Multiple-Interval Coverage for Resource Management of Passive Surveillance Systems

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Abstract

Passive surveillance systems (PSS) are used to detect and track various targets by processing the electromagnetic signals they release. The study and design of the resource management algorithm for these systems revealed several phenomena and combinatorial problems with crucial theoretical properties. In this article, we first prove the completeness of the newly introduced algorithm used to generate receiver settings that determine which frequency bands the PSS monitors. Next, we formulate a new optimization problem called multiple-interval coverage (MIC), which is used to determine how often each of the generated settings must be used by the PSS. We show that the MIC problem is closely related to the multicover problem, which is an extension of the well-known set cover problem. The uniqueness of MIC stems from the fact that both covered elements and covers are multiple-intervals. We propose a notation to distinguish between different variants of the problem and prove that some of them can be solved in polynomial time. Finally, we prove that the MIC problem is NP-hard even when restricted to 2-interval covers.

1 Introduction

A passive surveillance system (PSS) is a complex electronic support measure that is used to detect, track, and identify various targets hundreds of kilometers away. The system accomplishes this by intercepting the electromagnetic signals emitted by the targets. In order to intercept these signals, the system has multiple receivers that observe the frequencies at which the targets' emitters broadcast. This is in contrast to radars that emit the signals they intercept.

In order for the PSS to properly accomplish all of the required tasks, it is essential to correctly manage its receivers. The field of study concerned with this is called PSS resource management (PSSRM) (Hashmi et al. 2023). We studied the problem of determining which frequency bands the PSS receivers should observe in order to *cover* all frequency bands of interest where the emitters of tracked targets broadcast. While developing a two-step algorithm to solve this problem, which is explained in Section 3, we discovered several phenomena that may be relevant to other disciplines and whose theoretical properties have never been studied.

This article first focuses on the newly introduced algorithm, called left-right, which is used to construct the frequency bands to be observed by the receivers, called covers. We prove that the cover construction using this algorithm is sufficient and that no additional covers are necessary. Then, we formulate a variant of the multicover problem (Hochbaum 1997) called multiple-interval coverage (MIC), in which the covers and frequency bands of targets are multiple-intervals. This problem is used to determine how often each constructed cover should be used by the system. To easily differentiate between different types of MIC problems, we introduce a new notation. Next, we prove that single-interval MIC problems can be solved in polynomial time when the targets are not nested. We also prove that the same holds for the covers. In the final section of the paper, we demonstrate that all MIC problems whose covers can include 2-intervals are NP-hard.

2 Preliminaries

The interval defined as $s = [p, q] = \{v \in \mathbb{R} \mid p \leq v \leq q\}$, where $p, q \in \mathbb{R}$ and $p < q$, will be called *single-interval*. The leftmost point p of the single-interval s will be indicated as $l(s)$. Similarly, the rightmost point q will be written as $r(s)$. The size of the single-interval is equal to $q - p$.

The union of $k \in \mathbb{N}_{>0}$ pairwise disjoint single-intervals $s^1, \ldots, s^k$ is called the *k-interval*. The set of all k-intervals is written as $\mathbb{I}_k$. Similarly, $\bigcup_{l=1}^{k} \mathbb{I}_l$ is denoted by $\mathbb{I}_{\leq k}$.

Multiple-interval is an arbitrary k-interval. The set of all multiple-intervals is symbolized as $\mathbb{I}$. Consider a multiple-interval z; the set of single-intervals it consists of is denoted by $\mathcal{S}(z)$. The *shape* of a multiple-interval is an odd-length vector of positive real numbers that describes the sizes of its constituent single-intervals and the spaces between them, from left to right. For example, consider multiple-interval $z = [-3, 2] \cup [4, 5] \cup [8, 12]$, then $\mathcal{S}(z) = \{[-3, 2], [4, 5], [8, 12]\}$ and its shape is $(5, 2, 1, 3, 4)$. Furthermore, we say that the multiple-interval z_1 is *covered* by multiple-interval z_2 or that z_2 *covers* z_1 if $z_1 \subseteq z_2$.

