showed that, in a specific spatial voting setting, transitive delegations decrease an expected loss measuring how well the votes represent the voters. On the other hand, Kahng, Mackenzie, and Procaccia (2018), studied a binary election with a ground truth. In their model, no "local" procedure (i.e., a procedure working locally on the SN to organize delegations) can guarantee that LD is, at the same time, never less accurate and sometimes strictly more accurate than direct voting. Caragiannis and Micha (2019) further discussed the possible flaws of local delegation rules and showed that organizing the delegations to maximize the probability of electing the ground truth is NP-hard.

Other issues on LD are related to the number of delegations a guru should get. On the one hand, gurus should have incentives to obtain delegations. In the work of Kotsialou and Riley (2018), voters have both preferences over candidates and over possible gurus. Given the preferences over

gurus, a *delegation rule function* decides who should get the delegations, and then, a *voting rule function* decides who wins the election given the preferences and the voting power of each guru. Focusing on two delegation rules named *depth first delegation rule* and *breadth first delegation rule*, the authors showed that the latter one guarantees that a guru is always better of w.r.t. the outcome of the election when receiving a delegation whereas it is not the case for the former one. This shows that incentivising participation can be a concern in LD. On the other hand, another possible pitfall of LD is that some agents may amass an enormous voting power. This issue was addressed by Gözl et al. (2018) who considered the problem of minimizing the maximal weight of a guru given some delegation constraints. The authors designed a $(1 + \log(n))$ -approximation algorithm (where n denotes the number of voters) and proved that approximating this problem within a factor $\frac{1}{2} \log_2(n)$ is NP-hard. Lastly, the authors gave empirical evidence that allowing each voter to specify multiple possible delegation options instead of one induces a large decrease of the maximum voting power of a guru.

Another possible pitfall of LD is the loss of rationality arising if voters should vote on different but connected issues. In this case, if a voter has different gurus for these different issues, the resulting vote might violate some rationality constraint (Christoff and Grossi 2017). For instance, in the work of Brill and Talmon (2018) each voter should provide a complete ranking over candidates. To do so, each voter may delegate different binary preference queries to different proxies. The delegation process may then yield incomplete or intransitive preference orders. Notably, the authors proved that deciding if such a ballot can be completed to obtain complete and transitive preferences while respecting the constraints induced by the delegations is NP-hard.

Lastly, several authors have investigated the stability of LD's delegation process. In the LD setting studied by Bloembergen, Grossi, and Lackner (2019), voters are connected in an SN and can only delegate to their neighbors in the network. The election is on a binary issue for which some voters should vote for the 0 answer and the others should vote for the 1 answer. Each voter only knows in a probabilistic way what is the correct choice for her, as well as for the others. This modeling leads to a class of games, called *delegation games* in which each voter aims at maximizing the accuracy of her vote/delegation. The authors proved the existence of pure Nash equilibria in several types of delegation games and gave upper and lower bounds on the price of anarchy, and the gain (i.e., the difference between the accuracy of the group after the delegation process and the one induced by direct voting) of such games. Following this line of research, Escoffier, Gilbert, and Pass-Lanneau (2019a) considered a more general type of delegation games in which voters have a complete preference order over who could be their guru, and each voter aims at being represented by the best possible one. The authors showed that an equilibrium may not exist in this type of delegation games. In fact, the existence of such an equilibrium is NP-hard to decide even if the SN is complete or is of bounded degree, and it is W[1]-hard when parameterized by the treewidth of the SN. Then, the authors showed that such an equilibrium is guaranteed to

exist, whatever the preferences of the voters, if and only if the SN is a tree, and provided dynamic programming procedures to compute some equilibria with desirable properties.

Aim of our work. The previous work of Escoffier, Gilbert, and Pass-Lanneau (2019a) considered unrestricted preference profiles and studied how the structure of the SN impacts the equilibria of the delegation game. As noted above, the authors showed that deciding the existence of an equilibrium is NP-hard, even when the SN is a complete graph. Moreover, they proved that in *any* SN which is not a tree, there exists a preference profile with no equilibrium. Thus, the guarantee of the existence of an equilibrium relies on the strong and unrealistic assumption that the SN is a tree.

This work looks in the other direction: we study how the structure of the preference profile impacts the equilibria of the delegation game. Investigating structured preference domains is a well established line of research which makes it possible to better understand the algorithmic complexity of a computational social choice problem (Elkind, Lackner, and Peters 2016). As a first step in this direction, we further investigate the case where the SN is complete. In this case, any voter can delegate to any other voter without requiring the transitive nature of delegations; however, the transitivity of delegations is an essential feature for the study of dynamics and convergence to a stable state. The study of LD's delegation process in a complete SN is motivated by the fact that communities where LD is likely to be implemented (e.g., for inner decision-making in political parties) are often highly connected. More importantly, results obtained for complete graphs, such as hardness results or non-convergence of dynamics, can also apply for SNs that are not complete but may contain a clique as a subgraph. While SNs are not cliques, they are typically composed of small cliques (e.g., communities) that are connected in a sparse way.

