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Qualitative Spatial Logic over 2D Euclidean Spaces Is Not Finitely Axiomatisable

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Abstract

Several qualitative spatial logics used in reasoning about geospatial data have a sound and complete axiomatisation over metric spaces. It has been open whether the same axiomatisation is also sound and complete for 2D Euclidean spaces. We answer this question negatively by showing that the axiomatisations presented in (Du et al. 2013; Du and Alechina 2016) are not complete for 2D Euclidean spaces and, moreover, the logics are not finitely axiomatisable.

1 Introduction

The spatial logics LNF (a logic of NEAR and FAR for buffered points), LNFS (a logic of NEAR and FAR for buffered geometries), LBPT (a logic of part and whole for buffered geometries) were introduced in (Du et al. 2013; Du and Alechina 2016) to aid in debugging matches between different geospatial datasets (Du et al. 2015). They are sound and complete for metric spaces. The axiomatisation was used to develop a reasoner which together with a truth maintenance system could pinpoint minimal reasons for contradictions. Contradictions arise from interactions between two kinds of statements: statements that two geospatial objects are the same, and statements that two objects in the same dataset are near each other or far from each other. The first kind of statements are treated as assumptions that can be withdrawn if they lead to inconsistency. For example, suppose that in the process of matching two datasets S_1 and S_2 two assumptions are generated, that a_1 in S_1 is the same as a_2 in S_2, and b_1 in S_1 is the same as b_2 in S_2. A contradiction would arise if a_1 and b_1 are near each other in S_1 and a_2 and b_2 are far from each other in S_2. The notions of near and far are expressed in a qualitative language where the semantics uses a metric distance measure and a positive constant σ to define what it means to be near (within 2σ distance) and far (more than 4σ). The intuition behind the semantics is that errors of measurement may position the same object in two different datasets in different locations, but the difference between these two locations should be within some margin of error σ (depending on the precision of positioning available to the dataset creators). Assumptions of matching are generated only for objects which are within σ distance of each other. Objects are defined to be near if they can conceivably be in the same location (if both are shifted towards each other by at most σ) and are far if even after such shifting they are not near each other. In spite of somewhat arbitrary definitions, the resulting qualitative spatial logic was successfully used for debugging matches as reported in (Du et al. 2015).

However, 2D Euclidean spaces are a much more appropriate semantics for geospatial data, as geospatial data is usually represented using geometries or coordinates and visualized as a map. The question is open whether a more precise debugging tool could be developed using the same approach. In other words, is the axiomatisation presented in (Du et al. 2013; Du and Alechina 2016) still complete with respect to 2D Euclidean spaces, and if not, what are the missing axioms?

In this paper, we use recent results from graph theory (Atminas and Zamaraev 2018) to answer this question. Unfortunately, the answer is negative: with respect to 2D Euclidean spaces, a finite complete axiomatisation of LNF (LBPT, LNFS) does not exist.

2 Background

In (Du et al. 2013; Du and Alechina 2016), the languages $L(LNF)$ and $L(LNFS)$ are defined as

$$\phi, \psi := BEQ(a,b) \mid NEAR(a,b) \mid FAR(a,b) \mid \neg\phi \mid \phi \wedge \psi$$

where a, b are individual names. $\phi \rightarrow \psi \equiv_{def} \neg(\phi \wedge \neg\psi)$. BEQ stands for 'BufferedEqual'. $L(LBPT)$ is defined as

$$\phi, \psi := BPT(a,b) \mid NEAR(a,b) \mid FAR(a,b) \mid \neg\phi \mid \phi \wedge \psi$$

$L(LBPT)$ contains BPT instead of BEQ. BPT stands for 'BufferedPartOf'.

$L(LNF)$, $L(LNFS)$ and $L(LBPT)$ are all interpreted over models based on a metric space. Every individual name involved in an LNF formula is mapped to a point, whilst every individual name in an LNFS/LBPT formula is mapped to an arbitrary geometry or a non-empty set of points.

Definition 1 (Metric Space) *A* metric space *is a pair* (Δ, d)*, where* Δ *is a non-empty set (of points) and* d *is a metric on* Δ*, i.e. a function* $d : \Delta \times \Delta \longrightarrow \mathbb{R}_{\geq 0}$*, such that for any* $x, y, z \in \Delta$*, the following axioms are satisfied:*

1. *identity of indiscernibles:* $d(x, y) = 0$ *iff* $x = y$*;*

2. *symmetry:* $d(x, y) = d(y, x)$*;*
3. *triangle inequality:* $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2 (Metric Model of LNF) *A metric model M of LNF is a tuple (Δ, d, I, σ), where (Δ, d) is a metric space, I is an interpretation function which maps each individual name to an element in Δ, and $\sigma \in \mathbb{R}_{>0}$ is a margin of error. The notion of $M \models \phi$ (ϕ is true in the model M) is defined as follows:*

$M \models BEQ(a, b)$ *iff* $d(I(a), I(b)) \in [0, \sigma]$;
$M \models NEAR(a, b)$ *iff* $d(I(a), I(b)) \in [0, 2\sigma]$;
$M \models FAR(a, b)$ *iff* $d(I(a), I(b)) \in (4\sigma, \infty)$;
$M \models \neg\phi$ *iff* $M \not\models \phi$;
$M \models \phi \wedge \psi$ *iff* $M \models \phi$ *and* $M \models \psi$,

where a, b are individual names, ϕ, ψ are formulas in $L(LNF)$.

Definition 3 (Metric Model of LNFS/LBPT) *A metric model M of LNFS/LBPT is a tuple (Δ, d, I, σ), where (Δ, d) is a metric space, I is an interpretation function which maps each individual name to a non-empty set of elements in Δ, and $\sigma \in \mathbb{R}_{>0}$ is a margin of error. The notion of $M \models \phi$ is defined as follows:*

$M \models BPT(a, b)$ *iff*
$\forall p_a \in I(a)\ \exists p_b \in I(b) : d(p_a, p_b) \in [0, \sigma]$;
$M \models BEQ(a, b)$ *iff*
$\forall p_a \in I(a)\ \exists p_b \in I(b) : d(p_a, p_b) \in [0, \sigma]$ *and*
$\forall p_b \in I(b)\ \exists p_a \in I(a) : d(p_a, p_b) \in [0, \sigma]$;
$M \models NEAR(a, b)$ *iff*
$\exists p_a \in I(a)\ \exists p_b \in I(b) : d(p_a, p_b) \in [0, 2\sigma]$;
$M \models FAR(a, b)$ *iff*
$\forall p_a \in I(a)\ \forall p_b \in I(b) : d(p_a, p_b) \in (4\sigma, \infty)$,

where a, b are individual names, ϕ is a formula in $L(LNFS)$/$L(LBPT)$.