3 Problem Statement

We consider a PSS with a single receiver that is used to monitor the frequency bands of interest. The frequency bands monitored by the receiver at a given time can be described

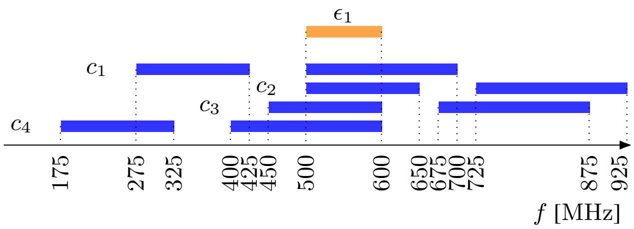

Figure 1: The four covers, c_1, c_2, c_3, c_4, generated by LRA for emitter $\epsilon_1 = [500, 600]$ and shape $\delta = (150, 75, 200)$.

by a multiple-interval called a *cover*. The possible shapes of this cover depend on the construction characteristics of the PSS and are stored in set of shapes $\Delta \subseteq \bigcup_{k=1}^{\infty} \mathbb{R}_{>0}^{2k-1}$. For the PSS to function properly, each cover must have a *usage* value that determines how often it must be used. Depending on the properties of the PSS, the usage must be either *continuous* or *discrete*.

We assume that the PSS is tracking $m \in \mathbb{N}_{>0}$ *targets* $t_1, \ldots, t_m$. Each target t_j is represented by a multiple-interval and therefore $t_j \in \mathbb{I}$. Each single-interval of this target $s \in \mathcal{S}(t_j)$ represents a frequency band in which one of the target's emitters is broadcasting. The target also has *demand* $d_j \in \mathbb{Q}_{\geq 0}$ that describes how often the target must be *measured*. We say that target t_j is measured at a given time if the receiver has cover c such that $\exists s \in \mathcal{S}(t_j) : s \subseteq c$, i.e., at least one of the target's emitters is covered by c.

The goal of the PSSRM algorithm is to construct covers and determine their usage, while ensuring that all targets are sufficiently measured w.r.t. d_j. In practice, this must be made quickly because situations can change rapidly. Existing targets may disappear suddenly, while new ones may appear.

This problem can be solved in many ways. Inspiration can come from existing solutions to related problems, such as those presented in Section 4. Another possible approach is to break the problem down into two steps. In the first step, we construct promising covers using the left-right algorithm (LRA). Then we compute the usage of the constructed covers. Now, we will explain both steps in more detail.

1. *Cover construction using LRA.* LRA uses the set of all emitters $\mathcal{E} = \bigcup_{j=1}^{m} \mathcal{S}(t_j)$ and the set of cover shapes Δ as input to construct covers. For each shape $\delta \in \Delta$ and for each single-interval $\epsilon \in \mathcal{E}$, the algorithm constructs all δ-shaped covers such that the leftmost point of one of its single intervals is equal to $l(\epsilon)$. The same is done for the rightmost points and $r(\epsilon)$. In this way, LRA constructs n covers, $c_1, \ldots, c_n$. An example can be seen in Figure 1.
2. *Cover usage computation using MIC.* During this step, we compute how often each of the constructed covers should be used by the receiver so that all targets are sufficiently measured. This computation is equivalent to solving the following MIC problem.

   The MIC problem consists of m targets and n covers. Each cover c_i also has weight w_i and usage x_i. The weights enable MIC to express a wider range of possible problems. For example, they can be used to represent the cost of using different covers. The demand of target t_j is satisfied if the sum of the usages of the covers that measure the target is greater than or equal to d_j. The task is to determine the usage of each cover such that the demand of all targets is satisfied and the weighted cover usage is minimized. This can be formulated as

$$\min_x \sum_{i=1}^{n} w_i x_i \tag{1}$$

subject to:

$$\sum_{\substack{i \in \{1,\ldots,n\}: \\ (\exists s \in \mathcal{S}(t_j):\, s \subseteq c_i)}} x_i \geq d_j \quad \forall j \in \{1, \ldots, m\} \tag{2}$$

$$x_i \in \mathcal{X} \quad \forall i \in \{1, \ldots, n\}, \tag{3}$$

where $\mathcal{X}$ is the domain of cover usage. Figure 2 shows an instance of the MIC problem.