In contrast to the hardness results obtained in (Escoffier, Gilbert, and Pass-Lanneau 2019a), we show that several classical structures of preference (e.g., single-peaked preferences) make it possible to guarantee the existence of an equilibrium. This provides much more positive results than in the unrestricted preference case. We then focus on convergence issues towards equilibria, as well as computational issues for computing equilibria with special properties.

3 Notations, Settings and Results Overview

Notations and Nash-Stable Delegation Functions

Following the notations of Escoffier, Gilbert, and Pass-Lanneau (2019a), we denote by $\mathcal{N} = \{1, \dots, n\}$ a set of voters. In this former work, the voters were connected in an SN such that each voter could only delegate directly to their neighbors in the network. In this work, we assume that any voter can delegate directly to any other voter. This is equivalent to considering a complete SN. Hence, each voter i can either vote herself, delegate to another voter j , or abstain. We denote by $d : \mathcal{N} \rightarrow \mathcal{N} \cup \{0\}$ a delegation function such that $d(i) = i$ if voter i declares intention to vote, $d(i) = j$ with $j \in \mathcal{N} \setminus \{i\}$ if i delegates to j , and $d(i) = 0$ if i declares intention to abstain. The *set of gurus* $\text{Gu}(d)$ is defined

as the set of voters that vote themselves given the delegation function d , i.e., $\text{Gu}(d) = \{i \in \mathcal{N} \mid d(i) = i\}$. Delegations are transitive which means that if $d(i) = j$, $d(j) = k$, and $d(k) = k$, then i is represented by k . In the end, the voter who votes for i , called the *guru* of i and denoted by $\text{gu}(i, d)$, is the voter in $\text{Gu}(d) \cup \{0\}$ attained by following the chain of delegations starting from i . In the former example $\text{gu}(i, d) = \text{gu}(j, d) = \text{gu}(k, d) = k$. Note that these successive delegations may also end up in a circuit. In this case, we consider that all voters in the chain of delegations abstain, as none of them take the responsibility to vote.

Each voter i has a preference order \succ_i over who could be their guru in $\mathcal{N} \cup \{0\}$, which is a linear order over $\mathcal{N} \cup \{0\}$. For every voter $i \in \mathcal{N}$, and for every $j, k \in \mathcal{N} \cup \{0\}$ we have that $j \succ_i k$ if i prefers to delegate to j (or to vote if $j = i$, or to abstain if $j = 0$) rather than to delegate to k (or to vote if $k = i$, or to abstain if $k = 0$). We say that voter i is an *abstainer* in P if she prefers to abstain rather than to vote, i.e., if $0 \succ_i i$; she is a *non-abstainer* otherwise. The set of abstainers is denoted by \mathcal{A} . The collection of these preference orders defines a *preference profile* (or profile for short) $P = \{\succ_i \mid i \in \mathcal{N}\}$.

Example 1 ((Escoffier, Gilbert, and Pass-Lanneau 2019a)). Consider the following profile with 3 voters:

$$\begin{aligned} 1 : 2 \succ_1 1 \succ_1 3 \succ_1 0 \\ 2 : 3 \succ_2 2 \succ_2 1 \succ_2 0 \\ 3 : 1 \succ_3 3 \succ_3 2 \succ_3 0 \end{aligned}$$

In this example, each voter i prefers to delegate to $(i \bmod 3) + 1$ rather than to vote directly and each voter prefers to vote rather than to abstain.

A delegation function d is *Nash-stable* for voter i if

$$\text{gu}(i, d) \succ_i g \quad \forall g \in (\text{Gu}(d) \cup \{0, i\}) \setminus \{\text{gu}(i, d)\}.$$

A delegation function d is *Nash-stable* if it is Nash-stable for every voter in \mathcal{N} . Hence, in a Nash-stable delegation function, no voter has an incentive to change unilaterally her delegation. A Nash-stable delegation function is also called an *equilibrium* in the sequel. Unfortunately, as noted by Escoffier, Gilbert, and Pass-Lanneau (2019a) such an equilibrium may not exist as Example 1 admits no equilibrium. In fact, sets of gurus of equilibria correspond to the kernels of a particular digraph as proven by Escoffier, Gilbert, and Pass-Lanneau (2019a) and as explained in the next subsection.

Delegation-Acceptability Digraph and Existence of Equilibria

Let $\text{Acc}(i) = \{j \in \mathcal{N} \mid j \succ_i i \text{ and } j \succ_i 0\}$ be the set of voters to which voter i would rather delegate to than to abstain or vote directly. A necessary condition for a delegation function to be Nash-stable is that $\text{gu}(i, d) \in \text{Acc}(i)$ for every voter i who delegates to another voter. Otherwise, voter i would change her delegation to abstain or vote directly. We refer to $\text{Acc}(i)$ as the set of *acceptable gurus* for i . Note that in an equilibrium d , if $\text{gu}(i, d) = 0$ then i must be an abstainer (otherwise i would rather vote herself); similarly if $\text{gu}(i, d) = i$ then i must be a non-abstainer.

