Consensus in Opinion Formation Processes in Fully Evolving Environments

Vincenzo Auletta
University of Salerno, Italy.
auletta@unisa.it

Angelo Fanelli
CNRS, (UMR-6211), France.
angelo.fanelli@unicaen.fr

Diodato Ferraioli
University of Salerno, Italy.
dferraioli@unisa.it

Abstract

(Friedkin and Johnsen 1990) modeled opinion formation in social networks as a dynamic process which evolves in rounds: at each round each agent updates her expressed opinion to a weighted average of her innate belief and the opinions expressed in the previous round by her social neighbors. The stubbornness level of an agent represents the tendency of the agent to express an opinion close to her innate belief.

Motivated by the observation that innate beliefs, stubbornness levels and even social relations can co-evolve together with the expressed opinions, we present a new model of opinion formation where the dynamics runs in a co-evolving environment. We assume that agents' stubbornness and social relations can vary arbitrarily, while their innate beliefs slowly change as a function of the opinions they expressed in the past. We prove that, in our model, the opinion formation dynamics converges to a consensus if reasonable conditions on the structure of the social relationships and on how the personal beliefs can change are satisfied. Moreover, we discuss how this result applies in several simpler (but realistic) settings.

Introduction

It is well known that how individuals form their opinions and how they express them in a social context is strongly influenced by their social relations. For example, social pressure can suggest an individual to hide her personal (unpopular) belief and publicly express an opinion that is more conform to the opinions expressed by the majority of her neighbours. On the other hand, she can be firmly tied to her personal belief and not willing to deviate from it, whatever the opinions expressed by her friends are.

Thus, the formation of the opinions in a social context can be modelled as a dynamic process, where opinions publicly expressed by individuals depend on both their personal innate beliefs and on the opinions expressed by other individuals they interact with. Understanding how opinions form, how they diffuse in a social network and how the network can influence this process is a fundamental issue both for the Artificial Intelligence and Social Science communities.

The study of opinion diffusion in social networks dates back to 70s, with the seminal paper on Threshold Model by Granovetter (Granovetter 1978) and the papers of DeGroot (DeGroot 1974) and Lehrer-Wagner (Lehrer and Wagner 1981) on consensus. In the last decades this line of research received great attention and several models have been proposed for describing the process of opinion formation and diffusion (see (Easley and Kleinberg 2010; Jackson 2008; Shakarian et al. 2015) for detailed surveys). Many works model the social influence by simply considering agents that follow the majority (Berger 2001; Feldman et al. 2014; Tamuz and Tessler 2013; Mossel, Neeman, and Tamuz 2014; Auletta, Ferraioli, and Greco 2018).

A classical model that describes how opinions form in a social context has been proposed in (DeGroot 1974) (in the following denoted as DG model). Here, each agent has an opinion on a specific issue of interest. The process is iterative with discrete time and at each round each agent updates her opinion to a weghted sum of all the opinions expressed by her neighbors (including herself). The author gives necessary and sufficient conditions on the structure of the graph describing the social relationships among the agents under which the proposed dynamics converges to a consensus.

A different model, that takes into account both the innate beliefs and the public opinions of the agents, has been proposed in (Friedkin and Johnsen 1990) (in the following denoted as FJ model). They assume that each agent has a (innate) belief and a publicly expressed opinion and these two elements are not necessarily the same. The process of formation and expression of the opinions proceeds in rounds: at each round all the agents make an averaging between their personal beliefs and the opinions expressed by agents with whom they have social relationships. The trade-off between the innate belief and the social pressure of the opinions expressed by her neighbours is weighted by the agent's stubbornness level, that is, the scaling factor used to counterbalance the cost that the agent incurs for disagreeing with the society and the cost she incurs for disagreeing with her innate belief. (Bindel, Kleinberg, and Oren 2011) prove that this repeated averaging process can be interpreted as a best-response play in a naturally defined game that leads to a unique equilibrium. (Ghaderi and Srikant 2014) determine the convergence time of this process to a stable state. Variations of the FJ model have been studied by (Fotakis, Palyvos-Giannas, and Skoulakis 2016), that assume agents update their opinions by consulting only a small (possibly random) subset of their neighbours, and by (Mossel, Sly, and Tamuz 2014), that assume agents update their opinions according to a Bayes rule that takes into account both their beliefs and the neighbours opinions.

A discrete version of the FJ model with binary opinions has been proposed in (Chierichetti, Kleinberg, and Oren 2018) in the setting of *Discrete Preference Games* (see also (Ferraioli, Goldberg, and Ventre 2016)). These games have also been considered to characterize the social graphs where the opinion hold by a minority of agents in the initial profile may become a majority in a stable state (Auletta et al. 2015; 2017b; 2017a). (Yildiz et al. 2013), instead, discuss a discrete model where there are some stubborn agents, that never deflect from their innate beliefs but they can influence other agents. Recently, there models have been also generalized in order to represent through propositional logic more complex relationships among agents (Grandi, Lorini, and Perrussel 2015; Auletta et al. 2016; Acar, Greco, and Manna 2017).

We observe that, although the opinion formation process in the DG and FJ models and their discrete versions is dynamic and evolves over time, the environment where dynamics runs is essentially static and decisions that agents take in each round are based on three fundamental ingredients that are assumed to be fixed: their personal innate beliefs; their stubbornness levels; their social relations, including both the set of the neighbours and the weights they put on the opinions of their neighbours.

However, our real-life experience shows that the environment is not fixed but it co-evolves together with the opinions. Indeed, we can modify our social relationships, getting to know new people, reinforcing or reducing interactions with people, changing our trust on them; our stubbornness levels may also change over time, for example for the social pressure on reaching an equilibrium; even our more consolidated beliefs may change due to the prolonged interactions with other social neighbours. We remark that all these processes run simultaneously and can interwind in very complex ways.

