Voting in Divisible Settings: A Survey

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Abstract

Voting is one of the most prominent applications of preference aggregation and computational social choice. While much of the literature focuses on models involving discrete candidates, there has been a growing interest in voting over divisible resources, such as budget, space, and time. In this survey, we review existing work on voting in divisible settings, including fundamental models of budget aggregation, fair mixing, and cake sharing. We also establish connections among these models, highlight unifying themes across different frameworks, and suggest directions for future research.

1 Introduction

Computational social choice is a field that studies algorithms and procedures for aggregating individual preferences into a collective decision (Brandt et al. 2016). This area of research is inherently interdisciplinary, drawing upon concepts from mathematics, economics, political science, and computer science. Not only does the field continue to produce exciting theoretical research every year—with its findings presented at leading AI conferences including AAAI—but the solutions that it developed have also significantly improved a variety of practical decision-making processes.

Among the most prominent applications of computational social choice is *voting*, in which individual preferences are combined to produce a shared outcome for all agents (Kilgour 2010; Zwicker 2016; Faliszewski et al. 2017; Lackner and Skowron 2023). Traditionally, voting has been studied in *discrete* or *indivisible* settings. In a typical voting model, there is a set of candidates, and each voter submits a ballot, which may take the form of a ranking or an approval set, among others. The objective of a voting rule is to aggregate these ballots into a collective choice, for instance, a single candidate (*single-winner voting*) or a subset of candidates (*multiwinner voting*). Examples include electing a public official, appointing committee members within an organization, or selecting a venue for a graduation party.

While multiwinner voting typically involves selecting a predetermined number of candidates, an important extension is *participatory budgeting*, where candidates (i.e., projects) may have differing costs (Aziz and Shah 2021). Participatory budgeting allows residents of a city or community to vote on the public projects to receive funding, and has been adopted in numerous countries. Naturally, the total cost of the selected projects must not exceed the available budget. Much of the work on participatory budgeting also deals with the discrete setting, where each project is implemented either fully or not at all (Rey, Schmidt, and Maly 2025).

The emphasis of the voting literature on discrete settings stands in contrast to *fair division*, another major topic in computational social choice (Thomson 2016). Fair division studies the aggregation of individual preferences into *private* outcomes, such as the assignment of resources or tasks that each agent receives privately. The allocation of *continuous* or *divisible* resources has long been a central topic in fair division, a problem also known as *cake cutting* (Procaccia 2016). The cake serves as a metaphor for any divisible resource such as land or time, and the objective is to ensure fairness with respect to the agents' preferences. Some of the most notable results in fair division pertain to cake cutting, for example, the existence of a bounded protocol for finding an *envy-free* cake division, where no agent prefers another agent's piece over her own (Aziz and Mackenzie 2016).

Although voting may appear more naturally suited to discrete settings at first glance, a number of voting applications in fact involve divisibility and have received increasing attention in recent years. For example, consider a workshop organizer tasked with dividing time between presentations, discussions, and social events. Participants may be asked to vote on their preferred time distribution, and the organizer would then aggregate these preferences into a final schedule. This type of problem is known as *budget aggregation*. More generally, the aggregation of preferences into a distribution is referred to as *fair mixing*, with the case of approval preferences (i.e., each voter approves each candidate or not) receiving particular attention. Another divisible voting model is *cake sharing*, where there is a heterogeneous cake and only a subset of the cake of a specified size can be chosen. As in cake cutting, the cake can represent time or space, which must be reserved for a group of agents to share.

In this survey, we provide an overview of voting in divisible settings, with an emphasis on recent developments in this growing area of research. Besides summarizing key results for each fundamental model, we explore the connections among these models, highlight unifying themes, and identify directions for future research in the area.

2 Budget Aggregation

In addition to modeling the allocation of time as discussed earlier, *budget aggregation* can also represent how funds are distributed across various public projects. From this viewpoint, it can be seen as a divisible variant of participatory budgeting (Aziz and Shah 2021, Sec. 4).