Definition 1 ((Escoffier, Gilbert, and Pass-Lanneau 2019a)). The delegation-acceptability digraph is the digraph $G_P^* = (\mathcal{N} \setminus \mathcal{A}, A_P)$ where $A_P = \{(i, j) \mid j \in \text{Acc}(i)\}$.

Stated differently, there exists an arc from non-abstainer i to non-abstainer j if and only if i accepts j as a guru.

Example 2. Consider the following profile P on 4 voters $\{1, 2, 3, 4\}$. Its delegation-acceptability digraph G_P^* is represented in Figure 1. This example also appears in (Escoffier, Gilbert, and Pass-Lanneau 2019b) (see proof of Theorem 8).

$$\begin{aligned} 1 : 2 \succ_1 1 \succ_1 3 \succ_1 4 \succ_1 0 \\ 2 : 3 \succ_2 4 \succ_2 2 \succ_2 1 \succ_2 0 \\ 3 : 2 \succ_3 1 \succ_3 3 \succ_3 4 \succ_3 0 \\ 4 : 3 \succ_4 4 \succ_4 2 \succ_4 1 \succ_4 0 \end{aligned}$$

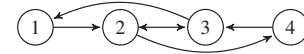


Figure 1: Delegation-acceptability digraph in Example 2.

Given a digraph $G = (V, A)$, a subset of vertices $K \subseteq V$ is *independent* if there is no arc between two vertices of K . It is *absorbing* if for every vertex $u \notin K$, there exists $k \in K$ such that $(u, k) \in A$ (then we say that k absorbs u). A *kernel* of G is an independent and absorbing subset of vertices.

Theorem 1 ((Escoffier, Gilbert, and Pass-Lanneau 2019a)). Given a profile P and a subset $K \subseteq \mathcal{N} \setminus \mathcal{A}$ of voters, there exists an equilibrium d such that $\text{Gu}(d) = K$ if and only if K is a kernel of G_P^* .

For instance, in Example 2 the only kernel of G_P^* is $\{1, 4\}$ which corresponds to the equilibrium where 1 and 4 vote, 2 delegates to 4 and 3 delegates to 1.

Note that given a set K which is a kernel of G_P^* , it is straightforward to retrieve an equilibrium d such that $\text{Gu}(d) = K$. Indeed, the delegation function where every voter in K votes, and every voter not in K delegates to her preferred voter in K , is Nash-stable. Hence, surprisingly, given any equilibrium d , there exists an equilibrium d' such that $\text{gu}(i, d) = \text{gu}(i, d')$ for every voter i and where each voter delegates directly to her guru in d' . However, the transitivity property of delegations is key to our setting because it is at the root of the instability of the delegation process.

The problem of determining if an equilibrium exists is equivalent to the problem of determining if a digraph admits a kernel which is NP-complete (Chvátal 1973). Interestingly, we show in the sequel that for several natural structured profiles (e.g., single-peaked profiles) an equilibrium always exists. For these structured profiles we will investigate if we can compute equilibria verifying particular desirable properties, and tackle convergence issues.

Problems Investigated

Optimization and Decision Problems on Equilibria.

The first question is whether a given profile admits an equilibrium. As stated before, we will show that an equilibrium always exists for the considered structures of preference.

Then, given a voter $i \in \mathcal{N} \setminus \mathcal{A}$, problem **MEMB** aims at deciding if there exists an equilibrium for which i is a guru.

MEMB

INSTANCE: A profile P and a voter $i \in \mathcal{N} \setminus \mathcal{A}$.

QUESTION: Is there an equilibrium d s.t. $i \in \text{Gu}(d)$?

Moreover, we will try to find equilibria that minimize some objective functions. First, problem **MINDIS** tries to find an equilibrium that satisfies most the voters, where the *dissatisfaction* of a voter i w.r.t. a delegation function d is given by $\text{rk}(i, d) - 1$ where $\text{rk}(i, d)$ is the rank of $\text{gu}(i, d)$ in the preference order of i . Second, problem **MINMAXVP** tries to avoid that a guru would amass too much voting power, where the voting power $\text{vp}(i, d)$ of a guru i w.r.t. a delegation function d is defined as $\text{vp}(i, d) = |\{j \in \mathcal{N} | \text{gu}(j, d) = i\}|$. Last, problem **MINABST** tries to determine an equilibrium d minimizing the number of people who abstain, i.e., $|\{i \in \mathcal{N} | \text{gu}(i, d) = 0\}|$.