In the last years several works were presented that study the opinion formation process in settings where opinions and the social graph co-evolve under the effects of the mutual influence between the agents. (Bhawalkar, Gollapudi, and Munagala 2013) introduce game-theoretic models of opinion formation in social networks and they investigate the existence and the efficiency of stable states both for the discrete and the continuous process in these models. (Bilò, Fanelli, and Moscardelli 2016) consider opinion formation games in a setting where opinions and social relationships co-evolve in a cross-influencing manner and give bounds on the price of anarchy and price of stability which depend only on the individuals' stubbornness. (Ferraioli and Ventre 2017) study opinion formation games in a setting where the social pressure to reach an agreement makes the agents' stubbornness decrease with time. They characterize the graphs for which consensus is guaranteed and study the complexity of checking whether a graph satisfies such a condition.

A different but related approach to describe the coevolution of the opinions and the social graph is given by the Bounded Confidence Model of (Hegselmann and Krause 2002), where the structure of the social graph reflects the affinity of the agents' opinions.

All the discussed papers consider environments where only some of the ingredients we are considering can evolve. In this paper, instead, we want to study how opinions form and are publicly expressed in a social context when the environment fully evolves and both the structure of the social graph, the agents' stubbornness levels and even their innate beliefs can change over time. We remark that, at the best of our knowledge, this is the first work that considers fully evolving environments, in particular with respect to the evolution of the innate beliefs.

As in (Friedkin and Johnsen 1990) we assume that the process works in rounds and at each round agents take an average between her innate beliefs and the opinions expressed by her neighbours. However, the dynamics is divided in epochs of finite (maybe different) length: for the whole length of an epoch the belief, the stubbornness level and the set of social relations of each agent are fixed; at the beginning of each epoch agents can change her stubbornness level, belief and social relations.

Our main contribution is the proof that in the general model, where the environment fully evolves with opinions, under reasonable conditions on the structure of the social graphs used in each epoch and on the recall of the agents in updating their beliefs, the opinion formation dynamics is ergodic and converges to a consensus. We also show that if we assume the social graph fixed, then, just as in the simpler DG model, it is sufficient that the graph is strongly connected and aperiodic to guarantee the convergence of the dynamics to a consensus.

In order to show the versatility of our model, we present several simple but realistic settings that are only partially evolving, and discuss how they can be casted in our model by conveniently setting the length of the epochs, the belief update rule or how the stubbornness levels may change. We show how results on the convergence to consensus for these settings can be derived from our general result.

The Model

In this section we present a model of opinion formation in a social context where the environment fully co-evolves with the expressed opinions, i.e., the innate beliefs and the stubbornness levels of the agents as well as their social relations change over time.

For every integer $k \geq 1$, we denote by $[k]$ the set $\{1, \ldots, k\}$. We say that a matrix has dimension k, for any $k \geq 1$, if it is made of k rows and k columns. For a matrix A of dimension k, we denote by A_{ij} the entry of A at the i-th row and the j-th column and by $\langle A \rangle_i$ the sum of the entries in the i-th row, i.e., $\langle A \rangle_i = \sum_{j=1}^{k} A_{ij}$. We say that A is *stochastic* if $\langle A \rangle_i = 1$, for every $i \in [k]$. We say that A is *positive* if $A_{ij} > 0$, for every $i, j \in [k]$. We denote by $\mathbb{I}_k$ the *identity* matrix of dimension k.

We are given a set of $n \geq 2$ agents $[n]$, each one holding an opinion in $[0, 1]$. The opinion formation process evolves in rounds and at every round $t \geq 0$, every agent i holds an *innate belief* $g_i(t) \in [0, 1]$ and an *expressed opinion* $z_i(t) \in [0, 1]$. The innate belief represents a memory of the opinions expressed by the agent in former times. Formally, for every agent i, $g_i(0)$ is given by the instance, whereas $g_i(t)$ at round $t \geq 1$ is a function of the opinions expressed by agent i during the previous rounds, i.e., $z_i(0), z_i(1), \ldots, z_i(t-1)$. For the initial round $t = 0$ we assume $z_i(0) = g_i(0)$, whereas the opinion $z_i(t)$ expressed by agent i at time $t \geq 1$ is a *convex combination* of her innate belief $g_i(t-1)$ and the opinions $z_j(t-1)$ expressed by her neighbours at round $t-1$. The weight assigned in the convex combination to $g_i(t-1)$ quantifies i's self-confidence at time t, while the weight given to each $z_j(t-1)$ quantifies how much the opinion expressed by agent j at time $t-1$ influences i at time t. We next provide a more detailed description of how $z_i(t)$ and $g_i(t)$ are updated at each round.

We assume that our formation process works in a fully dynamic environment, where the structure of the social relationships evolves over time. However, we assume that it changes less frequently than the expressed opinions. To model this, we divide the process into time intervals called *epochs*. More specifically, an epoch is a sequence of consecutive rounds, in which the weights assigned to $g_i(t-1)$ and $z_j(t-1)$ remain unchanged. Every epoch $\ell \geq 0$ starts at round $\rho_\ell + 1$ and consists of a number, denoted by $h(\ell) \geq 1$, of consecutive rounds. Let $R(\ell)$ be the set of rounds belonging to epoch ℓ, i.e., $R(\ell) = \{\rho_\ell + 1, \cdots, \rho_\ell + h(l)\}$. We set $\rho_0 = 0$ and, by definition, we have $\rho_{\ell+1} = \rho_\ell + h(\ell)$. Notice that round 0 does not belong to any epoch. For every epoch ℓ and for every agent i, we denote by $w_{ii}^{(\ell)} \geq 0$ the *stubbornness* of i in this epoch, that is the weight the agent puts on her innate belief. Moreover, for every pair of agents (i, j), with $i \neq j$, we denote by $w_{ij}^{(\ell)} \geq 0$ the strength by which the opinion of i is influenced by j during the epoch ℓ (set $w_{ij}^{(\ell)} = 0$ if i is not influenced by j). We write $W_i^{(\ell)}$ to denote the summation $\sum_{j=1}^{n} w_{ij}^{(\ell)}$ and assume that $W_i^{(\ell)} > 0$.