We now describe the formal model of budget aggregation. Let $N = [n]$ be the set of *agents*, where we use $[z]$ to denote $\{1, 2, \ldots, z\}$ for each positive integer z. There is a fixed budget to be distributed among a set $C = \{c_1, c_2, \ldots, c_m\}$ of *candidates*. By scaling if necessary, we may assume without loss of generality that the budget is 1. Denote by Δ^m the set of all m-tuples $\mathbf{x} = (x_1, \ldots, x_m)$ such that $x_j \geq 0$ for each $j \in [m]$ and $\sum_{j\in[m]} x_j = 1$. Each agent $i \in N$ submits her *preferred distribution* $\mathbf{s}_i = (s_{i,1}, \ldots, s_{i,m}) \in \Delta^m$ of the budget, and has a *utility function* $u_i : \Delta^m \to \mathbb{R}$ such that for each $\mathbf{x} \in \Delta^m$, the value $u_i(\mathbf{x})$ indicates agent i's satisfaction with the distribution $\mathbf{x}$. The most common way to define an agent's utility function is to use the ℓ_1 distance from her preferred distribution, i.e., $u_i(\mathbf{x}) := -\|\mathbf{x} - \mathbf{s}_i\|_1 = -\sum_{j\in[m]} |x_j - s_{i,j}|$ for any $i \in N$ and $\mathbf{x} \in \Delta^m$. An *instance* $\mathcal{I}$ consists of the candidates along with the agents and their preferred distributions. An *aggregation rule* F maps each instance $\mathcal{I}$ to an outcome distribution $F(\mathcal{I}) \in \Delta^m$.

What properties might one want an aggregation rule to satisfy? One of the most frequently studied properties is *truthfulness*, also known as *strategyproofness*. An aggregation rule is said to be truthful if agents can never benefit by misreporting their preferences. More formally, for every pair of instances $\mathcal{I}$ and $\mathcal{I}'$ that differ only in the preferred distribution of a single agent i, truthfulness requires that $u_i(F(\mathcal{I})) \geq u_i(F(\mathcal{I}'))$, where u_i is agent i's utility function in instance $\mathcal{I}$. Achieving truthfulness alone is trivial—for example, any rule that simply outputs a fixed distribution regardless of the input is truthful. However, such a rule is clearly undesirable, as it does not take into account the agents' preferences.

As a more sensible aggregation rule, Lindner, Nehring, and Puppe (2008) and Goel et al. (2019) considered the *utilitarian rule* (UTIL), which returns the distribution that maximizes the utilitarian welfare, defined as the sum of the agents' utilities, subject to a certain tie-breaking choice.[1] These authors showed that under ℓ_1 utilities, UTIL is a truthful aggregation rule. However, Freeman et al. (2021) observed that UTIL has a tendency to overweight majority preferences. As an example, consider an instance with $m = 2$ candidates and $n = 99$ agents such that 50 agents prefer the distribution $(1, 0)$ and 49 agents prefer the distribution $(0, 1)$. For this instance, UTIL returns the distribution $(1, 0)$, which leaves almost half of the agents maximally unhappy.

In an effort to obtain truthful aggregation rules that are "more proportional", Freeman et al. (2021) introduced the class of *moving-phantom rules*. To understand these rules, it is instructive to focus first on the case with only $m = 2$ candidates. In this special case, each distribution $(x_1, x_2) = (x_1, 1 - x_1)$ can be specified by a single number x_1. Accordingly, budget aggregation with $m = 2$ is equivalent to a basic model of *facility location*, where each agent submits a preferred location within the interval $[0, 1]$ and an aggregation rule decides on the outcome location (Chan et al. 2021). Given this equivalence, ℓ_1 preferences belong to the domain of *single-peaked preferences*, a well-studied preference domain in the voting literature. Moulin (1980) introduced a class of rules that, for any set of n values (i.e., locations) reported by the agents, add a fixed set of $n + 1$ "phantoms" and return the median of the resulting $2n + 1$ values. For example, if n is odd and $(n + 1)/2$ phantoms are placed at 0 while the remaining $(n + 1)/2$ phantoms are placed at 1, then the rule simply returns the median of the n original values. Moulin showed that these "phantom rules" are truthful under single-peaked preferences, and moreover, under mild assumptions, they are the only truthful rules.

When $m \geq 3$, inserting fixed phantoms no longer works, as it may lead to outcomes that violate the unit-budget constraint. To address this, the moving-phantom rules proposed by Freeman et al. (2021) replace the $n + 1$ fixed phantoms with $n + 1$ "phantom functions" $f_0, \ldots, f_n : [0, 1] \to [0, 1]$, each weakly increasing and satisfying $f_k(0) = 0$ and $f_k(1) = 1$ for all $k \in \{0, 1, \ldots, n\}$.[2] Given an instance with preferred distributions $(\mathbf{s}_i)_{i\in N}$, a moving-phantom rule finds a value $t^* \in [0, 1]$ such that the sum of the coordinate-wise medians is equal to 1, i.e.,

$$\sum_{j\in[m]} \text{med}(s_{1,j}, \ldots, s_{n,j}, f_0(t^*), \ldots, f_n(t^*)) = 1.$$