Problems **MINDIS**, **MINMAXVP** and **MINABST**

INSTANCE: A profile P .

SOLUTION: A Nash-stable delegation function d .

MEASURE to minimize:

– **MINDIS:** $\sum_{i \in \mathcal{N}} (\text{rk}(i, d) - 1)$.

– **MINMAXVP:** $\max_{i \in \text{Gu}(d)} \text{vp}(i, d)$.

– **MINABST:** $|\{i \in \mathcal{N} | \text{gu}(i, d) = 0\}|$.

Convergence of Iterative Delegations. As we focus on instances where an equilibrium always exists, a natural question is whether a dynamic delegation process necessarily converges. As classically done in game theory (see (Nisan et al. 2011)), we consider dynamics where iteratively one voter has the possibility to change her delegation/vote.

In a dynamics, we are given a starting delegation function d_0 and a token function $T : \mathbb{N}^* \rightarrow \mathcal{N}$ which specifies that voter $T(t)$ has the token at step t : she can change her delegation. This gives a sequence of delegation functions $(d_t)_{t \in \mathbb{N}}$ where for any $t \in \mathbb{N}^*$, if $j \neq T(t)$ then $d_t(j) = d_{t-1}(j)$. A dynamics is said to converge if there is a t^* such that $d_t = d_{t^*}$ for all $t \geq t^*$. Given d_0 and T , a dynamics is called a *better response dynamics* or *Improved Response Dynamics* (IRD) if for all t , $T(t)$ chooses a move that strictly improves her outcome if any, otherwise does not change her delegation; it is called a *Best Response Dynamics* (BRD) if for all t , $T(t)$ chooses $d_t(i)$ so as to maximize her outcome. Note that a BRD is also an IRD. We will assume, as usual, that each voter has the token infinitely many times. A classical way of choosing such a function T is to consider a permutation σ over the voters in \mathcal{N} , and to repeat this permutation over time to give the token (if $t = r \bmod n$ then $T(t) = \sigma(r)$). These dynamics are called *permutation dynamics*.

The last problems that we investigate, denoted by **IR-CONV** and **BR-CONV**, can be formalized as:

IR-CONV (resp. BR-CONV)

QUESTION: Does a dynamic delegation process under IRD (resp. BRD) necessarily converges whatever the profile P , initial delegations d_0 , and token function T ?

Table 1: Synthesis of Results.

Problems	Preference Structures		
	Single-Peaked	Symmetrical	Distance-Based
IR-CONV	Not Always	Not Always	Not Always
BR-CONV	Not Always	Always	Always
MEMB	$O(n^2)$	Always Exists	NP-Complete
MINDIS	$O(n^3)$	NP-Hard	NP-Hard
MINMAXVP	$O(n^3)$	NP-Hard	NP-Hard
MINABST	$O(n^3)$	NP-Hard	NP-Hard

Summary of the Results

Our purpose is to investigate the aforementioned problems under restricted preferences. In Section 4, we study *single-peaked profiles*, where agents are ordered on a line and they prefer gurus that are “close” to them on this axis. In Section 5, we investigate *symmetrical profiles*, where all pairs of voters always accept each other as guru, or reject each other. Finally, as classically done in the framework of spatial preferences (Bogomolnaia and Laslier 2007), we consider in Section 6 that voters are embedded in a metric space. They accept as possible gurus voters that are close to them in this space. We denote these profiles as *distance-based profiles*.

For each of these preference structures, we first show that an equilibrium always exists. Our results for problems **IR-CONV**, **BR-CONV**, **MEMB**, **MINDIS**, **MINMAXVP** and **MINABST** are presented in Table 1. Due to lack of space, all missing proofs can be found in an extended version of this paper (Escoffier, Gilbert, and Pass-Lanneau 2019c).

4 Single-Peaked Preferences

Definition

In this section, we consider that voters can be ordered on a line; this ordering $<$ may represent, e.g., the political positions of the voters on a left-right ladder. We assume that voters are indexed w.r.t. this ordering and we identify them with their index in $\{1, \dots, n\}$. A profile is *single-peaked for voter $i \in \mathcal{N}$* if for every $j, k \in \mathcal{N}$,

$$(i < j < k \text{ or } k < j < i) \implies j \succ_i k.$$

A profile is *single-peaked* if it is single-peaked for all voters. For instance, the profile given in Example 2 is single-peaked w.r.t. the axis $1 < 2 < 3 < 4$.

In a single-peaked (SP) profile, if a voter delegates to a guru on her left (and similarly on her right), she prefers to delegate to the closest possible. Note that in i 's preference list, we allow i (vote) and 0 (abstention) to be in any position (differently from single-peakedness traditional definition). It represents the fact that voter i prefers to delegate to close gurus, but then, beyond a given threshold on her left (resp. right), she prefers to abstain or vote herself rather than to delegate to a guru that is too far from her opinions.