By Definition 3, $BEQ(a, b)$ can be defined as $BPT(a, b) \wedge BPT(b, a)$.

The notions of validity and satisfiability in metric models are standard. A formula is satisfiable if it is true in some metric model. A formula ϕ is valid ($\models \phi$) if it is true in all metric models (hence if its negation is not satisfiable). The logic LNF/LNFS/LBPT is the set of all valid formulas in the language $L(LNF)$/$L(LNFS)$/$L(LBPT)$ respectively.

The following calculus (also referred to as LNF) is sound and complete for LNF with respect to metric spaces (Du et al. 2013):

Axiom 0 All tautologies of classical propositional logic

Axiom 1 $BEQ(a, a)$;

Axiom 2 $BEQ(a, b) \to BEQ(b, a)$;

Axiom 3 $NEAR(a, b) \to NEAR(b, a)$;

Axiom 4 $FAR(a, b) \to FAR(b, a)$;

Axiom 5 $BEQ(a, b) \wedge BEQ(b, c) \to NEAR(c, a)$;

Axiom 6 $BEQ(a, b) \wedge NEAR(b, c) \wedge BEQ(c, d) \to \neg FAR(d, a)$;

Axiom 7 $NEAR(a, b) \wedge NEAR(b, c) \to \neg FAR(c, a)$;

MP Modus ponens: $\phi,\ \phi \to \psi \vdash \psi$.

The following calculus (also referred to as LNFS) is sound and complete for LNFS with respect to metric spaces (Du and Alechina 2016):

Axiom 0 – Axiom 6, MP as above

Axiom 8 $NEAR(a, b) \wedge BEQ(b, c) \wedge BEQ(c, d) \to \neg FAR(d, a)$.

The following calculus (also referred to as LBPT) is sound and complete for LBPT with respect to metric spaces (Du and Alechina 2016):

Axioms 0, 3, 4, MP as above

Axiom 9 $BPT(a, a)$;

Axiom 10 $BPT(a, b) \wedge BPT(b, c) \to NEAR(c, a)$;

Axiom 11 $BPT(b, a) \wedge BPT(b, c) \to NEAR(c, a)$;

Axiom 12 $BPT(b, a) \wedge NEAR(b, c) \wedge BPT(c, d) \to \neg FAR(d, a)$;

Axiom 13 $NEAR(a, b) \wedge BPT(b, c) \wedge BPT(c, d) \to \neg FAR(d, a)$.

The notion of derivability $\Gamma \vdash \phi$ in LNF/LNFS/LBPT calculus is standard. A formula ϕ is derivable if $\vdash \phi$. A set Γ is LNF/LNFS/LBPT-inconsistent if for some formula ϕ it derives both ϕ and $\neg\phi$.

3 2D Euclidean Semantics

In this section, we define the 2D Euclidean semantics for LNF, LNFS and LBPT. Compared to metric spaces, 2D Euclidean spaces are more appropriate for geospatial or map data, as the locations of spatial objects are usually described using coordinates or geometries.

Definition 4 (2D Euclidean Space) *The 2D Euclidean space $(\mathbb{R}^2, d)$ is the metric space over $\mathbb{R}^2$ such that for any pair of points $p = (p_x, p_y)$, $q = (q_x, q_y)$ of $\mathbb{R}^2$, $d(p, q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$.*

Definition 5 (2D Euclidean Model of LNF) *A 2D Euclidean model M of LNF is a tuple $(\mathbb{R}^2, d, I, \sigma)$, where $(\mathbb{R}^2, d)$ is a 2D Euclidean space, I is an interpretation function which maps each individual name to an element in $\mathbb{R}^2$, and $\sigma \in \mathbb{R}_{>0}$ is a margin of error. The notion of $M \models \phi$ is defined as in Definition 2.*

Definition 6 (2D Euclidean Model of LNFS/LBPT) *A 2D Euclidean model M of LNFS/LBPT is a tuple $(\mathbb{R}^2, d, I, \sigma)$, where $(\mathbb{R}^2, d)$ is a 2D Euclidean space, I is an interpretation function which maps each individual name to a non-empty set of elements in $\mathbb{R}^2$, and $\sigma \in \mathbb{R}_{>0}$ is a margin of error. The notion of $M \models \phi$ is defined as in Definition 3.*

The notions of validity and satisfiability in 2D Euclidean models are standard. With respect to 2D Euclidean models, the three calculi in Section 2 are sound but not complete. The proof of soundness is by an easy induction on the length of the derivation of ϕ. To prove each of these calculi is not complete, below we show that there exists a consistent formula with respect to that calculus that is not satisfied in any 2D

Euclidean model (so its negation is valid in 2D Euclidean models, but not derivable in the calculus).

In this paper, an angle refers to a positive angle greater than or equal to 0° and less than or equal to 180°.

Proposition 1 (from Euclid's Elements) *Let a, b, c be different points in a 2D Euclidean space that are not in the same straight line. If $d(a, c) > d(a, b)$, then the angle $\angle abc$ is greater than the angle $\angle acb$.*

Lemma 1 *Let a, b, c be different points in a 2D Euclidean space. If $d(a, b) \in [0, 1]$, $d(b, c) \in [0, 1]$ and $d(a, c) \in (1, \infty)$, then the angle $\angle abc > 60°$.*

Proof. If a, b, c are in the same straight line, then $\angle abc = 180° > 60°$, as $d(a, c) > 1$.

Suppose a, b, c are not in the same straight line. Since $d(a, c) > 1 \geq d(a, b)$, by Proposition 1, $\angle abc > \angle acb$. Similarly, $\angle abc > \angle bac$. Since $\angle abc + \angle acb + \angle bac = 180°$, $\angle abc > 60°$. □

Theorem 1 *There exists a finite consistent set of LNF formulas which are not satisfied in any 2D Euclidean model.*

Proof. Consider a set of LNF formulas Σ over 7 individual names $a_0, \ldots, a_6$. For every $i \in \mathbb{N}$, $1 \leq i \leq 6$, $BEQ(a_0, a_i) \in \Sigma$. For every $i \in \mathbb{N}$, $j \in \mathbb{N}$, $1 \leq i < j \leq 6$, $\neg BEQ(a_i, a_j) \in \Sigma$. No other formulas are in Σ. It is clear that no contradiction can be derived using the LNF axioms, hence Σ is consistent. Below we will show that Σ is not satisfied in any 2D Euclidean space by contradiction.