The rule then returns the distribution $\mathbf{x}$ given by $x_j = \text{med}(s_{1,j}, \ldots, s_{n,j}, f_0(t^*), \ldots, f_n(t^*))$ for each $j \in [m]$. By definition, $\mathbf{x}$ is a valid distribution, and although multiple values of t^* may exist, every such value yields the same distribution $\mathbf{x}$. An important moving-phantom rule put forward by Freeman et al. is the *independent markets* (IM) rule, defined by setting $f_k(t) = \min(kt, 1)$ for all $k \in \{0, \ldots, n\}$ and $t \in [0, 1]$. Figure 1 illustrates an example of IM applied to an instance with $n = m = 3$.

Remarkably, Freeman et al. (2021) proved that, regardless of the phantom functions, a moving-phantom rule always ensures truthfulness. The intuition behind the proof is as follows. When an agent i reports a different distribution $\widehat{\mathbf{s}}_i$ instead of her true preferred distribution $\mathbf{s}_i$, the effect on the aggregate distribution returned by a moving-phantom rule can be decomposed into two stages. In the first stage, we hold the phantoms fixed as in the truthful instance, and update the medians according to i's report $\widehat{\mathbf{s}}_i$. In the second stage, we update the phantoms to account for the change from $\mathbf{s}_i$ to $\widehat{\mathbf{s}}_i$. Freeman et al. showed that the first-stage effect can only hurt agent i, and while the second-stage effect could potentially benefit the agent, the magnitude of the latter effect never exceeds that of the former.

[1] Namely, among all distributions that maximize the utilitarian welfare, UTIL outputs the distribution $\mathbf{x}$ that minimizes the ℓ_2 distance to the uniform distribution $(1/m, \ldots, 1/m)$. This is equivalent to maximizing the entropy of the distribution.

[2] In fact, Freeman et al. (2021) showed that it is sufficient that there exists a permutation $\sigma : \{0, \ldots, n\} \to \{0, \ldots, n\}$ such that $f_{\sigma(k)}(1) \geq k/n$ for all $k \in \{0, \ldots, n\}$; see their Proposition 3 and the remarks thereafter.

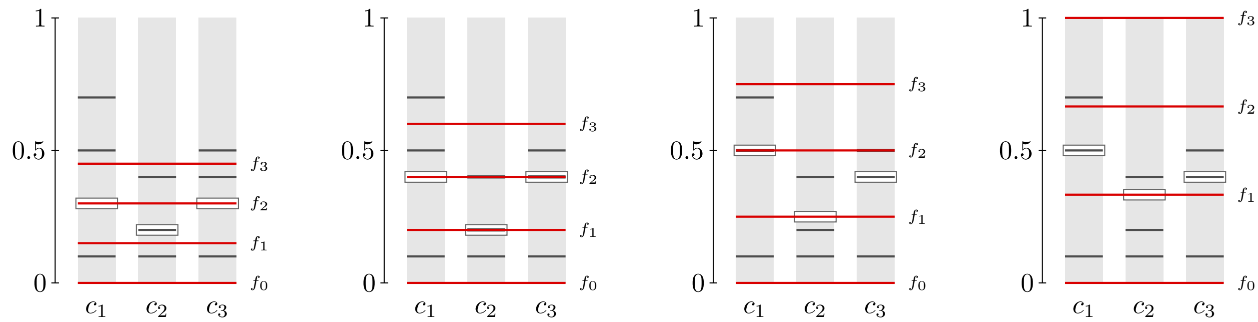

Figure 1: An illustration of the independent markets (IM) moving-phantom rule on an instance with $n = m = 3$. The agents' preferred distributions are $(0.7, 0.2, 0.1)$, $(0.5, 0.1, 0.4)$, and $(0.1, 0.4, 0.5)$. Each column corresponds to a candidate, the black bars indicate the agents' preferred amounts on that candidate, the red horizontal lines indicate the phantom functions $f_0, \dots, f_3$ parameterized by t, and the white rectangles denote the resulting medians in the respective columns. The values of t in the four snapshots are $0.15, 0.2, 0.25$, and $1/3$, respectively. Normalization occurs at $t^* = 0.2$ (second snapshot), and IM returns the distribution $\mathbf{x} = (0.4, 0.2, 0.4)$.