We are ready to formally define how the expressed opinions are updated at each round. For the initial round $t = 0$, we assume $z_i(0) = g_i(0)$, whereas the opinion $z_i(t)$ expressed by agent i at time $t \in R(\ell)$ (notice that $t \geq 1$) is updated according to the following rule

$$z_i(t) = \frac{1}{W_i^{(\ell)}} \left(w_{ii}^{(\ell)} g_i(t-1) + \sum_{j \neq i} w_{ij}^{(\ell)} z_j(t-1) \right).$$

Let us rewrite the updating rule in a more compact way. Let $\mathbf{z}(t)$ and $\mathbf{g}(t)$ denote respectively the vectors of the expressed opinions and the innate beliefs of all the agents at time t, i.e., $\mathbf{z}(t) = (z_1(t), z_2(t), \ldots, z_n(t))$ and $\mathbf{g}(t) = (g_1(t), g_2(t), \ldots, g_n(t))$. For every epoch ℓ, let $E^{(\ell)}$ be the matrix of dimension n such that $E_{ij}^{(\ell)} = \frac{w_{ij}^{(\ell)}}{W_i^{(\ell)}}$ for every $i \neq j$, and $E_{ii}^{(\ell)} = 0$ for every $i \in [n]$. Moreover, let $S^{(\ell)}$ be the matrix of dimension n such that $S_{ij}^{(\ell)} = 0$ for every $i \neq j$, and $S_{ii}^{(\ell)} = \frac{w_{ii}^{(\ell)}}{W_i^{(\ell)}}$ for every $i \in [n]$. Then, for each round t we have

$$\mathbf{z}(t) = \begin{cases} \mathbf{g}(0) & \textit{if } t = 0, \\ S^{(\ell)}\mathbf{g}(t-1) + E^{(\ell)}\mathbf{z}(t-1) & \textit{if } t \in R(\ell). \end{cases} \quad (1)$$

Notice that, $S^{(\ell)} + E^{(\ell)}$ is clearly a stochastic matrix, for every epoch ℓ.

We assume that the innate beliefs also may be updated at the beginning of each epoch, but they remain unchanged during an epoch. We next describe how they are updated. Formally, for every epoch ℓ and agent i, we have $g_i(\rho_\ell + 1) = g_i(\rho_\ell + 2) = \cdots = g_i(\rho_\ell + h(l))$. We denote by $b_i(\ell)$ the innate belief hold by agent i during the epoch ℓ and by $\mathbf{b}(\ell)$ the vector of innate beliefs of all agents during the epoch ℓ, i.e., $\mathbf{b}(\ell) = (b_1(\ell), b_2(\ell), \ldots, b_n(\ell))$. Therefore, in (1) we can set $\mathbf{g}(t-1)$ equal to $\mathbf{b}(\ell)$, thus obtaining that, for every $t \in R(\ell)$,

$$\mathbf{z}(t) = S^{(\ell)}\mathbf{b}(\ell) + E^{(\ell)}\mathbf{z}(t-1). \quad (2)$$

We look at a particular form of evolution of the profile of innate beliefs $\mathbf{b}(\ell)$. For the initial epoch $\ell = 0$ we assume $b_i(0) = z_i(0)$ (equivalently $b_i(0) = g_i(0)$), whereas, for every epoch $\ell \geq 1$, we define $b_i(\ell)$ as a *convex combination* of all the opinions expressed by i in former times. In particular, for every epoch $\ell \geq 1$ and for every round $t \in [0, \rho_\ell]$, let $c_{it}^{(\ell)} \geq 0$ be the weight that agent i assigns, during the epoch ℓ, to the opinion she expressed at time t and assume $\sum_{t=0}^{\rho_\ell} c_{it}^{(\ell)} = 1$. Then we have that $b_i(\ell) = \sum_{t=0}^{\rho_\ell} c_{it}^{(\ell)} z_i(t)$, for every epoch $\ell \geq 1$. Let us rewrite compactly the updating rule for $\mathbf{b}(\ell)$. For every epoch $\ell \geq 1$ and round $t \in [0, \rho_\ell]$, let $C^{(\ell,t)}$ be a matrix of dimension n such that $C_{ii}^{(\ell,t)} = c_{it}^{\ell}$ for every $i \in [n]$ and $C_{ij}^{(\ell,t)} = 0$ for every $j \neq i$. Notice that, for every epoch $\ell \geq 1$, $\sum_{t=0}^{\rho_\ell} C^{(\ell,t)} = \mathbb{I}_n$. Then we have

$$\mathbf{b}(\ell) = \begin{cases} \mathbf{z}(0) & \textit{if } \ell = 0, \\ \sum_{t=0}^{\rho_\ell} C^{(\ell,t)}\mathbf{z}(t) & \textit{if } \ell \geq 1. \end{cases} \quad (3)$$

DG and FJ Models. Let us conclude this section by looking more closely to the relations between our model and the well-known FJ (Friedkin and Johnsen 1990) and DG (DeGroot 1974) models. The FJ model corresponds to the limiting case in which epochs have *infinite* length. Indeed, in this case our model never updates the social relationship among agents and their innate beliefs, and thus our update rule turns out to be equivalent to the one described in (Friedkin and Johnsen 1990). Specifically, by using the notation introduced above, the latter can be expressed as

$$\mathbf{z}(t) = \begin{cases} \mathbf{g}(0) & \textit{if } t = 0, \\ S^{(0)}\mathbf{g}(0) + E^{(0)}\mathbf{z}(t-1) & \textit{if } t \geq 1. \end{cases}$$

Since the FJ model is well-studied, we henceforth do not consider this limiting case of our model and we assume that epochs have *finite* length.

The DG model also can easily casted in our framework. Indeed, in this model the social relationships are never updated, but the innate beliefs are updated at every round and are set equal to the opinions expressed in the previous round. This can be easily achieved by assuming that epochs consist of a single round, i.e., $h(\ell) = 1$ for every ℓ, and assuming that $C^{(\ell,t)} = \mathbb{I}_n$ if $t = \rho_\ell$ and $C^{(\ell,t)} = 0$ otherwise, from which we aobtain the desired belief update rule:

$$\mathbf{b}(\ell) = \begin{cases} \mathbf{z}(0) & \textit{if } \ell = 0, \\ \mathbf{z}(\rho_\ell) & \textit{if } \ell \geq 1. \end{cases}$$

Preliminaries

The main contribution of this section is stated by Theorem 1, which, for every epoch ℓ and $j \in [h(\ell)]$, expresses $\mathbf{z}(\rho_\ell + j)$ as the product of a stochastic matrix depending of the pair (ℓ, j), which we denote by $T^{(\ell,j)}$, and $\mathbf{z}(0)$.