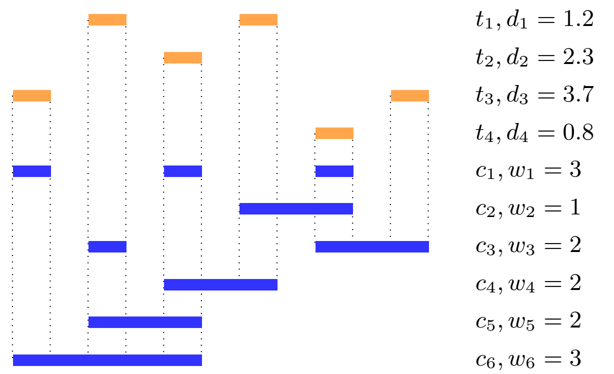

Figure 2: An instance of MIC consisting of four targets (orange rectangles) and six covers (blue rectangles). The usage domain is $\mathbb{N}$. Usage $x = (0, 0, 4, 3, 0, 0)$ is a feasible solution with the objective equal to 14. The optimal solution has an objective value of 11, and is given by $x^* = (2, 0, 1, 0, 0, 1)$.

4 Related Work

The multicover problem (MC) is a well-known extension of the set cover problem where each element must be covered by at least a given number of sets. This problem is known to be NP-hard (Hochbaum 1997). Hua et al. (2010) created multiple exact exponential-time algorithms to solve the problem optimally. Chekuri, Clarkson, and Har-Peled (2012) studied MC in geometric settings. They improved the approximation bounds for problems where the elements are points and the covers are sets of bounded VC-dimension, half-spaces in three dimensions, unit cubes, or so-called well-behaved shapes. Raman and Ray (2022) present an approximation algorithm for a similar problem involving non-piercing regions in the plane. These regions can be, for example, disks, pseudodiscs, or squares. In recent years, researchers have studied a variant of MC called partial set

MC (PSMC), in which only a fraction of the elements must be covered (Shi et al. 2019; Ran et al. 2020a,b). Ran et al. (2022) focused on PSMC in a geometric context. More precisely, their study involved points on the plane with unit squares as covers. The PSMC problem was also applied to optimize a wireless sensor network used for target tracking (Ran et al. 2021).

Hochbaum and Levin (2006) formulate the task of labor scheduling as MC. In this problem, the elements correspond to time periods, and the covers represent different work shifts. These shifts can be represented by binary vectors. The authors prove that when these vectors have the consecutive ones property, the problem can be solved in polynomial time. On the other hand, when the vectors consist of more than one consecutive block of ones, the problem is NP-hard. Ding, Fu, and Zhu (2011), motivated by the use in DNA sequencing, study a variant of MC where elements correspond to a sequence of numbers without repetition, the covers are multiple-intervals, and the demand is always equal to one. They provide an approximation algorithm, prove that the problem is NP-complete when the covers are 2-intervals, and show that for k-intervals, $k \geq 3$, the problem is APX-hard. Van Bevern et al. (2015) consider the same problem and manage to improve the approximation results. The papers mentioned in this paragraph address problems similar to the newly proposed MIC problem, but they are not equivalent. This is because the covered elements are numbers, not multiple-intervals. Nevertheless, thanks to this similarity, we show that some of the results proven by Hochbaum and Levin (2006) can be applied to MIC.

Another closely related topic is the study of PSSRM. However, there is very little existing literature focusing on this topic. An exception is the series of three articles (Kulmon, Suja, and Benko 2023; Suja and Kulmon 2024; Suja, Kulmon, and Benko 2025) that all address similar PSSRM problems. Their goal was to jointly optimize both the target search and tracking by determining which predetermined frequency bands to observe and when. The authors formulated this task as an optimization problem with simple constraints and a multi-criteria objective function. Kulmon, Suja, and Benko (2023) solve the problem directly using the Non-dominated Sorting Genetic Algorithm II (NSGA-II) (Deb et al. 2002). Suja and Kulmon (2024) focus on transforming the problem using goal programming and then solving it using genetic algorithm (GA). Finally, Suja, Kulmon, and Benko (2025) further studied and improved the individual objective functions and their interaction, resulting in a parameterless problem formulation. Jiang et al. (2018) studied PSSRM with a focus on target tracking. They formulated the problem as one with an extensive objective function that balances multiple aspects, such as tracking accuracy, resource utilization, and track priority, and solved it using GA.

Due to the similarity of radars and PSSs, radar resource management (RRM) is closely connected to PSSRM. Briheche et al. (2018a) studied the optimization of radar search patterns. They approximated the patterns as rectangles, which transformed the problem into a grid cover problem, a special case of a geometric set cover problem. Briheche et al. (2018b) also studied the theoretical complexity of this problem. The authors showed that the line and circle cover problem can be solved in polynomial time. In contrast, the rectangular grid cover problem was proved to be NP-hard.