Suppose $I(a_0), \ldots, I(a_6)$ are points in a 2D Euclidean space. By Definition 5, for every $i \in \mathbb{N}$, $1 \leq i \leq 6$, we have $d(I(a_0), I(a_i)) \in [0, \sigma]$, and for every $i \in \mathbb{N}$, $j \in \mathbb{N}$, $1 \leq i < j \leq 6$, we have $d(I(a_i), I(a_j)) \in (\sigma, \infty)$. Hence, $I(a_0), \ldots, I(a_6)$ are different points. By Lemma 1, $\angle I(a_i)I(a_0)I(a_j) > 60°$ (σ is a scaling factor).

Consider relative positions of $I(a_0)$ and the other 6 points. The other 6 points should be located within a circle centered at $I(a_0)$ with radius σ. Consider any angle $\angle I(a_i)I(a_0)I(a_j)$, by Lemma 1, it should be greater than 60°. Each $\angle I(a_i)I(a_0)I(a_j)$ where there is no $I(a_k)$ such that $\angle I(a_i)I(a_0)I(a_k)$ is smaller, defines a segment of a circle centered at $I(a_0)$. The sum of the angles of all those segments is greater than 360°, which is impossible in a 2D Euclidean space. □

Theorem 2 *There exists a finite consistent set of LBPT formulas which are not satisfied in any 2D Euclidean model.*

Proof. Consider a set of LBPT formulas Σ over 13 individual names $a_0, \ldots, a_6, b_1, \ldots, b_6$. For every $i \in \mathbb{N}$, $1 \leq i \leq 6$, $BPT(a_0, b_i) \in \Sigma$, $BPT(b_i, a_i) \in \Sigma$. For every $i \in \mathbb{N}$, $j \in \mathbb{N}$, $1 \leq i < j \leq 6$, $\neg NEAR(a_i, a_j) \in \Sigma$. No other formulas are in Σ. It is clear that no contradiction can be derived using the LBPT axioms, hence Σ is consistent. Below we will show that Σ is not satisfied in any 2D Euclidean space by contradiction.

Suppose $I(a_0), \ldots, I(a_6), I(b_1), \ldots, I(b_6)$ are nonempty sets of points in a 2D Euclidean space. Take any point $p_0 \in I(a_0)$. Then for every $i \in \mathbb{N}$, $1 \leq i \leq 6$, since $BPT(a_0, b_i) \in \Sigma$, by Definition 6, there exists a point $q_i \in I(b_i)$ such that $d(p_0, q_i) \in [0, \sigma]$; and since $BPT(b_i, a_i) \in \Sigma$, there exists a point $p_i \in I(a_i)$ such that $d(q_i, p_i) \in [0, \sigma]$. Hence, $d(p_0, p_i) \in [0, 2\sigma]$. For every $i \in \mathbb{N}$, $j \in \mathbb{N}$, $1 \leq i < j \leq 6$, since $\neg NEAR(a_i, a_j) \in \Sigma$, by Definition 6, we have $d(p_i, p_j) \in (2\sigma, \infty)$. Hence, $p_0, \ldots, p_6$ are different points. By Lemma 1, $\angle p_i p_0 p_j > 60°$ (2σ is a scaling factor).

Consider relative positions of p_0 and the 6 points $p_1, \ldots, p_6$. These 6 points should be located within a circle centered at p_0 with radius 2σ. Consider any angle $\angle p_i p_0 p_j$, by Lemma 1, it should be greater than 60°. Each $\angle p_i p_0 p_j$ where there is no p_k such that $\angle p_i p_0 p_k$ is smaller, defines a segment of a circle centered at p_0. The sum of the angles of all those segments is greater than 360°, which is impossible in a 2D Euclidean space. □

Theorem 3 *There exists a finite consistent set of LNFS formulas which are not satisfied in any 2D Euclidean model.*

Proof.[sketch] Consider a set of LNFS formulas Σ over 13 individual names $a_0, \ldots, a_6, b_1, \ldots, b_6$. For every $i \in \mathbb{N}$, $1 \leq i \leq 6$, $BEQ(a_0, b_i) \in \Sigma$, $BEQ(b_i, a_i) \in \Sigma$. For every $i \in \mathbb{N}$, $j \in \mathbb{N}$, $1 \leq i < j \leq 6$, $\neg NEAR(a_i, a_j) \in \Sigma$. No other formulas are in Σ. It is clear that no contradiction can be derived using the LNFS axioms, hence Σ is consistent. Note that $BEQ(a, b) \equiv BPT(a, b) \wedge BPT(b, a)$. The rest of the proof is similar to that of Theorem 2 and is omitted due to limited space. □

4 LNF is Not Finitely Axiomatisable

In this section, we will first briefly introduce relevant background from graph theory, then show that LNF is not finitely axiomatisable with respect to 2D Euclidean spaces.

4.1 Background on Unit Disk Graphs

A graph is a unit disk graph, if its vertices can be represented as points in a 2D Euclidean space such that there is an edge between two vertices iff the distance between their corresponding points is at most m, where m is a positive constant (Atminas and Zamaraev 2018). Unit disk graphs are useful in many applications, for example wireless networks, and have been studied actively in recent years (Breu and Kirkpatrick 1998; McDiarmid and Müller 2013; da Fonseca et al. 2015).

Definition 7 *Let $G = (V, E)$ be a graph and $(\mathbb{R}^2, d)$ be a 2D Euclidean space. A unit disk graph realization of G is a function f from V to $\mathbb{R}^2$, such that for some positive real number m and every pair of vertices $u, v \in V$,*

- *$d(f(u), f(v)) \leq m$ iff $uv \in E$; and*
- *$d(f(u), f(v)) > m$ iff $uv \notin E$.*

A graph is a unit disk graph iff it has a unit disk graph realization. The problem of determining whether a graph has a unit disk graph realization is called 'the recognition problem of unit disk graphs'.

The class of unit disk graphs is closed under vertex deletion, or equivalently, closed under induced subgraphs (Atminas and Zamaraev 2018). A graph g is a *forbidden induced subgraph* for the class of unit disk graphs $\mathcal{X}$, if none of the graphs in $\mathcal{X}$ contains g as an induced subgraph. For the class of unit disk graphs $\mathcal{X}$, a forbidden induced subgraph g is minimal, if the graph obtained by deleting a vertex from g is not a forbidden induced subgraph for $\mathcal{X}$ (i.e. the resulting graph is a unit disk graph).