Before to formally define $T^{(\ell,j)}$ and state the theorem, we start with a reworking of the equations (2) and (3).

For every epoch ℓ and $j \in [h(\ell)]$, by recursively applying (2), we obtain

$$\begin{aligned}
&\mathbf{z}(\rho_\ell + j) = S^{(\ell)}\mathbf{b}(\ell) + E^{(\ell)}\mathbf{z}(\rho_\ell + j - 1) \\
&= S^{(\ell)}\mathbf{b}(\ell) + E^{(\ell)}[S^{(\ell)}\mathbf{b}(\ell) + E^{(\ell)}\mathbf{z}(\rho_\ell + j - 2)] \\
&= \left(S^{(\ell)} + E^{(\ell)}S^{(\ell)}\right)\mathbf{b}(\ell) \\
&\quad + \left(E^{(\ell)}\right)^2 [S^{(\ell)}\mathbf{b}(\ell) + E^{(\ell)}\mathbf{z}(\rho_\ell + j - 3)] \\
&= \cdots \\
&= \left[S^{(\ell)} + \left(\sum_{i=1}^{j-1}\left(E^{(\ell)}\right)^i\right)S^{(\ell)}\right]\mathbf{b}(\ell) + \left(E^{(\ell)}\right)^j\mathbf{z}(\rho_\ell).
\end{aligned}$$

If we define $A^{(\ell,j)} = \left[S^{(\ell)} + \left(\sum_{i=1}^{j-1}\left(E^{(\ell)}\right)^i\right)S^{(\ell)}\right]$ and $B^{(\ell,j)} = \left(E^{(\ell)}\right)^j$ then the previous equality becomes

$$\mathbf{z}(\rho_\ell + j) = A^{(\ell,j)}\mathbf{b}(\ell) + B^{(\ell,j)}\mathbf{z}(\rho_\ell). \tag{4}$$

Moreover, for every epoch ℓ and $j \in [h(\ell)]$, (3) can be explicitly rewritten as

$$\mathbf{b}(\ell) = C^{(\ell,0)}\mathbf{z}(0) + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)} C^{(\ell,\rho_r+k)}\mathbf{z}(\rho_r + k). \tag{5}$$

We now give a formal definition of $T^{(\ell,j)}$ and state the theorem.

Definition 1. *For every epoch ℓ and $j \in [h(\ell)]$, let $T^{(\ell,j)}$ be a matrix of dimension n recursively defined as follows*

- *if $\ell = 0$ then*

$$T^{(0,j)} = A^{(0,j)} + B^{(0,j)},$$

- *if $\ell \geq 1$ then*

$$\begin{aligned}
T^{(\ell,j)} \;&=\; A^{(\ell,j)}\left[C^{(\ell,0)} + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)} C^{(\ell,\rho_r+k)}T^{(r,k)}\right] \\
&\quad + B^{(\ell,j)}T^{(\ell-1,h(\ell-1))}.
\end{aligned}$$

Theorem 1. *For every epoch ℓ and $j \in [h(\ell)]$, the profile of opinions at time $\rho_\ell + j$ can be expressed as*

$$\mathbf{z}(\rho_\ell + j) = T^{(\ell,j)}\mathbf{z}(0),$$

and $T^{(\ell,j)}$ is a stochastic matrix.

In order to prove the theorem we employ the following lemma.

Lemma 1. *For every epoch $\ell \geq 0$ and $j \in [h(\ell)]$, the matrix $A^{(\ell,j)} + B^{(\ell,j)}$ is stochastic.*

Proof Sketch. If $j = 1$, then $A^{(\ell,j)} + B^{(\ell,j)} = S^{(\ell)} + E^{(\ell)}$, that is stochastic by construction. For larger values of j, the claim follows by a simple inductive argument. $\square$

We also need the following technical proposition, that immediately follows by the definition of stochastic matrix.

Proposition 1. *For every pair of matrices F and G of dimension $k \geq 1$, if G is stochastic then $\langle FG\rangle_i = \langle F\rangle_i$, for every $i \in [k]$.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. The proof is by induction on ℓ.

Let us prove the claim for $\ell = 0$. Since $\mathbf{b}(\ell) = \mathbf{z}(0)$ and $\mathbf{z}(\rho_\ell) = \mathbf{z}(0)$, from (4) and Definition 1, we obtain

$$\begin{aligned}
\mathbf{z}(\rho_0 + j) =& A^{(0,j)}\mathbf{b}(0) + B^{(0,j)}\mathbf{z}(\rho_0) \\
=& \left(A^{(0,j)} + B^{(0,j)}\right)\mathbf{z}(0) = T^{(0,j)}\mathbf{z}(0).
\end{aligned}$$

Moreover, by Lemma 1, it follows that $T^{(0,j)}$ is a stochastic matrix, as desired.