Many recent RRM articles (Sun et al. 2022; Zhang, Liu, and Yang 2023; Shi et al. 2020, 2022, 2024) consider the task of radar-to-target assignment in multi-radar systems. They described this task using various objective functions that predict future tracking accuracy, which they typically optimized using multistage algorithms. Task scheduling also plays an important role in RRM. Due to the ever-changing situational conditions, the used algorithms must be fast. Consequently, most existing methods rely on hand-crafted heuristics and basic algorithms, such as list scheduling, earliest deadline first (EDF), and earliest start time (EST) (Hashmi et al. 2023; Qu, Ding, and Moo 2019, 2020), while others use machine learning techniques (Shaghaghi and Adve 2017, 2018; Shaghaghi, Adve, and Ding 2019; Gaafar et al. 2019). Unlike our approach, none of the presented PSSRM or RRM methods optimize the observed frequency bands to improve resource management.

5 Completeness of Left-Right Algorithm

Now, we will focus on the first step of the algorithm described in Section 3. This step uses LRA to construct the covers. The following theorem proves that the cover construction using LRA is sufficient and that no additional covers are necessary. In other words, LRA constructs covers in such a way that all possible emitter combinations are covered by at least one of them. We call this property the completeness of LRA.

Theorem 1 (Completeness of LRA)**.** *For any set of m single-interval emitters $\mathcal{E}$ and vector $\delta \in \mathbb{R}_{>0}^{2k-1}$, $k \in \mathbb{N}_{>0}$, if there exists cover c^* with shape δ such that it covers all emitters, LRA constructs cover c with shape δ that also covers all emitters.*

Proof. Since the shape's length is $2k-1$, cover c^* must be k-interval. The proof can disregard any single-interval $s \in \mathcal{S}(c^*)$ that does not cover an emitter, since they will not play any role in the proof. We denote this newly created cover as c^{**}. Similarly, when two emitters, ϵ_1 and ϵ_2, are covered by single-interval $s \in \mathcal{S}(c^{**})$, the two emitters can be combined into single-interval emitter $\epsilon_{1,2} = [\min\{l(\epsilon_1), l(\epsilon_2)\}, \max\{r(\epsilon_1), r(\epsilon_2)\}]$. This is possible because when emitter $\epsilon_{1,2}$ is covered, both ϵ_1 and ϵ_2 are also covered. Figure 3 shows both of these simplifications in the upper part. Therefore, we can only consider cases in which $m = k$ and each emitter is covered by a different single-interval belonging to c^{**}.

We can move c^{**} to the left as long as each emitter remains covered by its corresponding single-interval from c^{**}. Inevitably, we must stop the movement when some of the rightmost points of cover's single-intervals reach the rightmost point of its emitter. The cover made by this movement is denoted as c^l. The same happens when we move c^{**} to the right, creating a cover called c^r. The lower part of Figure 3 shows these movements and the covers they created. The red circles highlight the points, $l(\epsilon_{3,4})$ and $r(\epsilon_5)$, that prevent the covers from moving further.

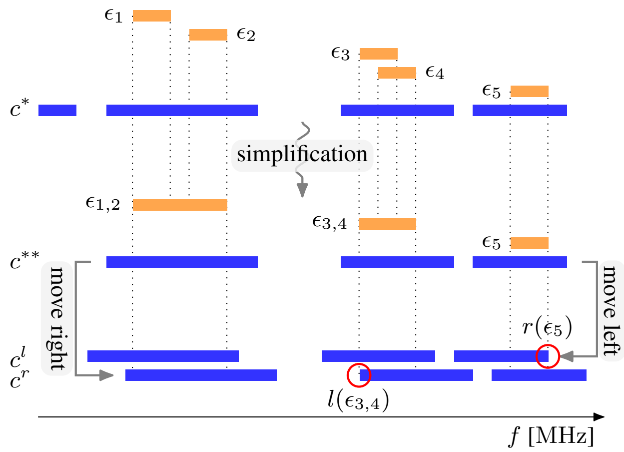

Figure 3: The removal of redundant single-intervals from cover c^*, the unification of emitters covered by the same single-interval from $\mathcal{S}(c^{**})$, and the construction of c^l and c^r through the movement of c^{**}. Orange rectangles represent the targets, while blue rectangles represent the covers.