SP preferences are one of the most well-known restrictions of preferences in social choice theory. They were introduced by Black (1948) who showed that they solve the Condorcet paradox in the sense that a weak Condorcet winner always exists with SP preferences. Furthermore, SP electorates have many desirable properties: they induce a simple characterization of strategy proof voting schemes (Moulin

1980); they are easily recognizable (Bartholdi III and Trick 1986; Doignon and Falmagne 1994; Escoffier, Lang, and Öztürk 2008); and they often lead to more desirable complexity results (e.g., in multi-winner elections, where the goal of the election is to elect a committee representing best a set of voters (Betzler, Slinko, and Uhlmann 2013)).

Existence of Equilibrium

We now establish that the existence of an equilibrium is guaranteed for an SP profile P . A digraph G is an *interval catch digraph* (Prisner 1994) with vertex-set $\mathcal{N} = \{1, \dots, n\}$ if for every $i \in \mathcal{N}$, there exists $l_i, r_i \in \mathcal{N}$ such that $l_i \leq i \leq r_i$ and the out-neighborhood of i in G is the subset $\{l_i, \dots, r_i\} \setminus \{i\}$. These digraphs are naturally related to SP profiles by the following proposition.

Proposition 1. *If P is an SP profile, then its delegation-acceptability digraph G_P^* is an interval catch digraph.*

Indeed, note first that if we remove abstainers from profile P , the remaining profile is still single-peaked. Then by defining l_i (resp. r_i) the smallest (resp. largest) voter that i accepts as guru, it is easy to check that G_P^* is an interval catch digraph. For instance, the digraph G_P^* of Figure 1 is clearly the interval catch digraph defined by $l_1 = 1, r_1 = 2, l_2 = 2, r_2 = 4, l_3 = 1, r_3 = 3, l_4 = 3$ and $r_4 = 4$.

By Theorem 1, deciding the existence of an equilibrium is equivalent to deciding the existence of a kernel in the delegation-acceptability digraph. Prisner (1994) showed that a kernel in an interval catch digraph always exists and is computable in $O(n^2)$ time. This leads to a polynomial algorithm for computing an equilibrium for an SP profile.

Theorem 2. *An SP profile always admits an equilibrium. Furthermore, an equilibrium can be computed in $O(n^2)$.*

Equilibria and Optimization

Theorem 2 addresses the question of computing one equilibrium. We now provide an additional characterization of sets of gurus of equilibria, that will be a convenient tool for solving other decision or optimization problems on equilibria.

Let us define an auxiliary digraph $G^{\text{aux}} = (V^{\text{aux}}, A^{\text{aux}})$ associated with G_P^* as follows. The vertex-set V^{aux} contains the set of voters $\{1, \dots, n\}$, plus a source s and a sink t . For $i < j$, the arc-set A^{aux} contains the arc (i, j) if the pair $\{i, j\}$ is a kernel of the subgraph of G_P^* induced by $\{i, \dots, j\}$. It contains the arc (s, j) (resp. the arc (i, t)) if the singleton $\{j\}$ (resp. $\{i\}$) is a kernel of the subgraph of G_P^* induced by $\{1, \dots, j\}$ (resp. $\{i, \dots, n\}$).

For illustration purposes, the auxiliary digraph of the profile from Example 2 is given in Figure 2. The two successors of source s are 1 and 2: indeed the singletons $\{1\}$ and $\{2\}$ are kernels of the subgraph induced by $\{1\}$ and $\{1, 2\}$ respectively. Vertices 3 or 4 do not absorb vertex 1, hence they are not successors of s . Between two vertices of $\{1, \dots, 4\}$ the only arc in G^{aux} is $(1, 4)$, because all other pairs of vertices are neighbors, while $\{1, 4\}$ is a kernel of G_P^* .

The importance of the auxiliary digraph is given by the following proposition.



Figure 2: Auxiliary digraph G^{aux} of G_P^* for the profile P of Example 2.

Proposition 2. *There is a one-to-one correspondence between sets of gurus of equilibria for profile P , and s - t paths in the auxiliary digraph of G_P^* .*

The proof of Proposition 2 relies on a technical lemma on kernels of interval catch digraphs. We obtain a one-to-one correspondence between sets of gurus of equilibria, kernels of G_P^* , and s - t paths of the auxiliary digraph. Using this result, it is possible to solve problems **MEMB**, **MINDIS**, **MINMAXVP** and **MINABST** by transforming them into path problems in the auxiliary digraph. The results we obtain are given in the following theorem.