Consider now the case $\ell \geq 1$ and suppose, by inductive hypothesis, that $\mathbf{z}(\rho_r + k) = T^{(r,k)}\mathbf{z}(0)$ for every $r \in [0, \ell - 1]$ and $k \in [h(r)]$. Then we have

$$\begin{aligned}
&\mathbf{z}(\rho_\ell + j) = A^{(\ell,j)}\mathbf{b}(\ell) + B^{(\ell,j)}\mathbf{z}(\rho_\ell) && (6) \\
&= A^{(\ell,j)}\left(C^{(\ell,0)}\mathbf{z}(0) + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)} C^{(\ell,\rho_r+k)}\mathbf{z}(\rho_r + k)\right) \\
&\quad + B^{(\ell,j)}\mathbf{z}(\rho_{\ell-1} + h(\ell - 1)) && (7) \\
&= \Bigg[A^{(\ell,j)}\left(C^{(\ell,0)} + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)} C^{(\ell,\rho_r+k)}T^{(r,k)}\right) \\
&\quad + B^{(\ell,j)}T^{(\ell-1,h(\ell-1))}\Bigg]\mathbf{z}(0) = T^{(\ell,j)}\mathbf{z}(0). && (8)
\end{aligned}$$

where (6) follows (4), (7) from (5) and the fact that $\rho_\ell = \rho_{\ell-1} + h(\ell - 1)$, and (8) from the inductive hypothesis and Definition 1. It remains to show that, for every epoch $\ell \geq 1$ and every $j \in h[\ell]$, we have that $T^{(\ell,j)}$ is stochastic, i.e., $\left\langle T^{(\ell,j)}\right\rangle_i = 1$ for every $i \in [n]$. For every epoch $\ell \geq 1$, let $D^{(\ell)} = C^{(\ell,0)} + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)}\left(C^{(\ell,\rho_r+k)}T^{(r,k)}\right)$. We first show that $D^{(\ell)}$ is stochastic. For every $i \in [n]$, we have

$$\begin{aligned}
\left\langle D^{(\ell)}\right\rangle_i &= \left\langle C^{(\ell,0)}\right\rangle_i + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)}\left\langle C^{(\ell,\rho_r+k)}T^{(r,k)}\right\rangle_i \\
&= \left\langle C^{(\ell,0)}\right\rangle_i + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)}\left\langle C^{(\ell,\rho_r+k)}\right\rangle_i && (9) \\
&= \left\langle C^{(\ell,0)}\right\rangle_i + \sum_{t=1}^{\rho_\ell}\left\langle C^{(\ell,t)}\right\rangle_i \\
&= \sum_{t=0}^{\rho_\ell}\left\langle C^{(\ell,t)}\right\rangle_i = \langle \mathbb{I}_n\rangle_i = 1, && (10)
\end{aligned}$$

where (9) follows from Proposition 1 and the fact that $T^{(r,k)}$ is stochastic by inductive hypothesis, and (10) from the definition of the set of matrices $C^{(\ell,t)}$ for $t \in [0, \rho_\ell]$. We are now ready to prove that $T^{(\ell,j)}$ is stochastic. For every $i \in [n]$, we have

$$\begin{aligned}
&\left\langle T^{(\ell,j)} \right\rangle_i = \\
&\quad \sum_{k=1}^{n} \left[\left(A^{(\ell,j)} D^{(\ell)} \right)_{ik} + \left(B^{(\ell,j)} T^{(\ell-1,h(\ell-1))} \right)_{ik} \right] = \\
&\quad \sum_{k=1}^{n} \sum_{h=1}^{n} \left(A_{ih}^{(\ell,j)} D_{hk}^{(\ell)} + B_{ih}^{(\ell,j)} T_{hk}^{(\ell-1,h(\ell-1))} \right) = \\
&\quad \sum_{h=1}^{n} \left(A_{ih}^{(\ell,j)} \sum_{k=1}^{n} D_{hk}^{(\ell)} + B_{ih}^{(\ell,j)} \sum_{k=1}^{n} T_{hk}^{(\ell-1,h(\ell-1))} \right) = \\
&\quad \sum_{h=1}^{n} \left(A_{ih}^{(\ell,j)} \left\langle D^{(\ell)} \right\rangle_h + B_{ih}^{(\ell,j)} \left\langle T^{(\ell-1,h(\ell-1))} \right\rangle_h \right) = \\
&\quad \sum_{h=1}^{n} \left(A_{ih}^{(\ell,j)} + B_{ih}^{(\ell,j)} \right) = \qquad (11) \\
&\quad \left\langle A^{(\ell,j)} + B^{(\ell,j)} \right\rangle_i = 1,
\end{aligned}$$

where (11) follows from the fact that are stochastic both $D^{(\ell)}$, as proved above, and $T^{(\ell-1,h(\ell-1))}$, by inductive hypothesis, while the last equality follows from Lemma 1. □

Convergence to Consensus

In this section we present the main contribution of this work, claimed by Theorem 2. Informally, this theorem states that under reasonable conditions related to structural properties of the social graphs and to the recall of the agents, the opinion formation process, described by $\mathbf{z}(t)$, converges to a consensus. We will also show that these conditions are in some way necessary for consensus, and we will discuss what happens whenever they do not hold. Before to formally state the theorem, let us introduce some useful concepts.

For every round t, let $G(t)$ denote the *social influence graph* at t, which characterizes the interpersonal influences among the agents. The influence graph associated to round $t \in R(\ell)$ is formally defined as a directed graph having as nodes the set of agents $[n]$ and as edges the set of ordered pairs $\left\{(i,j) \colon i \neq j \in [n], w_{ij}^{(\ell)} > 0\right\}$. Notice that all the influence graphs associated to the rounds in the same epoch are the same. Now, let us consider the *sequence of social influence graphs* $\mathcal{G} = \{G(t)\}_{t \geq 0}$. Given any pair of agents $x \neq y$, an *influence path* of length $k \geq 1$ from x to y in $\mathcal{G}$ is a sequence of agents $(x = \omega_0, \ldots, \omega_k = y)$ such that (ω_j, ω_{j-1}) is an arc of $G(j)$, for every $j \in [k]$. Intuitively, this means that the influence of x on y goes through the first k rounds of the dynamics; therefore the opinion expressed by y at round k is positively influenced by the initial opinion of x expressed at round 0, i.e., $g_i(0)$. For every integer $k \geq 1$ and pair of agents $x, y \in [n]$, we denote by $\mathcal{P}_{xy}$ the set of all the influence paths from x to y (of any length) in $\mathcal{G}$.