Theorem 1 (Freeman et al. 2021)**.** *In budget aggregation under ℓ_1 utilities, every moving-phantom rule is truthful.*

Moving-phantom rules form a large class of truthful rules. Within this class, Freeman et al. (2021) identified IM as a particularly appealing rule. Indeed, IM satisfies a fairness notion called *single-minded proportionality*. This means that in any "single-minded instance"—that is, an instance where each agent's preferred distribution assigns the entire budget of 1 to some candidate—the rule returns the average of the agents' preferred distributions.[3] Moreover, the IM outcome can be interpreted as clearing prices in a market system, and as the unique equilibrium of a natural voting game. These interpretations allowed Freeman et al. to demonstrate that IM satisfies classic social choice properties such as *participation*—an agent cannot become worse off by participating in the rule—and *reinforcement*—if the rule returns the same distribution for two instances, then it also returns this distribution when both instances are combined.

Besides truthfulness and fairness, another fundamental desideratum of aggregation rules is economic efficiency. This is often captured by *Pareto optimality*, which states that, compared to the outcome of the rule, no other outcome makes at least one agent better off and none of the agents worse off. Freeman et al. (2021) showed that not only does IM fail Pareto optimality, but there is also an inherent tradeoff between Pareto optimality and single-minded proportionality, at least among moving-phantom rules. In particular, the only moving-phantom rule that satisfies Pareto optimality is UTIL, which can be formulated in terms of moving phantoms by letting the phantoms move from 0 to 1 one after another, with each phantom leaving 0 only after the previous one has reached 1. Brandt et al. (2026) extended the incompatibility between truthfulness, single-minded proportionality, and Pareto optimality beyond moving-phantom rules. In fact, their impossibility also holds for ℓ_∞ preferences, where $u_i(\mathbf{x}) := -\|\mathbf{x} - \mathbf{s}_i\|_\infty = -\max_{j\in[m]} |x_j - s_{i,j}|$.

Theorem 2 (Brandt et al. 2026)**.** *In budget aggregation under ℓ_1 or ℓ_∞ utilities, for any $n, m \geq 3$, no rule satisfies truthfulness, Pareto optimality, and single-minded proportionality.*

A stronger version of truthfulness is *group-truthfulness*, which demands that no group of agents can misreport their preferences in such a way that all agents in the group benefit. While the phantom rules of Moulin (1980) for $m = 2$ are group-truthful, Freeman et al. (2021) observed that moving-phantom rules do not satisfy this property in general. This suggests the following question.

Open problem 1. *In budget aggregation under ℓ_1 utilities, does there exist a rule that satisfies group-truthfulness and single-minded proportionality?*

In an effort to capture proportionality beyond single-minded instances, Caragiannis, Christodoulou, and Protopapas (2024) considered the ℓ_1 *disproportionality*, defined as the maximum ℓ_1 distance between the distribution output by a rule on some instance and the average distribution of that instance, where the maximum is taken across all instances.[4] Caragiannis et al. showed that for each $m \geq 2$, the ℓ_1 disproportionality of any moving-phantom rule is at least $1 - 1/m$, and for $m = 2$, IM achieves the tight bound of $1/2$. In addition, the ℓ_1 disproportionality of any truthful rule is at least $1/2$, so the IM guarantee is again tight. However, when $m = 3$, these authors proved that the ℓ_1 disproportionality of IM is higher than 0.68, which does not match the lower bound of $2/3$ for moving-phantom rules. They therefore proposed a different moving-phantom rule called the *piecewise uniform* rule, and showed that the ℓ_1 disproportionality of this rule is $2/3 + \varepsilon$ for $\varepsilon \leq 10^{-5}$, with the error term arising from solving a non-linear program.

Another interesting moving-phantom rule in the literature is the *ladder* rule, introduced by Freeman and Schmidt-Kraepelin (2024). This rule works by setting $f_k(t) =$

[3] Freeman et al. (2021) simply called this notion "proportionality". However, as the notion only applies to single-minded instances, it is rather weak compared to proportionality notions in other contexts such as fair division (Procaccia 2016).

[4] Caragiannis et al. (2024) called this measure the ℓ_1-*loss*.