As shown in Theorem 4 below, it was recently proved that there exist infinitely many minimal forbidden induced subgraphs for the class of unit disk graphs (UDGs) or infinitely many minimal non-UDGs (Atminas and Zamaraev 2018). Readers may refer to the proof of Theorem 4 in (Atminas and Zamaraev 2018).

In this paper, let K_n and C_n denote a complete n-vertex graph and a chordless cycle on n vertices respectively. The disjoint union of two graphs G_1 and G_2 is denoted as $G_1 + G_2$. The complement of a graph G is denoted as $\overline{G}$.

Theorem 4 *(Atminas and Zamaraev 2018) For every integer $k \geq 1$, $\overline{K_2 + C_{2k+1}}$ is a minimal non-UDG.*

4.2 Infinitely Many Independent Axioms

This section will show that, with respect to 2D Euclidean spaces, there exist infinitely many independent axioms in LNF, and hence a finite complete axiomatisation of LNF does not exist.

Theorem 5 *For every integer $k \geq 1$, the minimal non-UDG $\overline{K_2 + C_{2k+1}}$ can be described by an LNF formula which is satisfiable in a metric model.*

Proof. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph isomorphic to $K_2 + C_{2k+1}$, where $\mathcal{V} = \{u, w, c_1, \ldots, c_{2k+1}\}$ and $\mathcal{E} = \{c_i c_j \mid |i-j| = 1\} \cup \{uw, c_1 c_{2k+1}\}$. Then for every integer $k \geq 1$, the minimal non-UDG $\overline{K_2 + C_{2k+1}}$ enforces a finite set of distance constraints $\mathcal{S}$ which cannot be realized in a 2D Euclidean space. The set $\mathcal{S}$ can be constructed as follows, where I is an interpretation function which maps every vertex to a point in a 2D Euclidean space, and $\sigma = 1$.

1. $(d(I(u), I(w)) > 1) \in \mathcal{S}$;
2. for every pair of integers i, j such that $1 \leq i < j \leq 2k+1$, if $|i-j| = 1$ or $\{i, j\} = \{1, 2k+1\}$, then $(d(I(c_i), I(c_j)) > 1) \in \mathcal{S}$;
3. for every integer i such that $1 \leq i \leq 2k+1$, $(d(I(u), I(c_i)) \leq 1) \in S$ and $(d(I(w), I(c_i)) \leq 1) \in \mathcal{S}$;
4. for every pair of integers i, j such that $1 \leq i < j \leq 2k+1$, if $|i-j| \neq 1$ and $\{i, j\} \neq \{1, 2k+1\}$, then $(d(I(c_i), I(c_j)) \leq 1) \in \mathcal{S}$.

Every distance constraint of the form $d(I(p), I(q)) > 1$ can be translated to $\neg BEQ(p, q)$ by Definition 5. Every distance constraint of the form $d(I(p), I(q)) \leq 1$ can be translated to $BEQ(p, q)$. Hence, the finite set of distance constraints $\mathcal{S}$, which describes a minimal non-UDG of size $2k+3$, can be expressed as a conjunction of LNF formulas $\mathcal{A}_{2k+3}$.

Although $\mathcal{A}_{2k+3}$ is not satisfiable in a 2D Euclidean model, it is satisfiable in a metric model. Note that for every pair of vertices u, v in this forbidden graph, either $d(I(u), I(v)) \in [0, \sigma]$ or $d(I(u), I(v)) \in (\sigma, \infty)$ holds. Applying the consistency checking method from (Du and Alechina 2016), the corresponding set of distance constraints of the forbidden graph can be shown to be path-consistent. This involves showing that starting from intervals $[0, \sigma]$ and (σ, ∞) in the constraint set and applying composition and intersection rules, for every pair of constants in this set, their distance range cannot be strengthened to be empty by enforcing path-consistency. □

By Theorem 5, for every integer $k \geq 1$, the fact that the minimal non-UDG $\overline{K_2 + C_{2k+1}}$ cannot be realized in a 2D Euclidean space can be expressed as an axiom in LNF, which is $\mathcal{A}_{2k+3} \rightarrow \bot$. Below we will show that all axioms of the form $\mathcal{A}_i \rightarrow \bot$ are independent.

Definition 8 (Complement Graph Model) *A* complement graph model *for BEQ formulas is a tuple $M = (\mathcal{G}, \mathcal{I})$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph and $\mathcal{I}$ is an interpretation function which maps each individual name to a vertex $v \in \mathcal{V}$. The notion of $M \models \phi$ is defined as follows:*

$M \models BEQ(a, b)$ iff $(\mathcal{I}(a), \mathcal{I}(b)) \notin \mathcal{E}$

where a, b are individual names, ϕ is a formula in $L(LNF_{BEQ})$ (the language with only BEQ).

By Definition 8, $M \models \neg BEQ(a, b)$ iff $(\mathcal{I}(a), \mathcal{I}(b)) \in \mathcal{E}$.

Theorem 6 *Let $\mathcal{A}_{n_1} \rightarrow \bot$ and $\mathcal{A}_{n_2} \rightarrow \bot$ denote axioms corresponding to two minimal non-UDGs $\overline{K_2 + C_{2k_1+1}}$ and $\overline{K_2 + C_{2k_2+1}}$ respectively, where k_1, k_2 are different positive integers. There exists a complement graph model where $\mathcal{A}_{n_1} \rightarrow \bot$ is false and $\mathcal{A}_{n_2} \rightarrow \bot$ is true.*

Proof. Consider a complement graph model $M = (\mathcal{G}, \mathcal{I})$, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{u, w, c_1, \ldots, c_{2k_1+1}\}$ and $\mathcal{E} = \{c_i c_j \mid |i-j| = 1\} \cup \{uw, c_1 c_{2k_1+1}\}$. Then by Definition 8 and the construction of $\mathcal{A}_{n_1}$ in the proof of Theorem 5, $M \models \mathcal{A}_{n_1}$ ($\mathcal{A}_{n_1} \rightarrow \bot$ is false in M).