Note that both c^l and c^r will always be among the covers constructed by applying LRA to emitter set $\mathcal{E}$ and shape δ. Therefore, the theorem has been proven. □

6 Multiple-Interval Coverage

The rest of this paper focuses on the MIC problem, as defined by Equations (1)–(3) in the second step of the algorithm presented in Section 3. Studying the MIC problem in its entirety would be difficult, so we focus on specific variants of the problem. These can be created, for example, by restricting the multiple-intervals of targets and covers, or by changing the domain of weights, demands, or usages. It is reasonable to study these variants since they can represent different types of PSSs and targets. For example, the receiver of some PSS may be limited to monitoring frequencies that can only be described by 2-intervals. Similarly, it might be known that the tracked targets have at most four emitters The weights can be used to describe the varying energy consumption needed for different types of covers.

To be able to effectively distinguish between different variants of MIC, we introduce the following problem notation, which was inspired by the notation used in scheduling proposed by Graham et al. (1979):

$$\mathcal{T}, \mathcal{D} \mid \mathcal{C}, \mathcal{W} \mid \mathcal{X}. \tag{4}$$

In this notation, $\mathcal{T}$ is the target domain, $\mathcal{D}$ is the domain of targets' demands, $\mathcal{C}$ is the cover domain, $\mathcal{W}$ is the domain of covers' weights, and the meaning of $\mathcal{X}$ remains as defined before, i.e., it is the cover usage domain. An instance of the problem described by Equation (4) is tuple (t, d, c, w) such that $\forall j \in \{1, \ldots, m\} : t_j \in \mathcal{T}$, $\forall j \in \{1, \ldots, m\} : d_j \in \mathcal{D}$, $\forall i \in \{1, \ldots, n\} : c_i \in \mathcal{C}$, and $\forall i \in \{1, \ldots, n\} : w_i \in \mathcal{W}$. For example, problem $(\mathbb{I}_{\leq 2}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_{\leq 3}, \mathbb{N} \mid \mathbb{N})$ contains all instances where targets are single-intervals or 2-intervals, demands are non-negative rational numbers, covers are single-intervals, 2-intervals, or 3-intervals, weights and usages are natural numbers. Figure 2 shows an instance of this problem. We sometimes enclose our notation in parentheses to make it easier to read. Furthermore, it is obvious that some problems can be subsets of another problem. For example, $(\mathbb{I}_{\leq 2}, \mathbb{N} \mid \mathbb{I}_1, \{0, 1\} \mid \mathbb{N}) \subset (\mathbb{I}_{\leq 2}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_{\leq 3}, \mathbb{N} \mid \mathbb{N})$.

The problem notation can be further extended by adding optional information or constraints. One such constraint, which is used in the following proofs, is called "proper set". It can be used in conjunction with targets, meaning that no target can be a proper subset of another, i.e., $\nexists g, h \in \{1, \ldots, m\} : t_g \subset t_h$. This is denoted by

$$\mathcal{T}, \mathcal{D}, \text{proper set} \mid \mathcal{C}, \mathcal{W} \mid \mathcal{X}. \tag{5}$$

Or it can be used with covers with the same meaning, i.e., no cover can be a proper subset of another; $\nexists g, h \in \{1, \ldots, n\} : c_g \subset c_h$. This is denoted by

$$\mathcal{T}, \mathcal{D} \mid \mathcal{C}, \mathcal{W}, \text{proper set} \mid \mathcal{X}. \tag{6}$$

6.1 Continuous MIC

This problem is motivated by PSSs with continuous usage, as described in Section 3. Therefore, it is practical to consider an MIC problem with the cover usage domain equal to the non-negative rational numbers, $(\mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}, \mathbb{Q}_{\geq 0}, \mid \mathbb{Q}_{\geq 0})$. By substituting $\mathcal{X}$ with $\mathbb{Q}_{\geq 0}$ in Equations (1)–(3), it is evident that the continuous MIC can be formulated as a linear program (LP) and thus solved in polynomial time.

6.2 Discrete MIC

We study MIC with a usage domain equal to $\mathbb{N}$, $(\mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$, because of PSSs with discrete usage. We mentioned at the beginning of this section that it would be pointless to study the MIC problem in its entirety. To support this claim, we prove that discrete MIC is in terms of complexity equivalent to MC and thus NP-hard.