$\max(t - k/n, 0)$ for each $k \in \{0, \ldots, n\}$ and $t \in [0,1]$. Intuitively, one can think of the phantoms being uniformly spread in the interval $[-1, 0]$ initially and, as t increases, pulled upwards with equal speed until they are uniformly spread in $[0, 1]$. Freeman and Schmidt-Kraepelin showed that the ladder rule attains the optimal ℓ_∞ *disproportionality*—defined in an analogous manner as ℓ_1 disproportionality—among all moving-phantom rules; in particular, this disproportionality is $1/2 - 1/(2m)$. Moreover, these authors proved that the rule also yields the optimal ℓ_1 disproportionality of $2/3$ for $m = 3$, thereby closing the tiny gap left by Caragiannis, Christodoulou, and Protopapas (2024). Nevertheless, the optimal disproportionality in general remains intriguingly open.

Open problem 2. *In budget aggregation under ℓ_1 utilities, for each $m \geq 4$, what is the optimal ℓ_1 disproportionality among moving-phantom rules? For each $m \geq 3$, what is the optimal ℓ_1 or ℓ_∞ disproportionality among truthful rules?*

When $m = 2$, subject to mild conditions, phantom rules constitute the only truthful rules (Moulin 1980; Massó and de Barreda 2011; Freeman et al. 2021; Brandt et al. 2026). Are moving-phantom rules the only truthful rules for any m as well? This question was answered negatively by de Berg et al. (2024), who exhibited a rule that is truthful, continuous, anonymous, and neutral,[5] but is not a moving-phantom rule for any $m \geq 3$ and $n \geq 1$.[6] The same authors also extended the lower bound of $1 - 1/m$ (resp., $1/2 - 1/(2m)$) for the ℓ_1 (resp., ℓ_∞) disproportionality from moving-phantom rules to all truthful, continuous, anonymous, and neutral rules. Their work leaves the fascinating question of whether one can characterize the space of truthful rules.

Open problem 3. *In budget aggregation under ℓ_1 utilities, does there exist a characterization of all rules that satisfy truthfulness (possibly together with all or some of continuity, anonymity, and neutrality)?*

Despite its importance, truthfulness is far from the only desirable property in budget aggregation. Elkind et al. (2024) conducted an axiomatic analysis of aggregation rules with respect to several properties. They found that the *average rule*, which simply returns the average of the agents' preferred distributions, satisfies most of the properties despite failing truthfulness. These authors further cemented the case in favor of the average rule by providing two characterizations of it. Wagner and Meir (2024) adapted Thiele's rules from multiwinner voting to the continuous setting, and derived bounds on the welfare and fairness of the resulting rules. Garg et al. (2019) assumed ℓ_p utilities and proposed an algorithm that iteratively allows agents to move the current distribution towards their preferred distribution.

While ℓ_p utilities are natural and have received due interest from researchers, they do not represent agents' preferences in certain applications. As an example, for any $p \geq 1$, an agent who prefers a $(10\%, 40\%, 50\%)$ distribution of time among three countries at an international conference finds the distribution $(0\%, 45\%, 55\%)$ to be as good as $(5\%, 50\%, 45\%)$, even though the former distribution leaves the first country unrepresented. In light of this, Brandt et al. (2026) considered *Leontief utilities* drawn from the economics literature, where each agent's utility for a distribution is the minimum ratio among all candidates that the distribution preserves compared to the agent's preferred distribution—formally, $u_i(\mathbf{x}) := \min_{j \in [m]} x_j / s_{i,j}$. Brandt et al. showed that the *Nash product rule*, which chooses a distribution maximizing the agents' utilities, satisfies group-truthfulness, single-minded proportionality, and Pareto optimality. An important future direction is to identify and examine further utility domains in budget aggregation. Amster et al. (2025) empirically tested the utility functions of human participants, while Becker et al. (2025) studied the computation of equilibria under various utility models.

We end this section by discussing a few related lines of work. Intriligator (1973), Fishburn (1975), and Rice (1977) addressed a probabilistic social choice setting, where individual probability distributions are aggregated into a social probability distribution. However, these authors did not assume that agents have utility functions, so properties such as truthfulness were not considered.[7] Intriligator provided a characterization of the average rule, which was later corrected by Rice. Other similar models include probabilistic opinion pooling (Genest and Zidek 1986; Clemen 1989) and belief aggregation (Varloot and Laraki 2022). Specifically, in the model of Varloot and Laraki, the candidates are linearly ordered: for example, candidate c_j may represent an earthquake of magnitude j. Then, an agent who predicts the outcome c_3 with probability 1 would prefer the outcome c_5 with probability 1 to the outcome c_7 with probability 1, even though the ℓ_p distance is the same in both cases.