Let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be a graph isomorphic to $K_2 + C_{2k_2+1}$, where $\mathcal{V}' = \{u', w', c'_1, \ldots, c'_{2k_2+1}\}$ and $\mathcal{E}' = \{c'_i c'_j \mid |i-j| = 1\} \cup \{u'w', c'_1 c'_{2k_2+1}\}$. To show $\mathcal{A}_{n_2}$ is not satisfied in M, it is sufficient to show that it is impossible to define an interpretation function f from $\mathcal{V}'$ to $\mathcal{V}$ such that f preserves corresponding edges and non-edges: for any v_1, v_2 in $\mathcal{V}'$, if $(v_1, v_2) \in \mathcal{E}'$ then $(f(v_1), f(v_2)) \in \mathcal{E}$; and if $(v_1, v_2) \notin \mathcal{E}'$ then $(f(v_1), f(v_2)) \notin \mathcal{E}$. Suppose such a function f exists. Then f should map elements in $\{u', w'\}$ to those in $\{u, w\}$ and map elements in $\{c'_1, \ldots, c'_{2k_2+1}\}$ to those in $\{c_1, \ldots, c_{2k_1+1}\}$. If $k_2 > k_1$, then f would map two vertices in $\mathcal{V}'$ to the same vertex in $\mathcal{V}$, which cannot preserve some non-edges in $\mathcal{E}'$. If $k_1 > k_2$, no matter which elements f maps $c'_1, \ldots, c'_{2k_2+1}$ to, f cannot preserve some edges in $\mathcal{E}'$. Hence, $M \not\models \mathcal{A}_{n_2}$ ($\mathcal{A}_{n_2} \rightarrow \bot$ is true in M).

By Definition 8, every complement graph model corresponds to a set of BEQ and $\neg BEQ$ formulas, where BEQ is reflexive and symmetric. By the proof of Theorem 5, such set of BEQ and $\neg BEQ$ formulas is satisfiable in a metric model. Therefore, for every witness complement graph model (making one axiom true and the other false) shown above, there exists a metric model that satisfies the same set of BEQ and $\neg BEQ$ formulas. □

Theorem 7 *There is no axiom $\mathcal{A}\omega$ such that $\mathcal{A}\omega$ is valid in 2D Euclidean models and LNF+$\mathcal{A}\omega$ entail all of the forbidden subgraph axioms.*

Proof. The proof is based on an intuition that an axiom that is a formula over n constants cannot rule out forbidden graphs of size larger than n.

Suppose there exists an axiom $\mathcal{A}\omega$ such that $\mathcal{A}\omega$ is valid in 2D Euclidean models (hence is consistent with all the LNF axioms for metric spaces) and LNF+$\mathcal{A}\omega$ entail all of the forbidden subgraph axioms. Clearly, $\mathcal{A}\omega$ is a formula over some finite number of constants n.

Consider a metric model M corresponding to a minimal forbidden graph of the size n' such that n' is the size of the smallest size of minimal forbidden graphs with $n' > n$. Such a metric model exists by Theorem 5. Every submodel M^- of M of size at most n does not contain a minimal forbidden subgraph, hence the graph corresponding to M^- is realizable in a 2D Euclidean model. Hence $\mathcal{A}\omega$ is true in every submodel of M with n elements, because $\mathcal{A}\omega$ is valid in 2D Euclidean models. Since the truth of $\mathcal{A}\omega$ only depends on the interpretation of n constants, $M \models \mathcal{A}\omega$. However by assumption $M \models \mathcal{A}_{n'}$ hence $M \not\models \mathcal{A}_{n'} \to \bot$. Hence LNF + $\mathcal{A}\omega$ does not entail $\mathcal{A}_{n'} \to \bot$. □

5 LBPT and LNFS are Not Finitely Axiomatisable

In this section, we will show that LBPT is not finitely axiomatisable with respect to 2D Euclidean spaces. LBPT is more expressive than LNFS. The proofs for LNFS can be constructed similarly and are omitted due to limited space.

Recall that in Section 3, we provided a counter-example to the completeness of LBPT in the proof of Theorem 2. Let $BPT^m(a, b)$ denote a chain of m BPT formulas $BPT(a_0, a_1)$, $BPT(a_1, a_2)$, …, $BPT(a_{m-1}, a_m)$, where $m \in \mathbb{N}_{>0}$, $a = a_0$ and $b = a_m$. Then the counter-example can be formalized as the following axiom:

Axiom 14 $(\forall 0 < i \le 6 : BPT^2(a_0, a_i)) \wedge (\forall 0 < i < j \le 6 : \neg NEAR(a_i, a_j)) \to \bot$.

More counter-examples to the completeness of LBPT can be identified using Lemma 2, which generalizes Lemma 1.

Lemma 2 *Let a, b, c be different points in a 2D Euclidean space, g_1, g_2, g_3 be positive real numbers, $g_3 \ge \max(g_1, g_2)$ and $g_1 + g_2 > g_3$. If $d(a, b) \in [0, g_1]$, $d(b, c) \in [0, g_2]$ and $d(a, c) \in (g_3, \infty)$, then the angle $\angle abc > \cos^{-1}(\frac{g_1^2+g_2^2-g_3^2}{2g_1g_2})$.*

Lemma 2 can be proved by showing that in the region $D = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, g_1] \textit{ and } y \in [0, g_2]\}$, the maximal value of the function $f(x, y) = x^2 + y^2 - \frac{g_1^2+g_2^2-g_3^2}{g_1g_2}xy$ is g_3^2. Lemma 2 can also be proved by applying the Law of Cosines and then observing that shortening the shorter edges of a triangle or lengthening the longest edge will enlarge the angle opposite to the longest edge.

More counter-examples are formalized as follows. The expressions of complicated axioms could be simplified by introducing abbreviations of the conjunctions of LBPT formulas. Let $R_1(a, b)$ and $R_2(b, c)$ be LBPT formulas, their conjunction $R_1(a, b) \wedge R_2(b, c)$ is abbreviated as $(R_1 \circ R_2)(a, c)$. Using Lemma 2, the proofs of these axioms can be constructed similarly to that of Theorem 2.

Axiom 15 $(\forall\ 0 < i \le 6 : BPT(a_0, a_i)) \wedge (\forall\ 0 < i < j \le 6 : (BPT^3 \circ FAR)(a_i, a_j)) \to \bot$;

Axiom 16 $(\forall\ 0 < i \le 4 : BPT^2(a_0, a_i)) \wedge (\forall\ 0 < i < j \le 4 : (BPT \circ FAR)(a_i, a_j)) \to \bot$;

Axiom 17 $(\forall\ 0 < i \le 5 : BPT^3(a_0, a_i)) \wedge (\forall\ 0 < i < j \le 5 : FAR(a_i, a_j)) \to \bot$;

Axiom 18 $BPT(a_0, a_1) \wedge (\forall\ 1 < i \le 6 : BPT^2(a_0, a_i)) \wedge (\forall\ 0 < i < j \le 6 : \neg NEAR(a_i, a_j)) \to \bot$.