Theorem 3. *Given an SP profile P : the auxiliary digraph of G_P^* is computable in $O(n^2)$ time; problem **MEMB** is solvable in $O(n^2)$ time; problems **MINDIS**, **MINMAXVP** and **MINABST** are solvable in $O(n^3)$ time.*

Proof. (sketch) For **MINABST** we sketch the proof of the equivalence with a path problem in G^{aux} . Given d an equilibrium, with Proposition 2 the set $K = \text{Gu}(d)$ forms an s - t path in G^{aux} . We claim that the number of voters who abstain in d can be obtained by summing, on all pairs of successive gurus k, k' , the number $a_{k, k'}$ of voters between k and k' who prefer abstention over k and k' . Indeed because preferences are SP, any non-guru delegates to the closest guru on her left, or the closest guru on her right, or abstains. Thus G^{aux} can be labeled on arcs with values $a_{i, j}$, and **MINABST** can be solved by finding a shortest s - t path in G^{aux} . \square

Convergence of Dynamics

As Theorem 2 asserts that an equilibrium always exists in the SP case, it is worth considering convergence of dynamics in this setting. Unfortunately, such a convergence is not guaranteed. Indeed, Escoffier, Gilbert, and Pass-Lanneau (2019b) provide a best-response permutation dynamics that does not converge for the profile of Example 2 (see proof of Theorem 8). As this profile is SP, convergence of BRDs are not guaranteed for SP preferences.

5 Symmetrical Preference Profiles

Definition, Existence of Equilibrium and Membership Problem

In this section, we consider *symmetrical preferences* in the sense that $i \in \text{Acc}(j)$ iff $j \in \text{Acc}(i)$. As we will see in Section 6, this is a particular case of the more general distance-based profiles. For symmetrical profiles, the delegation-acceptability digraph has the arc (i, j) iff it has the arc (j, i) (it is symmetrical). Then the existence of an equilibrium is trivially guaranteed (take any maximal independent set of G_P^*), and for any $i \in \mathcal{N} \setminus \mathcal{A}$ there exists an

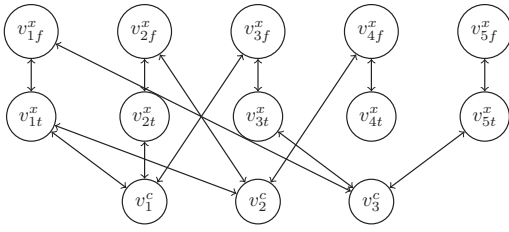


Figure 3: Gadget digraph $G_{U,C}$

equilibrium in which i is a guru (take a maximal independent set containing i).

Equilibria and Optimization

Though the existence of an equilibrium is trivial for symmetrical preference profiles, we now show that **MINDIS**, **MINMAXVP** and **MINABST** are computationally hard, in contrast with the results of the SP case. These results, as well as another hardness result in Section 6, are all based on a reduction from the **3-Satisfiability** (3-SAT) problem, known to be NP-complete (Garey and Johnson 1990), and use the same gadget digraph that we present now.

In the 3-SAT problem, we are given a set U of n_u binary variables and a collection C of n_c disjunctive clauses of 3 literals, where a literal is a variable or a negated variable in U . The objective is then to determine if there exists a truth assignment for U that satisfies all clauses in C . To an instance (U, C) of 3-SAT we associate the symmetrical digraph $G_{U,C}$ defined as follows:

- For each variable $x_i \in U$, we create two adjacent vertices v_{it}^x and v_{if}^x , called variables vertices, representing respectively the literals x_i and $\neg x_i$.
- For each clause $c_j \in C$ we create one vertex v_j^c , called clause vertex; v_j^c is adjacent to the three vertices corresponding to the three literals in c_j .

The following instance illustrates this construction:

$$U = \{x_1, x_2, x_3, x_4, x_5\},$$

$$C = \{(x_1 \vee x_2 \vee \neg x_3), (x_1 \vee \neg x_2 \vee \neg x_4), (\neg x_1 \vee x_3 \vee x_5)\}.$$

Figure 3 gives the corresponding gadget digraph $G_{U,C}$.

Observation 1. $G_{U,C}$ has a kernel containing no clause vertex if and only if (U, C) is satisfiable.

From this construction we derive the following results.

Theorem 4. Given a symmetrical profile P :

- it is NP-hard to decide whether there exists an equilibrium where no voter abstains, or not. Thus, in particular, **MINABST** is NP-hard.
- **MINDIS** is NP-hard even if there are no abstainers.
- **MINMAXVP** is NP-hard even if there are no abstainers.

Proof. We only prove the first item, which directly follows from Observation 1. Let us consider a 3-SAT instance with a set U of variables and a set C of clauses. We create a profile with $2n_u$ voters v_{it}^x and v_{if}^x , $i = 1, \dots, n_u$, and n_c voters v_j^c , $j = 1, \dots, n_c$. A voter v_j^c accepts to delegate to the 3

voters corresponding to the three literals in the clause (and they accept her by symmetry), and then v_j^c prefers to abstain. Moreover, v_{it}^x and v_{if}^x also accept to delegate to each other. Then they prefer to vote. Then an equilibrium where nobody (no voter v_j^c) abstains corresponds to a kernel in $G_{U,C}$ with no clause vertex. The result follows from Observation 1. \square

Convergence of Dynamics

We now focus on the question of convergence under BRD in the case of symmetrical profiles. Interestingly, while there might be cycles in the SP case, we show that under BRD the convergence is guaranteed for symmetrical profiles, and that this convergence occurs within a small number of steps.