3 Fair Mixing

In budget aggregation, both the input and output are in the form of distributions. However, the most common input formats in voting are *approval* preferences (also known as *dichotomous* or *binary* preferences) and *ranked* preferences (also known as *ordinal* preferences). The problem of computing a desirable output distribution in this setting has been referred to as *fair mixing* or *portioning*, with the case of approval preferences receiving significant attention. Approval preferences are not only simple—since each agent only needs to specify whether she approves each candidate or not—but are also expressive, allowing agents to approve as many (or as few) candidates as they wish.

In the model of fair mixing with approval preferences, each agent $i \in N$ submits her approval set $A_i \subseteq C$, which indicates the set of candidates that she approves. Agent i's utility for a distribution $\mathbf{x} \in \Delta^m$ is defined as $u_i(\mathbf{x}) := \sum_{j \,:\, c_j \in A_i} x_j$, that is, the total amount that $\mathbf{x}$ assigns to candidates approved by i. We remark that this model can be seen as a variant of budget aggregation under ℓ_1 utilities. To see this, observe that ℓ_1 utilities are equivalent to *overlap utilities* given by $u_i(\mathbf{x}) := \sum_{j \in [m]} \min(x_j, s_{i,j})$

[5] A rule is *anonymous* (resp., *neutral*) if its outcome does not depend on the identity of the agents (resp., candidates).

[6] Brandt et al. (2026) also proposed such a rule, though their rule coincides with a moving-phantom rule when $n > 1$.

[7] Intriligator (1973) considered a property called "Pareto optimality", but his definition is very different from the standard usage.

(Goel et al. 2019). Indeed, this equivalence holds because $\sum_{j\in[m]} |x_j - s_{i,j}| = 2 - 2\sum_{j\in[m]} \min(x_j, s_{i,j})$, which follows from the relation

$$\sum_{j\in[m]} |x_j - s_{i,j}| + 2\sum_{j\in[m]} \min(x_j, s_{i,j}) = \sum_{j\in[m]} (x_j + s_{i,j}) = \sum_{j\in[m]} x_j + \sum_{j\in[m]} s_{i,j} = 2.$$

Therefore, one can view fair mixing as a version of budget aggregation that lets each agent i choose a subset of candidates fully (i.e., set $s_{i,j} = 1$ for such candidates c_j) and derive utility $\min(x_j, s_{i,j}) = x_j$ for these candidates.

Bogomolnaia, Moulin, and Stong (2005) initiated the study of fair mixing with approval preferences. They introduced the concept of *individual fair share* (IFS),[8] which requires each of the n agents to receive utility at least $1/n$. These authors showed that no rule can simultaneously satisfy anonymity, neutrality, truthfulness, Pareto optimality, and IFS. However, they proved that the utilitarian rule satisfies all of these properties except IFS, whereas the *random priority* rule, which selects an ordering of the agents uniformly at random and maximizes the utilities lexicographically, satisfies a weaker version of Pareto optimality along with the four remaining properties.

Duddy (2015) subsequently noted that, while IFS guarantees fairness for individual agents, it does not do the same for groups. For example, consider an instance with $m = 2$ candidates and $n = 100$ agents, where 51 agents approve only c_1 and 49 approve only c_2. The outcome that assigns 0.99 to c_1 and 0.01 to c_2 satisfies IFS, but is intuitively unfair to the latter group. To address this, Duddy proposed a strengthening of IFS called *group fair share* (GFS),[9] which requires that for any subset of agents $N' \subseteq N$, the candidates approved by at least one member of N' must collectively receive a total amount of at least $|N'|/n$ in the outcome. Duddy showed that GFS is compatible with anonymity, neutrality, truthfulness, and Pareto optimality if and only if $n \leq 4$ or $m \leq 3$. Michorzewski, Peters, and Skowron (2020) quantified the loss of utilitarian welfare incurred by imposing fairness constraints such as IFS and GFS using the "price of fairness" framework.[10] Tang, Wang, and Zhang (2020) extended this analysis to egalitarian welfare, defined as the minimum utility obtained by any agent.