Definition 2. *We say that $\mathcal{G}$ is* ergodic *if there is a round t^* such that, for every round $t \geq t^* + 1$ and pair of agents $x, y \in [n]$, there is a path in $\mathcal{P}_{xy}$ of length t.*

Finally, let $M_i^{(\ell)} = \{t \in [0, \rho_\ell] \colon c_{it}^{(\ell)} > 0\}$ for every agent i and epoch ℓ. We define the *recall of i in ℓ* as $\mu_i^{(\ell)} = \max_{t \in M_i^{(\ell)}} |\rho_\ell - t|$. Informally, $\mu_i^{(\ell)}$ measures how far agent i is looking back in the past in order to shape her innate belief in epoch ℓ. We define the *recall of i* as $\mu_i = \max_{\ell \geq 0} \mu_i^{(\ell)}$. Observe that, if μ_i is finite then the opinion expressed by i at any round has a direct influence on the innate beliefs of i during only a finite number of future epochs and its influence vanishes in the long run.

We are now ready to state our main contribution.

Theorem 2. *If $\mathcal{G}$ is ergodic and every agent has finite recall then the profile of opinions $\mathbf{z}(t)$ converges to a consensus as t goes to infinity.*

To prove the theorem we need two technical lemmas. The first lemma states that the ergodicity of $\mathcal{G}$ implies that $T^{(\ell,j)}$ becomes positive in the long run.

Lemma 2. *If $\mathcal{G}$ is ergodic then there exists an epoch ℓ_0 such that $T^{(\ell,j)}$ is positive, for every $\ell \geq \ell_0$ and $j \in [h(\ell)]$.*

The second lemma states that if the agents form their innate beliefs by looking at only a finite number of previous epochs and if $T^{(\ell,j)}$ becomes positive in the long run then $T^{(\ell,j)}$ tends to a matrix where all the elements of every column $y \in [n]$ are equal to a constant in $[0,1]$; we denote this constant by π_y. Notice that, by Theorem 1, all these elements sum up to 1, i.e., $\sum_{y \in [n]} \pi_y = 1$.

Lemma 3. *If every agent has finite recall and there exists an epoch ℓ_0 such that $T^{(\ell,j)}$ is positive, for every $\ell \geq \ell_0$ and $j \in [h(\ell)]$, then $\lim_{\ell \to +\infty} T_{xy}^{(\ell,j)} = \pi_y$, for every pair $x, y \in [n]$ and $j \in [h(\ell)]$.*

Proof Sketch. Let ℓ_0 be such that for every $\ell \geq \ell_0$ and every $j \in [h(\ell)]$ we have that $T^{(\ell,j)}$ is positive and $C^{(\ell,0)} = 0$. The existence of ℓ_0 is guaranteed by Lemma 2 and the hypothesis of finite recall. We define $L(\ell)$ as the set containing all time steps t such that the beliefs at epoch ℓ depend on the opinions at step t, i.e., $L(\ell) = \{(r,k) \colon C^{(\ell,\rho_r+k)} \neq 0\}$. We also define $L^+(\ell) = L(\ell) \cup \{(\ell-1, h(\ell-1)\}$.

First, observe that if there is π_y such that $T_{wy}^{(r,k)} = \pi_y$ for every $(r,k) \in L^+(\ell)$ and every $w \in [n]$ then we can state that $T_{xy}^{(\ell,j)} = \pi_y$ for every $x \in [n]$.

Suppose now that such a π_y does not exist. Hence there is $(\hat{r}, \hat{k}) \in L^+(\ell)$ and $\hat{w} \in [n]$ such that $T_{\hat{w}y}^{(\hat{r},\hat{k}))} < \max_{(r,k) \in L^+(\ell)} \max_w \left\{T_{wy}^{(r,k)}\right\}$. Similarly, by Lemma 2, there must be $(\check{r}, \check{k}) \in L^+(\ell)$ and $\check{w} \in [n]$ such that $T_{\check{w}y}^{(\check{r},\check{k})} > \min_{(r,k) \in L^+(\ell)} \min_w \left\{T_{wy}^{(r,k)}\right\} \geq 0$. Then, we can prove that $0 \leq \min_{(r,k) \in L^+(\ell)} \min_w \left\{T_{wy}^{(r,k)}\right\} < T_{xy}^{(\ell,j)} < \max_{(r,k) \in L^+(\ell)} \max_w \left\{T_{wy}^{(r,k)}\right\}$. That is, rows in the $T^{(\ell,j)}$ matrices become closer each other as ℓ increases. Hence, their difference will eventually be 0, as desired. □

Armed with the previous two lemmas, we are ready to formally prove Theorem 2.

Proof of Theorem 2. By combining Lemma 2 and Lemma 3 we obtain that $\lim_{\ell\to\infty} T_{xy}^{(\ell,j)} = \pi_y$, for every pair $x, y \in [n]$ and $j \in [h(\ell)]$. Finally, by combining this latter fact with Theorem 1, we obtain that for every agent i, it holds that $\lim_{\ell\to\infty} z_i(\rho_\ell + j) = \lim_{\ell\to\infty} \sum_{k=1}^{[n]} T_{ik}^{(\ell,j)} z_k(0) = \sum_{k=1}^{[n]} \pi_k z_k(0) = z^*$, where $z^* \in [0,1]$ denotes the value of the consensus. □

We next focus on the conditions of Theorem 2: we show they are essentially tight, and we briefly discuss the behavior of the dynamics when they do not hold.

Finite Recall. The assumption of finite recall essentially states that the agents tend to forget opinions expressed too far in the past. Hence, it turns out to be a quite realistic assumption. On the other hand, suppose that there is an agent with infinite recall. Then there must be a round t such that the opinion expressed at that round influences the belief of that agent in infinitely many epochs. Hence, even if a consensus would be reached at round $t' > t$ such that $t \in R(\ell)$, there could be an epoch $\ell' > \ell$ such that the belief update at the beginning of this epoch breaks the consensus.

It remains open to understand whether and when, under the assumption of infinite recall, the process converges to a (non-consensus) fixed profile.

Ergodic Sequence of Graphs. The assumption that the sequence of social influence graphs is ergodic concerns the structure of the social relationships among agents. It implies, indeed, that, after a sufficently large number of rounds, the opinion of each agent is influenced by the initial opinions of all the other agents, i.e., $\mathbf{g}(0)$.