In the rest of this section, we will show that for LBPT, there exist infinitely many independent axioms in 2D Euclidean spaces.

Lemma 3 *Let r and c be positive real number constants. In a 2D Euclidean space, within a circle whose radius is r, it is impossible to have an infinite number of points such that for every pair of points p, q, $d(p, q) > c$.*

Proof. Suppose in a 2D Euclidean space, within a circle whose radius is r, there exist an infinite number of points such that for every pair of points p, q, $d(p, q) > c$. For every point p, let $R(p)$ denote the region of the circle centered at p with radius $\frac{c}{2}$. Then for every pair of points p, q, $R(p)$ and $R(q)$ are disjoint, because $d(p, q) > c$. Since the area of $R(p)$ is $\pi(\frac{c}{2})^2$, then the area of the union of all such regions for the infinite number of points is ∞. However, if all the points are within a circle whose radius is r, then the area of the union of all such regions is at most $\pi(r + \frac{c}{2})^2$. A contradiction exists. □

Following Lemma 3, we have Lemma 4.

Lemma 4 *Let r_1, r_2 and c be positive real number constants, $r_1 > r_2$. In a 2D Euclidean space, within a circle centered at a point o with radius r_1 and outside a circle centered at the same point o with radius r_2, it is impossible to have an infinite number of points such that for every pair of points p, q, $d(p, q) > c$.*

Theorem 8 *For every integer $m > 4$, there exists an integer n such that the following formula A_m is an axiom: $(\forall 0 < i \le n : BPT^m(a_0, a_i)) \wedge (\forall 0 \le i < j \le n : FAR(a_i, a_j)) \to \bot$.*

Proof. By Lemma 4, in a 2D Euclidean space, within a circle centered at a point o with radius $m\sigma$ and outside a circle centered at the same point o with radius 4σ, it is impossible to have an infinite number of points such that for every pair of points p, q, $d(p, q) > 4\sigma$. Hence, for any $m > 4$, there exists an integer k such that at most k points satisfying these specified constraints can be realized in a 2D Euclidean space. Let $n = k + 1$. Then it is impossible to have n points in a 2D Euclidean space such that all the specified constraints are satisfied. By Definition 6, the formula $\neg A_m = (\forall 0 < i \le n : BPT^m(a_0, a_i)) \wedge (\forall 0 \le i < j \le n : FAR(a_i, a_j))$ enforces the same constraints over n points in $I(a_1), \ldots, I(a_n)$ (the point o is in $I(a_0)$), hence, it implies $\bot$ (i.e. a contradiction in a 2D Euclidean space). □

For every integer $m > 4$, let $f(m)$ denote the smallest integer n such that $(\forall 0 < i \le n : BPT^m(a_0, a_i)) \wedge (\forall 0 \le i < j \le n : FAR(a_i, a_j)) \to \bot$ is valid in 2D Euclidean models.

Lemma 5 *For every integer $m > 4$, every positive integer k, $f(m) < f(m+5k)$.*

Proof. For any integer $m > 4$, since $f(m)$ denotes the smallest integer n such that $(\forall 0 < i \leq n : BPT^m(a_0, a_i)) \wedge (\forall 0 \leq i < j \leq n : FAR(a_i, a_j)) \rightarrow \perp$, following the proof of Theorem 8, in a 2D Euclidean space, within a circle C_1 centered at a point o with radius $m\sigma$ and outside a circle C_2 centered at the same point o with radius 4σ, there exist at most $f(m) - 1$ points such that for every pair of points $p, q, d(p, q) > 4\sigma$. Consider a circle C_3 centered at the same point o with radius $(m + 5k)\sigma$. Then all these $f(m) - 1$ points are also within it. In addition, at least one additional point p can be allocated within C_3 and outside C_1 such that the distance between p and any of these $f(m) - 1$ points is greater than 4σ (e.g. p is on the boundary of C_3). □

Next we show that there exist infinitely many *independent* axioms of the form A_m.

Theorem 9 *For every integer $m > 4$, the following two axioms are independent:*

1. $A_m = ((\forall 0 < i \leq f(m) : BPT^m(a_0, a_i)) \wedge (\forall 0 \leq i < j \leq f(m) : FAR(a_i, a_j)) \rightarrow \perp)$*;*
2. $A_{m+5k} = ((\forall 0 < i \leq f(m+5k) : BPT^{m+5k}(a_0, a_i)) \wedge (\forall 0 \leq i < j \leq f(m + 5k) : FAR(a_i, a_j)) \rightarrow \perp)$*, for every positive integer k.*

Proof. We will show that there exists a metric model where A_m is false and A_{m+5k} is true, and there exists a metric model where A_m is true and A_{m+5k} is false. Consider a metric model $M = (\Delta, d, I, \sigma)$, which interprets every individual name as a singleton. $\Delta = \{p_0, p_1^1, \ldots, p_1^m, \ldots, p_{f(m)}^1, \ldots, p_{f(m)}^m\}$. Recall that $BPT^m(a_0, a_i)$ is an abbreviation of the formula $BPT(a_0, a_i^1) \wedge BPT(a_i^1, a_i^2) \cdots \wedge BPT(a_i^{m-1}, a_i^m)$, where $a_i = a_i^m$. $I(a_0) = \{p_0\}$. For any pair of integers i, j such that $0 < i \leq f(m)$ and $0 < j \leq m$, $I(a_i^j) = \{p_i^j\}$. We construct a set of distance constraints S_1 from $\neg A_m$ following the steps below. Initially, $S_1 = \{\}$. For every pair of individual names a, b in A_m,

1. if $BPT(a, b)$ occurs in the antecedent of A_m, then add $d(I(a), I(b)) \in [0, \sigma]$ to S_1;
2. if $FAR(a, b)$ occurs in the antecedent of A_m, then add $d(I(a), I(b)) \in (4\sigma, \infty)$ to S_1;
3. if both $BPT(a, b)$ and $FAR(a, b)$ do not occur in the antecedent of A_m, then add $d(I(a), I(b)) \in (\sigma, 4\sigma]$ to S_1.

Using the approach in (Du and Alechina 2016), it can be shown that S_1 is path-consistent (defining $[0, \sigma]$, $(4\sigma, \infty)$ and $(\sigma, 4\sigma]$ as primitive intervals, then based on the composition rules and intersection rules, for every pair of constants in S_1, their distance range cannot be strengthened to be empty by enforcing path-consistency), and hence there exists such a metric model M satisfying $\neg A_m$. $\neg A_{m+5k}$ is not satisfied in M because by Lemma 5 $f(m+5k) > f(m)$, hence Δ does not contain a sufficient number of points for interpreting $f(m + 5k) + 1$ individual names which are all FAR from each other.