Aziz, Bogomolnaia, and Moulin (2020) highlighted three rules with notable axiomatic guarantees in this model, all of which satisfy anonymity, neutrality, and IFS. Firstly, the *egalitarian rule* (EGAL) selects a distribution that maximizes the minimum utility among all agents. If multiple such distributions exist, it breaks ties in a *leximin* manner: by maximizing the second-smallest utility, then the third-smallest, and so on. In addition to being Pareto optimal, EGAL satisfies a weakening of truthfulness known as *excludable truthfulness*, which ensures that an agent cannot benefit by misreporting her preference assuming that she is excluded from deriving utility from candidates she reportedly disapproves. Secondly, under the *conditional utilitarian rule* (CUT), each agent considers her approved candidates that are also approved by the largest number of other agents, and splits her "share" of $1/n$ uniformly among these candidates. CUT satisfies truthfulness and GFS, but fails Pareto optimality. Lastly, the *maximum Nash welfare rule* (MNW) returns a distribution that maximizes the product of the agents' utilities, a quantity known as Nash welfare.[11] While MNW satisfies Pareto optimality as well as strong fairness properties including GFS, it violates excludable truthfulness.

The trade-off between efficiency, truthfulness, and fairness is evident in the results of Bogomolnaia, Moulin, and Stong (2005) and Aziz, Bogomolnaia, and Moulin (2020). Brandl et al. (2021) strengthened the impossibility result of Bogomolnaia et al. by showing that IFS can be weakened to *positive share*, which only requires each agent to receive nonzero utility, and that anonymity and neutrality can be dropped altogether.

Theorem 3 (Brandl et al. 2021)**.** *In fair mixing with approval preferences, no rule satisfies truthfulness, Pareto optimality, and positive share.*

Interestingly, the proof of this result was obtained using a computer-aided technique and involves analyzing 386 instances.[12] The result holds for $n \geq 6$ agents and $m \geq 4$ candidates, and both bounds are tight. When anonymity and neutrality are assumed, Brandl et al. (2021) provided a simpler proof for the case of $n = 5$ and $m = 4$.

A model closely related to fair mixing is *donor coordination*, in which the resources to be distributed (e.g., money) are owned by the agents themselves (Brandl et al. 2022; Brandt et al. 2025). In this setting, a key goal of a distribution rule is to incentivize agents to contribute their resources to the shared pool rather than spending them on their own. Notably, an agent who withholds her contribution can still derive utility from the allocation of resources to her approved candidates. Brandl et al. (2022) formalized this goal of a distribution rule via the notion of *contribution incentive-compatibility*, which requires that for each agent, contributing her resource to the pool and benefiting from the distribution returned by the rule (with her preference) yields no less utility than spending her resource on her own and benefiting from the distribution that the rule chooses (without her preference). As their main result, Brandl et al. showed that MNW satisfies contribution incentive-compatibility. Since MNW violates truthfulness, or even its weakening called *monotonicity*—which demands that if an agent approves an additional project, the amount given to that project should not decrease—this raises the following question.

Open problem 4. *In fair mixing with approval preferences and private endowments, does there exist a rule that satisfies*

[8] Referred to as the *fair welfare share* in their original work.

[9] Duddy referred to this as the *proportional share*. Note that IFS imposes the same requirement only in the case where $|N'| = 1$.

[10] The price of fairness has been extensively studied in fair division, with respect to both utilitarian and egalitarian welfare (Caragiannis et al. 2012; Aumann and Dombb 2015; Bei et al. 2021).

[11] Aziz, Bogomolnaia, and Moulin (2020) referred to this rule as *max Nash product*.

[12] For an overview of computer-aided methods in social choice, we refer to the survey by Geist and Peters (2017).

Pareto optimality, monotonicity, and contribution incentive-compatibility?

Aziz et al. (2025) introduced the "maximum payment rule", which allows each agent to control an equal part of the budget. In each step, the rule identifies a candidate c_j approved by the maximum number of agents who have not spent their budget, and allocates the entire budget of these agents to c_j. Aziz et al. showed that the maximum payment rule satisfies consistency and monotonicity properties along with approximate fairness guarantees. Moreover, they investigated a broader class of sequential payment rules.

Besides approval preferences, fair mixing has also been studied in the context of ranked preferences. Airiau et al. (2023) considered a setting in which each agent submits a strict ranking over the candidates. They introduced "positional social decision schemes", which are rules that convert each input ranking into candidate scores and then select a distribution that maximizes a welfare function based on these scores. In addition to analyzing standard social choice properties, Airiau et al. examined computational aspects of these rules and identified score conversion schemes that guarantee rational-valued outcome distributions. With ranked preferences, the setting is formally equivalent to *probabilistic social choice*, a rich and well-established domain by itself (Brandt 2017). However, in probabilistic social choice, the outcome distribution is viewed as a random device for implementing a single outcome. As such, it is desirable for a rule to randomize as little as possible, and fairness notions become less relevant.