Note that our definition of ergodicity does not imply that agents remains connected to each other at every round t. In particular, $\mathcal{G}$ can be ergodic even if the evolution of the social relationships leads to the formation of distinct communities; ergodicity only requires that there has been a sufficiently large amount of time in which these different communities influenced each other. Hence, ergodicity, as defined above, turns out to be a milder requirement than just connectedness, often required for convergence to consensus (e.g., for the DG model). However, as we will discuss below, our definition relaxes to connectivity whenever the social relationships are assumed to be fixed.

Actually, as illustrated by the proof of Theorem 2, the ergodicity of $\mathcal{G}$ can be replaced with the weaker notion of positiveness of $T^{(\ell,j)}$. However, in the claim of the theorem we use the former condition because it explicitly reveals the structural properties of evolving social relationships sufficient to converge to consensus.

The positiveness of $T^{(\ell,j)}$ turns out to be necessary for the convergence to consensus. Indeed, the lack of positiveness leads the dynamics to two possible different outcomes: either the dynamics converges to a non-consensus stable outcome, or it does not converge at all.

The convergence to a stable outcome which is not a consensus occurs if there is a pair of nodes x, y such that $T_{xy}^{(\ell,j)} = 0$ for every ℓ sufficiently large. In this case, we can partition the agents into communities $N_1, \ldots, N_k$ such that all the members of the same community N_i will eventually influence each other, i.e., $T_{xy}^{(\ell,j)} > 0$, for every $x, y \in N_i$, $j \in [h(\ell)]$ and ℓ sufficiently large. Each community N_i will be classified as *recurrent* if $T_{xy}^{(\ell,j)} = 0$ for every ℓ sufficiently large and every $x \in N_i$ and $y \notin N_i$, and *transient* otherwise. That is, recurrent communities are the one that will eventually become disconnected by the rest of society. Then, we can apply Theorem 2 to every recurrent community and conclude that the opinion of its members will converge to a consensus (even if different consensi can be reached in different recurrent communities). Instead the opinion of an agent belonging to a transient community converges to a combination of the consensi of the recurrent communities.

The process does not convergence instead when $T^{(\ell,j)}$ is not positive and the condition in the previous paragraph does not hold. Formally, we have no convergence when, for every epoch ℓ_0, there are two epochs $\ell, \ell' \geq \ell_0$ such that $T_{xy}^{(\ell,j)} > 0$ but $T_{xy}^{(\ell',j')} = 0$, for every $x, y \in [n]$, $j \in [h(\ell)]$ and $j' \in [h(\ell')]$. This implies that the opinion of x changes infinitely often, therefore the convergence is unattainable.

Partially Evolving Environments

In this section, we analyze the opinion formation dynamics in some simpler (but realistic) settings where the environment only partially evolves.

Fixed Social Influence Graphs. Here we assume that the structure of the social relationships among the users never change. This implies that $\mathcal{G} = \{G(t)\}_{t\geq 0}$ is such that $G(t) = H$, for every $t \geq 0$. Interestingly, in this case, the property of ergodicity of $\mathcal{G}$ reduces to simple and more intuitive properties on H. In fact, as argued in (Levin and Peres 2017, Lemma 1.7), it is not hard to see that if H is *strongly connected* and *aperiodic* then $\mathcal{G}$ is ergodic. We say that H is *strongly connected* if for every pair of nodes $x, y \in [n]$, there is at least a directed path from x to y in G. We say that H is *aperiodic* if the maximum common divisor of the lengths of the cycles in the graph is 1 (i.e., it does not occur that all the cycles have a length that is a multiple of c for some $c > 1$). Therefore, we have the following theorem.

Theorem 3. *Let H be a strongly connected and aperiodic directed graph. If $G(t) = H$, for every $t \geq 0$, and every agent has finite recall then the profile of opinions $\mathbf{z}(t)$ converges to a consensus as t goes to infinity.*

For sake of presentation, we henceforth assume that the social influence graphs are fixed and they are all equal to H.

Belief as the Last Opinion of the Previous Epoch. Consider a setting where agents' stubborness levels are fixed and, at the beginning of each epoch, agents set their beliefs equal to the last opinion expressed in the previous epoch. Formally, for each epoch ℓ, we define $C^{(\ell,t)} = 1$ if $t = \rho_\ell$ and 0 otherwise.

By Theorem 3 the opinion formation dynamics converges to a consensus if the social influence graph H is strongly connected and aperiodic. Actually, it turns out that in this specific setting we can prove that the dynamics converges to a consensus even if H is strongly connected (but not necessarily aperiodic) and there is at least one agent with non-zero stubborness level. Moreover, the proof is much simpler than the general case.

Observe that the opinions announced in round $t \in R(\ell)$ are $\mathbf{z}(\rho_\ell + t) = E\mathbf{z}(\rho_\ell + t - 1) + S\mathbf{z}(\rho_\ell)$ where, since the social influence graph and the stubborness levels are fixed, $E = E^{(0)} = E^{(\ell)}$ and $S = S^{(0)} = S^{(\ell)}$ for each $\ell > 0$. Iterating, we have $\mathbf{z}(\rho_\ell + t) = E^t\mathbf{z}(\rho_\ell) + \sum_{r=0}^{t-1} E^r S\mathbf{z}(\rho_\ell)$ and $\mathbf{z}(\rho_{\ell+1}) = E^{h(\ell)}\mathbf{z}(\rho_\ell) + \sum_{r=0}^{h(\ell)-1} E^r S\mathbf{z}(\rho_\ell)$. For every integer $k \geq 1$, let $T^{\{k\}} = E^k + \sum_{r=0}^{k-1} E^r S$. Then, we achieve that $\mathbf{z}(\rho_{\ell+1}) = T^{\{h(\ell)\}}\mathbf{z}(\rho_\ell)$. By iterating on all the previous epochs and using $\rho_0 = 0$, we obtain that for every epoch $\ell \geq 0$

$$\mathbf{z}(\rho_{\ell+1}) = \left(\prod_{i=\ell}^{0} T^{\{h(i)\}}\right)\mathbf{z}(0). \tag{12}$$