Consider a metric model $M' = (\Delta', d', I', \sigma)$, which interprets every individual name as a singleton. $\Delta' = \{p_0, p_1^1, \ldots, p_1^{m+5k}, \ldots, p_{f(m+5k)}^1, \ldots, p_{f(m+5k)}^{m+5k}\}$. $I(a_0) = \{p_0\}$. For any pair of integers i, j such that $0 < i \leq f(m + 5k)$ and $0 < j \leq m + 5k$, $I(a_i^j) = \{p_i^j\}$. A set of distance constraints S_2 can be constructed from the antecedent of A_{m+5k} in a similar way to S_1. Again using the approach of (Du and Alechina 2016), we can show that S_2 is path-consistent, and hence there exists such a metric model M' satisfying $\neg A_{m+5k}$. By Definition 3, any metric model satisfying $\neg A_m$ should group points in Δ' into $mf(m) + 1$ sets (interpretations of the constants in A_m) such that they satisfy all the distance constraints obtained in Steps 1 and 2 for constructing S_1. It is clearly possible to find m sets of points in Δ' that are far from each other, since there are $m + 5k$ such sets in Δ'. However it is not possible to find sequences of sets/points which connect them to the interpretation of a_0 in m steps, by the construction of S_2 which requires distances between points not involved in a path to be greater than σ and paths between p_0 and p_i having length $m + 5k$. Hence, $\neg A_m$ is not satisfied in M'. □

By Theorem 9, the axioms A_{m+5k} are pairwise independent.

Theorem 10 *There is no axiom $\mathcal{A}$ such that $\mathcal{A}$ is valid in 2D Euclidean models and LBPT+$\mathcal{A}$ entail all the axioms A_{m+5k}, where m, k are non-negative integers and $m > 4$.*

Proof. Suppose there exists an axiom $\mathcal{A}$ such that $\mathcal{A}$ is valid in 2D Euclidean models (hence is consistent with all the LBPT axioms for metric spaces) and LBPT+$\mathcal{A}$ entail all of the axioms A_{m+5k}. Clearly, $\mathcal{A}$ is a formula over some finite number of individual names t. We construct a metric model M such that M satisfies $\neg A_m$ for some $f(m) > t$. Then we show that any property over at most t individual names which is true in M also true in some 2D Euclidean model. Hence all instances of $\mathcal{A}$ are true in M, because otherwise their negations would have been satisfiable in a 2D Euclidean model. Hence M satisfies all instances of $\mathcal{A}$ and $\neg A_m$: a contradiction with the assumption that $\mathcal{A}$ entails A_m.

The metric model M is designed to satisfy $\neg A_m$ using singleton interpretations of its constants, and otherwise be as like a Euclidean model as possible. Since $BPT^m(a_0, a_i)$ is an abbreviation of $BPT(a_0, a_i^1) \wedge BPT(a_i^1, a_i^2) \wedge ... \wedge BPT(a_i^{m-1}, a_i^m), a_i = a_i^m$, $\neg A_m$ is over $mf(m) + 1$ individual names. M has $mf(m) + 1$ points: p_0 in the centre, p_i^m $(1 \leq i \leq f(m))$ on a circle of radius $m\sigma$ around p_0, and $p_i^1, ..., p_i^{m-1}$ evenly spaced at distance σ from each other on a line from p_0 to p_i^m. For any pair of integers i, j such that $0 < i \leq f(m)$ and $0 < j \leq m$, $I(a_i^j) = \{p_i^j\}$. $d(p_0, p_i^m) = m\sigma$. This part of d in M is Euclidean and makes sure that $BPT^m(a_0, a_i)$ are satisfied in M (and also in a Euclidean model). The non-Euclidean part of d is that $d(p_i^m, p_j^m) > 4\sigma$ for all $i \neq j$, which makes $FAR(a_i, a_j)$ true (satisfying $\neg A_m$). Clearly not all of $FAR(a_i, a_j)$ can be satisfied in a 2D Euclidean model (see Theorem 8 proof) while also satisfying $BPT^m(a_0, a_i)$ and other LBPT-expressible properties of $a_0, \ldots, a_{f(m)}$. However we show that all LBPT formulas $\phi(b_1, .., b_t)$ using at most t individual names are satisfiable in a 2D Euclidean model. We reason by cases. Let us refer to

points p_i^m as 'circle' points and the rest as 'inner' or 'radius' points. $n = f(m)$.
Case 1: one of the circle points is not in any of $I(b_1), \ldots, I(b_t)$. Then that point p_i^m can be moved closer to its neighbour p_{i+1}^m and distances between the remaining points on the circle increased to spread them evenly on the circle with distances $> 4\sigma$. We assume that when we move apart points on the circle, we also move the lines they belong to, but the distances there were Euclidean to begin with. This will result in a 2D Euclidean model without any change in truth values of formulas over $b_1, \ldots, b_t$ since although $d(p_i^m, p_{i+1}^m)$ is less than 4σ now, p_i^m is not involved in truth conditions of any formulas over $b_1, \ldots, b_t$.
Case 2: each of $p_1, \ldots, p_n$ is in some $I(b_1), \ldots, I(b_t)$. Then because $n > t$, at least two of these points must belong to the same $I(b_j)$ (one or both of them may also occur in sets $I(b_l)$ for $j \neq l$). Case 2.1: two points in $I(b_j)$ are p_i^m and p_{i+1}^m (neighbours on the circle). The distance between them can be reduced and the rest of the circle points spread out as in Case 1, without affecting LBPT relationships. Suppose by contradiction that $FAR(b_j, b_l)$ holds in M and does not hold after we moved p_i^m and p_{i+1}^m closer than 4σ together, and spread out the remaining points. This means that all of b_j points are far from all of b_l points in M, and now they are not. The only way this can happen is if one of p_i^m or p_{i+1}^m is in $I(b_l)$. However if $I(b_j)$ and $I(b_l)$ share a point, then $FAR(b_j, b_l)$ can not hold in M. Similar argument can be made about all other possible relationships between names in LBPT.
Case 2.2.: no two neighbours on the circle are in the same $I(b_j)$. An example of a property which falls under this case is $\wedge_{j<t} FAR(b_j, b_{j+1})$ where each $I(b_j) = \{p_j^m, p_{j+2}^m\}$ (assuming $t \geq n/2$; otherwise sets $I(b_j)$ could be made larger), or a similar property of a set of sparse disjoint sets of points on the circle all being far from each other. A property can also involve b_j being $NEAR$ some set of inner points. In Case 2.2, the modification of M involves not moving a pair of neighbours closer to each other, but throwing one of the circle points away (and spreading the rest as before). If the property is stated entirely in terms of FAR relations, throwing points away will not make it false. If the property involves one of b_j sets containing circle points being $NEAR$ some r sets made of radius points, this cannot involve all n radii (since $t < n$). Hence there is one point p_i^m in one of b_j which is not needed to make a $NEAR$ statements true, and can be removed without affecting the truth of the property. □

6 Decidability of LNF over 2D Euclidean Spaces

The decidability of LNF over 2D Euclidean spaces is proved by translating LNF formulas to a sentence of elementary algebra. The basics of elementary algebra are as follows (Tarski 1951).