Theorem 2. *In terms of complexity, discrete MIC with demands and weights from* $\mathbb{N}$, $(\mathbb{I}, \mathbb{N} \mid \mathbb{I}, \mathbb{N} \mid \mathbb{N})$, *is equivalent to the MC problem.*

Proof. Hochbaum and Levin (2006) defines the MC problem as the optimal solution of the following optimization problem:

$$\min_x \sum_{i=1}^{n} \gamma_i x_i \tag{7}$$

subject to:

$$\sum_{i \in \{1, \ldots, n\}} a_{ji} x_i \geq b_j \quad \forall j \in \{1, \ldots, m\} \tag{8}$$

$$x_i \in \mathbb{N} \quad \forall i \in \{1, \ldots, n\}, \tag{9}$$

where a is a given binary matrix, $\forall i \in \{1, \ldots, n\} : \gamma_i \in \mathbb{N}$, and $\forall j \in \{1, \ldots, m\} : b_j \in \mathbb{N}$.

Consider an instance of $(\mathbb{I}, \mathbb{N} \mid \mathbb{I}, \mathbb{N} \mid \mathbb{N})$. Clearly, by setting $\forall i \in \{1, \ldots, n\} : \gamma_i = w_i$, $\forall j \in \{1, \ldots, m\} : b_j = d_j$

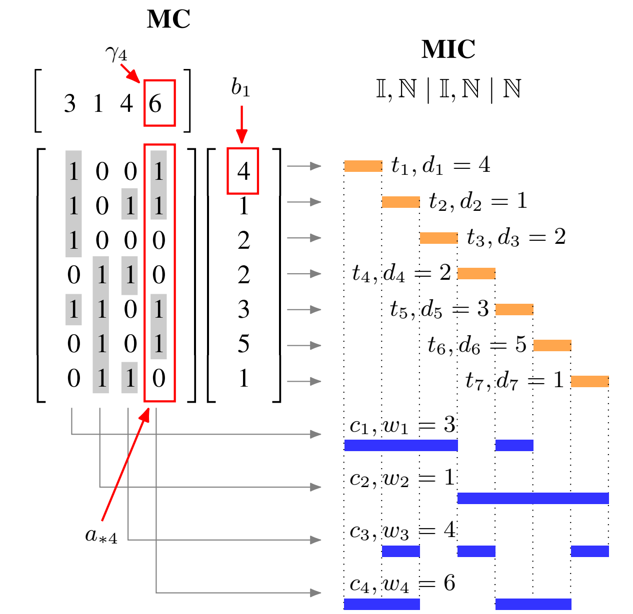

Figure 4: Example of a polynomial reduction from MC to $(\mathbb{I}, \mathbb{N} \mid \mathbb{I}, \mathbb{N} \mid \mathbb{N})$.

and $a_{ji} = 1$ if $\exists s \in \mathcal{S}(t_j) : \ s \subseteq c_i$ and $a_{ji} = 0$ otherwise, we get an instance of MC with the same optimal solution.

Conversely, consider an instance of MC. We build an equivalent instance of MIC by creating $\forall j \in \{1, \dots, m\}$: target $t_j = [j, j+1]$ with demand $d_j = b_j$. We then construct $\forall i \in \{1, \dots, n\}$: cover $c_i = \bigcup_{j \in \{k \in \{1,\dots,m\} \mid a_{ki}=1\}} t_j$ with weight $w_i = \gamma_i$. Obviously, cover c_i measures any target t_j such that $a_{ji} = 1$ and only these targets. Therefore, the optimal solution of the MIC instance gives an optimal solution of the MC instance. $\square$

Corollary 3. *Discrete MIC,* $(\mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$*, is NP-hard.*

Proof. This follows immediately from Theorem 2, and NP-hardness of MC (Hochbaum and Levin 2006), since $(\mathbb{I}, \mathbb{N} \mid \mathbb{I}, \mathbb{N} \mid \mathbb{N}) \subset (\mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$ $\square$

Next, we will prove that some discrete MIC problems, comprising of single-intervals, can be solved optimally in polynomial time. To accomplish this, we first prove the following lemma, which shows that problems with demands and weights from $\mathbb{Q}_{\geq 0}$ are equivalent to those involving demands and weights from $\mathbb{N}$.