Now observe that, by Lemma 1, $T^{\{k\}}$ is a stochastic matrix for every $k \geq 1$. Moreover, it is immediate to see that since the graph H is strongly connected, then the matrix E is *irreducible*, i.e., for every x, y there is an integer t such that $E^t(x, y) > 0$. Hence, it immediately follows that also E^k and $T^{\{k\}}$ are irreducible, for every $k \geq 1$. Moreover, it is immediate to see that if there is at least one agent x with non-zero stubborness level, then, for every k the matrix $P = E^k + S$ is *aperiodic*, i.e., the greatest common divisor of the element in the set $\{t \geq 1 \colon P^t(x, x) > 0\}$ is 1. Consequently, $T^{\{k\}}$ is aperiodic, too. Finally, observe that since all the epochs have finite length, there are always finitely many different matrices $T^{\{k\}}$ involved in the product of (12). These conditions (finiteness of the number of matrices involved in the product, and the fact that all these matrices are stochastic, irreducibile and aperiodic) are sufficient to imply that $\mathbf{z}(\rho_\ell)$ tends to a consensus as ℓ goes to infinity (see, e.g., (Coppersmith and Wu 2008, Theorem 5)).

We conclude the analysis of this setting, by observing that when all epochs have the same length, i.e. $h(\ell) = h$ for every ℓ, then (12) gives that $\mathbf{z}(\rho_\ell) = \left(T^{\{h\}}\right)^\ell \mathbf{z}(0)$. Thus, the evolution of opinions can be described through the evolution of a Markov chain with transition matrix $T^{\{h\}}$.

By similar arguments, but with different technicalities, we can also prove the convergence to consensus in settings where the agent belief in epoch ℓ depends only on the opinions expressed by the same agent in the epoch $\ell - 1$ (e.g., one could assume that the belief at epoch ℓ is a (discounted) average of opinions expressed in the epoch $\ell - 1$.

Social Pressure to Consensus. In (Ferraioli and Ventre 2017) a generalization of the FJ model is presented to model settings where there is a pressure on the agents on reaching a consensus within an upcoming deadline. In this model, the agents' stubborness proportionally reduces over time.

Even if this model is quite different from ours (e.g., they assume opinions are discrete), we can import some of their ideas in our framework. Assume that the stubborness level $w_{ii}^{(\ell)}$ decreases as ℓ increases, while the beliefs remain fixed, i.e. $C^{(\ell,t)} = 0$ for every $t \neq 0$, and $C^{(\ell,0)} = 1$. Notice that in this setting $\mu(0) = \infty$, and we cannot use Theorem 2. However, if for ℓ going to infinity, the stubborness levels go to 0, then, at the limit, the model turns out to be equivalent to the DG model, and hence convergence to consensus is guaranteed under the same hypothesis as the DG model.

Agents with Heterogeneous Epochs. In all previous settings we assumed that epochs are homogeneous among agents and at the beginning of each epoch all the agents update their beliefs and stubborness levels. However, our framework is powerful enough to allow also agents with heterogeneous epochs. We assume that for every agent i there is an infinite list of time steps $\left(t_i^{(1)}, t_i^{(2)}, \dots\right)$ at which i is allowed to update her belief and, possibly, her stubborness level. Assume, for sake of simplicity, that at these time steps i sets her belief to the last opinion expressed in the previous epoch. In order to model the dynamics in this setting, we define epochs as follows: let $\rho_0 = \min_i t_i^{(1)}$, and for each $\ell > 0$, let $\rho_\ell = \min_i \min_{r \colon t_i^{(r)} > \rho_{\ell-1}} \left\{t_i^{(r)}\right\}$. Hence, each epoch ends as soon as there is an agent that would like to update her belief. Moreover, for every epoch ℓ, we set $C_{ii}^{(\ell,\rho_\ell)} = 1$ if there is r such that $t_i^{(r)} = \rho_\ell$, and $C_{ii}^{(\ell,t^*)} = 1$ otherwise, where $\mathbf{z}(t^*)$ is the last opinion assumed as belief by i, i.e., $t^* = \max_{r \colon t_i^{(r)} < \rho_\ell} \left\{t_i^{(r)}\right\} - 1$. That is, if i is one of the agents that is supposed to change her belief at time step ρ_ℓ, then it sets the belief exactly as the last expressed opinion, otherwise it simply copies the last assumed belief.

Note that $h(\ell)$ and $\mu(\ell)$ are finite as long as each agent updates her belief at infinitely many time steps. We can prove that, in this setting, even if agents update their beliefs at different time steps, the dynamics still reaches a consensus if the social graph is strongly connected and aperiodic, and each agent updates her belief infinitely often.

Conclusions

In this paper we presented a new model of opinion formation in a fully evolving social environment when both the structure of the social relationships among the agents and their innate beliefs co-evolve with the expressed opinions. We proved that, under reasonable conditions on the structure of the sequence of influence graphs and on the recall of the agents in forming their beliefs, the opinion formation dynamics converges to a consensus.

Our results raise several interesting questions and suggest different research lines.

First of all, it would be interesting to study how much time the dynamics needs to reach a consensus. We run some preliminary experiments on very simple settings that give interesting results but a much more extended experimental activity is necessary to understand how the dynamics convergence time depends on the different parameters of the problem.

Another interesting question is to study the quality of the consensus reached by the dynamics with respect to the best agreement that could be reached by a central authority (e.g., a weighted average of the initial opinions).

We can also study our dynamics in a setting of information aggregation in social learning. Here, we assume that agents have noisy signals (their initial beliefs) with respect to a issue of interest and these signals are biased toward a groun truth. Through their social interactions, the agents can reach an agreement on this issue. It is well known that, due to information cascades, they can reach an agreement on a wrong value and conditions were given under which the consensus is reached on the right value in a static environment (Feldman et al. 2014; Mossel, Neeman, and Tamuz 2014; Mossel, Sly, and Tamuz 2014), It would be interesting to study this problem in an evolving enviroment to give conditions on the structure of the social influence graphs and on the evolution of the beliefs under which the dynamics reaches a consensus on the correct value or to bound how the learned value is far from the ground truth.

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