Given a dynamics with token function T , let us define rounds as follows. The first round is $[1, t_1]$ where t_1 is the smallest t such that each voter receives the token at least once in $[1, t]$. The k^{st} round is $[t_{k-1} + 1, t_k]$ where t_k is the smallest t such that each voter receives the token at least once in $[t_{k-1} + 1, t]$. For instance, in the case of permutation dynamics, we have $t_k = kn$.

Theorem 5. Given a symmetrical profile P , a BRD always converges in at most 3 rounds.

Intuitively, one can show that symmetry implies that when a voter decides to vote she will not change her mind later. Then after two rounds the set of gurus is fixed, and in the third round each non-guru chooses her best guru, leading to a Nash equilibrium.

We now show that convergence is *not* guaranteed under IRD, thus providing a notable difference between the two dynamics. This holds even if we start from the delegation function d_0 where all voters declare intention to vote, as shown by the following example.

Example 3. Let us consider the case of 4 voters, where $\text{Acc}(1) = \text{Acc}(3) = \{2, 4\}$, $\text{Acc}(2) = \text{Acc}(4) = \{1, 3\}$. They all prefer to vote than to abstain.

We give the token to $1, 2, 1, 3, 2, 4, 3, 1, 4, \dots$. Then the following is compatible with better response: $d_1(1) = 2$, $d_2(2) = 3$, $d_3(3) = 1$, $d_4(4) = 3$, $d_5(5) = 2$, $d_6(6) = 1$, $d_7(7) = 3$, $d_8(8) = 2$, $d_9(9) = 4$. At this point $d_9 = d_1$, so this is a cycle. Intuitively, each voter i delegates to its neighbor $i + 1$ (modulo 4); in the following step $i + 1$ delegates to $i + 2$, then we give the token back to i who is no more happy with her delegation and decides to vote herself.

6 Distance-Based Preference Profiles

Definition and Existence of Equilibrium

In this section, we assume that to each pair i, j of voters is associated a distance $\text{dist}(i, j) = \text{dist}(j, i) \in \mathbb{R}_+$. Then, each voter i has its own acceptability threshold $\rho_i \in \mathbb{R}_+$: she accepts as gurus the other voters that are at distance at most ρ_i from her:

$$\forall j \in \mathcal{N} \setminus \{i\}, \quad j \in \text{Acc}(i) \Leftrightarrow \text{dist}(i, j) \leq \rho_i.$$

We say that such a profile is DB (Distance-Based). Note that DB profiles may represent the case where voters are embedded in a metric space, as in spatial models of preferences (Bogomolnaia and Laslier 2007). They might be

points in \mathbb{R}^k ; they may also represent vertices of a given graph, the distance being the shortest path between vertices.

Any symmetrical profile is DB: indeed, consider the distance defined by $dist(i, j) = 1$ iff $j \in Acc(i)$ (or, equivalently for a symmetrical profile, $i \in Acc(j)$), and $dist(i, j) = 2$ otherwise, and set $\rho_i = 1$ for any voter i . This observation implies that **MINDIS**, **MINMAXVP** and **MINABST** are NP-hard in the case of DB profiles.

We now show that the existence of an equilibrium, which was trivially guaranteed in the symmetrical case, is also guaranteed in this more general case.

Theorem 6. *Any DB profile admits an equilibrium. Furthermore, an equilibrium can be computed in $O(n^2)$.*

Proof. We give an $O(n^2)$ procedure that builds an equilibrium for any DB profile. Build a set K of voters by using the following procedure. Let $S = \mathcal{N} \setminus \mathcal{A}$, and $K = \emptyset$. While S is not empty, add to K the voter i of S with smallest ρ_i value and remove i from S as well as all voters accepting i as guru. At the end of this process, we claim that $K = \{i_1, \dots, i_m\}$ is a kernel of G_P^* . It is absorbing: indeed each non-abstainer in $\mathcal{N} \setminus K$ has at some point been removed from S because it was absorbed by one element of K . It is also independent. Indeed, let us assume by contradiction that i_k accepts to delegate to i_l with $i_k, i_l \in K$. Then, necessarily i_k has been added to K before i_l . Otherwise, i_k would have been removed from S at the same time as i_l and would not have been added to K . Hence, $\rho_{i_k} \leq \rho_{i_l}$ and i_l accepts to delegate to i_k which is not possible by the same argument. This procedure builds K in $O(n^2)$ and the equilibrium induced by K can easily be build in $O(n^2)$. \square

We note that the proof does not rely on $dist$ being a distance: an equilibrium always exists as soon as $dist(i, j) = dist(j, i)$, even if the triangle inequality does not hold.