Lemma 4. *Problem* $(\mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$ *can be transformed in polynomial time into problem* $(\mathbb{I}, \mathbb{N} \mid \mathbb{I}, \mathbb{N} \mid \mathbb{N})$ *without loss of generality.*

Proof. Let us consider an instance $\mathcal{A} = (t, d, c, w)$ of $(\mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$ and the associated problem formulated by Equations (1)–(3). Once we assume that the variables x_i are integers, we can, without loss of generality, substitute d_j by $d'_j = \lceil d_j \rceil$ in Equation (2).

Since $w_i \in \mathbb{Q}_{\geq 0}$, it can be expressed as a ratio of two integer values, $w_i = \frac{p_i}{q_i}$. We can achieve an equivalent objective function by multiplying every w_i by the least common multiple $\alpha = \mathrm{lcm}(q_1, \dots, q_n)$, resulting in integer weights $w'_i = \alpha\, w_i$. The optimal value of the modified integer instance $\mathcal{A}' = (t, d', c, w')$ is the optimal value of instance $\mathcal{A}$ multiplied by α. The optimal solution x^* is the same in both cases. $\square$

Now, we will turn our attention to discrete single-interval MIC problems involving targets with the "proper set" property. We will prove that these problems can be solved in polynomial time.

Lemma 5. *For a set of m single-interval targets t with "proper set" property and arbitrary single-interval cover c_0, when set t is ordered into a sequence such that the targets' right endpoints are non-decreasing, the targets covered by c_0 form a consecutive subsequence.*

Proof. Assume that the targets from t are sorted so that their right endpoints are non-decreasing. This forms a sequence $(t_1, t_2, \dots, t_m)$ where:

$$\forall g, h \in \{1, \dots, m\}; g < h : r(t_g) \leq r(t_h). \tag{10}$$

Assume that the target at index i is the left-most one which is covered by c_0. If no such target exists, no targets are covered by c_0, and thus the consecutive subsequence is empty and the statement is true. When such target exists, either all of the following targets are covered by c_0 and form a consecutive subsequence $(t_i, \dots, t_m)$ or there exists a target at index j which is covered by c_0 while the target at index $j+1$ is not covered by c_0. Because t_{j+1} is not covered by c_0 we have that either $r(c_0) < r(t_{j+1})$ or $l(t_{j+1}) < l(c_0)$. Let us consider that the second inequality is true. Due to Equation (10) we have $r(t_j) \leq r(t_{j+1})$, while $l(c_0) \leq l(t_j)$ holds because t_j is covered by c_0. The last three mentioned inequalities can be combined:

$$l(t_{j+1}) < l(c_0) \leq l(t_j) < r(t_j) \leq r(t_{j+1}). \tag{11}$$

Equation (11) would imply that t_j is a subset of t_{j+1}, which cannot be true because of the theorem's assumption that the targets do not contain each other. Therefore, $l(c_0) \leq l(t_j)$ cannot be true, and consequently $r(c_0) < r(t_{j+1})$ must always hold. Then for any t_k where $j + 1 < k$ we have $r(c_0) < r(t_{j+1}) \leq r(t_k)$, as a result of the cover ordering, hence t_k cannot be covered by c_0. Consequently, $(t_i, \dots, t_j)$ is a consecutive subsequence of all targets covered by c_0. $\square$

Theorem 6. *Problem* $(\mathbb{I}_1, \mathbb{N}, \textit{proper set} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$ *is solvable in polynomial time.*

Proof. We describe the problem in matrix form by transforming the general formulation of the MIC problem described by Equations (1)–(3):

$$\min_x \ w^T x \tag{12}$$

subject to:

$$Ax \geq d \tag{13}$$

$$x \in \mathbb{Z}^n_{\geq 0}, \tag{14}$$

where $w \in \mathbb{Q}^n_{\geq 0}$, $d \in \mathbb{N}^m$, and A is an $m \times n$ binary matrix whose entry $A_{j,i}$ is one if cover c_i covers target t_j and zero otherwise. The order in which targets appear in matrix A is arbitrary. Following Lemma 5, the targets can be sorted by their right endpoints so that the matrix A consists of columns with the consecutive-ones property. Therefore, matrix A is totally unimodular (TU). This, combined with the fact that d consists of integers, means that the LP relaxation of this problem has an integer optimum and thus the problem can be solved in polynomial time. □

Corollary 7. *Problem* $(\mathbb{I}_1, \mathbb{Q}_{\geq 0}, \textit{proper set} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$ *can be solved in polynomial time.*

Proof. This follows from Lemma 4, which proves that the target demands can be rounded, and Theorem 6. □

The "proper set" condition may seem quite abstract and unlikely to occur in real life. Therefore, we will show that it is equivalent to a problem in which all targets have the same demand. This could occur, for example, when all targets have the same priority.