Finally, we remark that some authors have explored fair mixing beyond approval or ranked preferences (Fain, Goel, and Munagala 2016; Kroer and Peters 2025).

4 Cake Sharing

In both budget aggregation and fair mixing, there is a discrete set of candidates $C = \{c_1, \dots, c_m\}$. By contrast, in *cake sharing*, which is a public-good counterpart of cake cutting (cf. Section 1), the candidate itself is continuous. As is standard in cake cutting, this candidate is represented by the unit interval $C = [0, 1]$.

The model of cake sharing was introduced by Bei, Lu, and Suksompong (2025), who studied it for approval preferences, also known as "piecewise uniform utilities" in the cake-cutting literature. Specifically, each agent $i \in N$ submits an approval set $A_i \subseteq [0, 1]$ corresponding to a *piece of cake*, i.e., a union of finitely many intervals. Moreover, there is a given parameter $\alpha \in (0, 1)$. An aggregation rule maps any approval instance to a piece of cake of length at most α. The utility of an agent i for a piece of cake S is $u_i(S) := \ell(S \cap A_i)$, where $\ell(\cdot)$ denotes the length function. Observe that cake sharing is a generalization of fair mixing (with approval preferences). Indeed, one can let the interval $[(j-1)/m, j/m]$ represent candidate c_j for each $j \in [m]$, require each agent to approve any such interval either in its entirety or not at all, and set $\alpha = 1/m$ to ensure that the chosen fractions of the intervals sum up to 1. Similarly, cake sharing also generalizes budget aggregation (with ℓ_1 preferences), as we can let agent i approve the first $s_{i,j}$-fraction of the interval $[(j-1)/m, j/m]$; this allows us to assume that the output cake contains a prefix of each of these intervals.

Bei, Lu, and Suksompong (2025) considered both EGAL and MNW, defined analogously to their counterparts in fair mixing (Section 3). Both rules are Pareto optimal by definition. For EGAL, Bei et al. extended the fact that it satisfies excludable truthfulness from fair mixing to cake sharing.

Theorem 4 (Bei, Lu, and Suksompong 2025)**.** *In cake sharing with approval preferences,* EGAL *satisfies excludable truthfulness.*

As for MNW, the result of Aziz, Bogomolnaia, and Moulin (2020) from fair mixing implies that it violates excludable truthfulness in cake sharing. Bei et al. strengthened this result by showing that the violation remains even if each agent is only allowed to report a subset of her true desired piece. In contrast, MNW fares better than EGAL when it comes to fairness: it satisfies the notion of *average fair share* (AFS), which is stronger than the natural adaptation of *extended justified representation* (EJR) from multiwinner voting. Moreover, it is the only rule within the class of "welfare-maximizer rules" that satisfies either of these notions.

A promising future avenue is to explore cake sharing beyond approval preferences. In this context, a particularly intriguing fairness notion to investigate is the *core*. An outcome belongs to the core if no subset of agents $N' \subseteq N$ can choose a piece of cake of length $\alpha \cdot |N'|/n$ in such a way that no agent in the subset becomes worse off and at least one agent becomes better off. In fair mixing with approval preferences, Aziz, Bogomolnaia, and Moulin (2020) showed that MNW always returns an outcome in the core.

Open problem 5. *In cake sharing (not necessarily with approval preferences), is the core always non-empty?*

Lu et al. (2024) proposed a more general model which combines cake sharing with multiwinner voting by allowing both divisible and indivisible candidates. Again focusing on approval preferences, they presented adaptations of the EJR notion and multiwinner voting rules to their setting.

5 Conclusion

As we have seen throughout this survey, despite the predominant emphasis of the voting literature on discrete scenarios, voting in divisible settings constitutes a rich and distinct domain in its own right. The models of budget aggregation, fair mixing, and cake sharing are each fundamental and deserving of further study. In light of the relationships among these models, results established in one model can sometimes be translated to corresponding results in the others.

A recurring direction for future research is to investigate broader preference classes, such as those beyond approval or ℓ_1 preferences. Another common consideration in practical applications is the presence of constraints—for instance, there may be limits on the amount that each candidate can receive (Suzuki and Vollen 2024; Kroer and Peters 2025). Last but not least, it would be valuable to explore how insights from continuous contexts could be applied to discrete domains; an example is the adaptation of moving-phantom rules (Freeman et al. 2021) to discrete budget aggregation (Schmidt-Kraepelin, Suksompong, and Utke 2025).

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