Membership Problem

Given that an equilibrium always exist, we now focus on the problem **MEMB**. In the case of symmetrical preferences, any voter could be a guru. We show here a drastic difference in the case of DB preferences, as **MEMB** becomes NP-hard.

Theorem 7. ***MEMB** is NP-hard in the case of DB profiles, even if there are no abstainers.*

Proof. (sketch) We only give the way the reduction is built. Let us consider a 3-SAT instance with a set U of variables and a set C of clauses. We consider a graph made of:

- The undirected version of the graph $G_{U,C}$ associated to (U, C) (see Figure 3 in Section 5);
- Two adjacent vertices v_t and v_q ; v_t is also adjacent to all clause vertices v_j^c .

Each vertex of this graph is a voter (we have $2n_u + n_c + 2$ voters), and the distance between voters i and j is the shortest path (number of edges, the graph is unweighted) between the two vertices representing i and j in the graph. The acceptability threshold is 1 for all voters except v_q which has a threshold of 2; they all prefer to vote than to abstain. We can show that the 3-SAT instance is satisfiable iff the DB profile

induced by the corresponding distance admits a Nash-stable delegation function in which v_q is a guru. \square

Convergence of Dynamics

Since an equilibrium always exists, we consider now the convergence of dynamics. Example 3 shows that, under IRD, the convergence is not guaranteed in the case of symmetrical profiles. Therefore, it is the same in the case of DB profiles. We now extend Theorem 5 and show that under BRD the convergence is guaranteed under DB profiles.

Theorem 8. *Given a DB profile, a BRD always converges.*

Proof. (sketch) Let us recall that each voter has the token infinitely many times. Consider a DB profile P , and a BRD with a starting delegation d_0 and a token function T . We assume that voters are numbered $1, 2, \dots, n$ in such a way that $\rho_i \leq \rho_{i+1}$, $i = 1, \dots, n - 1$.

Let us define G as the set of voters which are gurus (vote) infinitely many times in the dynamics: $G = \{i_1, \dots, i_s\}$ with $i_1 \leq i_2 \leq \dots \leq i_s$. Note that, obviously, G contains no abstainers. Since voters in $\mathcal{N} \setminus G$ are guru finitely many times, let us consider a step t_0 such that, for any $t \geq t_0$, no voter in $\mathcal{N} \setminus G$ are guru (they always delegate or abstain). Let t_1 be the first time $t > t_0$ such that i_1 has the token and decides to vote. Since i_1 decides to vote at t_1 , no voter in $Acc(i_1)$ is a guru. Then, while i_1 is a guru, no voter $j > i_1$ in $Acc(i_1)$ ever becomes a guru: indeed, since ρ_i are in non-decreasing order, if $j \in Acc(i_1)$ then $i_1 \in Acc(j)$. While i_1 is a guru j does not decide to vote. Also, no voter $j < i_1$ in $Acc(i_1)$ ever becomes a guru: indeed, these are in $\mathcal{N} \setminus G$ and since $t_1 \geq t_0$ we know that they always delegate or abstain.

Then no voter in $Acc(i_1)$ becomes a guru, so i_1 will be a guru forever. By recursively defining t_k as the first time $t > t_{k-1}$ such that i_k has the token and decides to vote, we can show using similar arguments that i_k remains a guru forever after time t_k . Thus, at time t_s : voters in G are guru forever, and voters in $\mathcal{N} \setminus G$ never become gurus. From t_s we only have to wait for another round to reach an equilibrium. \square

We note again that no specific assumption on function $dist$ was made, except that $dist(i, j) = dist(j, i)$.

7 Conclusion and Future Work

We have investigated the stability of the delegation process in liquid democracy when voters have restricted types of preference on the agent representing them. Interestingly, while the existence of an equilibrium of this process is NP-hard to decide when preferences are unrestricted (Escoffier, Gilbert, and Pass-Lanneau 2019a), we have showed that various natural structures of preference, namely single-peaked, symmetrical and distance-based preferences, guarantee the existence of an equilibrium. For these structures of preference, we have obtained positive and negative results which surprisingly differ for the different structures of preference studied. An interesting direction would be to determine to what extent the positive results we got, such as existence of equilibria, remain valid when the social network is no more complete. As a first insight, the result for symmetrical preferences does not hold for any social network, since

there might be no equilibrium already if the social network is a cycle. (Consider 6 voters and as social network a cycle $v_1, a_1, v_2, a_2, v_3, a_3$. Voters a_1, a_2, a_3 are abstainers. Voters v_1, v_2 and v_3 accept each other as possible gurus. For every $i = 1, 2, 3$, a_i only accepts v_i , and v_i accepts a_i by symmetry. One can check that there is no equilibrium.) A natural question is then to find classes of social networks in which an equilibrium is guaranteed to exist under some classical preference restrictions.

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