In elementary algebra, a variable is one of the symbols $x, x_1, x_2, ..., y, y_1, y_2, ..., z, z_1, z_2, ...$, ranging over the set of real numbers. An algebraic constant is one of the three symbols $1, 0, -1$. Every variable or algebraic constant is an algebraic term. If α and β are algebraic terms, then $\alpha \times \beta$, $\alpha + \beta$ are algebraic terms. If α and β are algebraic terms, then $\alpha = \beta$, $\alpha > \beta$ are atomic formulas. Every atomic formula is a formula. If ϕ and ψ are formulas, then $\neg\phi$, $\exists x : \phi$, $\phi \wedge \psi$ are formulas. A formula containing no free variables is called a sentence.

As defined in (Du and Alechina 2016), a distance constraint is a statement of the form $d(p, q) \in g$, where p, q are constants representing points, g is a non-negative interval.

Lemma 6 *For a non-empty set of distance constraints D over n constants, there is a sentence of elementary algebra ϕ of size polynomial in the size of D, such that ϕ is true iff D is satisfiable in a 2D Euclidean space.*

Proof. By Definition 4, every constant p in D is a point (p_x, p_y) in a 2D Euclidean space. For any pair of constants p, q, $d(p, q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$. From a distance constraint $d(p, q) \in g$, it is easy to obtain an expression representing the range of $(p_x - q_x)^2 + (p_y - q_y)^2$, which is referred to as a squared distance constraint expression. For example, from $d(p, q) \in (\sigma, 2\sigma]$, we obtain $((p_x - q_x)^2 + (p_y - q_y)^2 > \sigma^2) \wedge ((p_x - q_x)^2 + (p_y - q_y)^2 \leq 4\sigma^2)$. Let ϕ be an expression of the form $\exists p_x \exists p_y ... \exists q_x \exists q_y : C$, such that a constant p is in D iff $\exists p_x \exists p_y$ is in ϕ, and a distance constraint is in D iff its corresponding squared distance constraint expression is in C. C is the conjunction of all such squared distance constraint expressions. Following the definitions in elementary algebra, ϕ is a sentence. Since D is over n constants, the number of constants in ϕ is $2n$. By Definition 4, ϕ is true iff D is satisfiable in a 2D Euclidean space. □

Theorem 11 *(Tarski 1951) There is a decision method for the class of all true sentences of elementary algebra.*

Theorem 11 is for the general decision problem for the first order theory of the reals, where existential quantifiers and universal quantifiers are allowed. A special case of the general problem is when all the quantifiers are existential, which is often referred to as the existential theory of the reals.

Theorem 12 *(Canny 1988) The existential theory of the reals is decidable in PSPACE.*

Theorem 13 (Decidability & Complexity) *The satisfiability problem for a finite set of LNF formulas in a 2D Euclidean space is decidable in PSPACE.*

Proof. By Definition 5, each of BEQ, NEAR and FAR can be rewritten as a distance constraint, then a finite set of LNF formulas Σ can be rewritten as a set of distance constraints D. If D is empty, then Σ is satisfiable. Otherwise, by Lemma 6, there is a sentence of elementary algebra ϕ of size polynomial in the size of D such that ϕ is true iff D is satisfiable in a 2D Euclidean space. ϕ only involves existential quantifier $\exists$. By Theorem 12, the satisfiability problem for a finite set of LNF formulas in a 2D Euclidean space is decidable in PSPACE. □

The question whether LNFS and LBPT are decidable over 2D Euclidean spaces is still open.

7 Discussion and Future Work

Qualitative spatial logics LNF, LNFS and LBPT were developed for validating matches between spatial objects represented in different datasets. The rationale is that cases where problematic matches exist are formalized as logical contradictions. With respect to metric spaces, sound and complete axiomatizations were provided and could be used to detect mistakes in matches (Du et al. 2015; Du and Alechina 2016). With respect to 2D Euclidean spaces, as shown in the previous sections, these axiomatizations are not complete and the spatial logics are not finitely axiomatisable. Since the LNF satisfiability problem is decidable over 2D Euclidean spaces, we consider the possibility of verifying matches by checking whether a set of LNF formulas is satisfiable or not.

There exist several software tools that support checking whether a sentence in elementary algebra is true. Most of them use quantifier elimination methods for the reals (Tarski 1951; Collins 1975; Sturm 2017). The first real quantifier elimination method was published by Tarski (Tarski 1951). In 1970s, Collins introduced the quantifier elimination procedure over reals based on cylindrical algebraic decomposition (CAD) (Collins 1975), which runs in doubly exponential time. It was proved that real quantifier elimination takes at least double exponential time (Davenport and Heintz 1988; Weispfenning 1988). The CAD algorithm was implemented in QEPCAD (Collins and Hong 1991), Mathematica (Strzebonski 2000) and the computer logic system Redlog (Dolzmann and Sturm 1997). The software QEPCAD B (Brown 2003) extends and improves the QEPCAD.

To check whether using these systems for debugging matches is feasible, we experimented with Redlog (a part of the computer algebra system Reduce, Free PSL version, revision 4726, 16 August 2018) and QEPCAD B (v.1.69, 16 March 2012) on a 2.4 GHz Intel Core i7, 8 GB 1600 MHz DDR3 MacBook Pro. An LNF formula was translated to a sentence of elementary algebra and checked for satisfiability using Redlog and QEPCAD B. However, even a simple formula with 3 names required more than 10 minutes on Redlog and caused QEPCAD B to ran out of memory. We conclude that currently the use of Redlog and QEPCAD B and similar tools for validating matches between spatial objects is not practical. Instead we plan to concentrate on developing specialized reasoning mechanisms for the qualitative spatial logics over 2D Euclidean spaces and consider adding new primitives to the logics (such as direction) to facilitate debugging.

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