Lemma 8. *Problem* $(\mathbb{I}_1, \{v\} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$*, in which the demands are all equal to* $v \in \mathbb{Q}_{\geq 0}$*, can be transformed into problem* $(\mathbb{I}_1, \{v\}, \textit{proper set} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$ *without loss of generality.*

Proof. The transformation algorithm takes instance of the original problem $\mathcal{A} = (t, (v)_{j=1}^m, c, w)$ and removes targets that are covered by any other target:

$$t' = \{t_j \mid j \in \{1, \ldots, m\}, (\nexists k \in \{1, \ldots, m\})[t_j \subset t_k]\}, \tag{15}$$

producing instance $\mathcal{B} = (t', (v)_{j=1}^{|t'|}, c, w)$.

First, we show that any feasible solution to instance $\mathcal{A}$, denoted as $x^{\mathcal{A}}$, is also a feasible solution to instance $\mathcal{B}$. To prove this, we must show that all targets in t' are satisfied by $x^{\mathcal{A}}$. We know that all targets in t are satisfied by $x^{\mathcal{A}}$, because it is a feasible solution to instance $\mathcal{A}$. Furthermore, by the definition of t' described by Equation (15), it holds that $t' \subseteq t$. Therefore, every target from t' is satisfied by $x^{\mathcal{A}}$.

Next, we show that any feasible solution to instance $\mathcal{B}$, denoted as $x^{\mathcal{B}}$, is also a feasible solution to instance $\mathcal{A}$. To prove this, we have to show that every target in t is satisfied by $x^{\mathcal{B}}$. We know that all targets from t' are satisfied by $x^{\mathcal{B}}$, because it is a feasible solution of instance $\mathcal{B}$. The remaining targets are those that were removed during the transformation, we will denote them by $t'' = (t \setminus t')$. From Equation (15) it follows that every target t''_j from t'' is a subset of some other target t'_k from t', which is, as already mentioned, satisfied by $x^{\mathcal{B}}$. Since we assume that all targets have the same demand and $t_j \subset t_k$, t_j must also be satisfied by $x^{\mathcal{B}}$. Consequently, every target from t'' is satisfied by $x^{\mathcal{B}}$. Hence, any feasible solution to problem $\mathcal{B}$ is also a feasible solution to problem $\mathcal{A}$.

Since every feasible solution of one instance is also a feasible solution of the other instance and the weights of the covers are the same for both instances, the optimal solutions must also be the same. □

Corollary 9. *Problem* $(\mathbb{I}_1, \{v\} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$ *can be solved in polynomial time.*

Proof. This follows from Lemma 4, Lemma 8, and Theorem 6. □

This corollary concludes our study of the discrete single-interval MIC problem with a focus on targets. We will now prove similar properties concerning the covers. Unfortunately, the existing proofs cannot be reused because of the difference between the concepts of "being covered" and "covering others."

Lemma 10. *For a set of* n *single-interval covers* c *with "proper set" property and arbitrary single-interval target* t_0*, when set* c *is ordered into a sequence such that the covers' right endpoints are non-decreasing, the covers from* c *covering* t_0 *form a consecutive subsequence.*

Proof. Assume that the covers from c are sorted so that their right endpoints are non-decreasing. This forms a sequence $(c_1, c_2, \ldots, c_n)$ with the same property as described in Equation (10). Assume that the cover at index i is the left-most one that covers t_0. If no such cover exists, then no cover covers t_0, so the consecutive subsequence is empty and the statement is true. When such a cover exists, either all of the following covers cover t_0 and form a consecutive subsequence $(c_i, \ldots, c_n)$ or there exists a cover at index j, which covers t_0, while the cover at index $j+1$ does not cover t_0. Since c_{j+1} does not cover t_0, we have either $l(t_0) < l(c_{j+1})$ or $r(c_{j+1}) < r(t_0)$. The second option is not possible. Due to the ordering, we have $r(c_j) \leq r(c_{j+1})$. By joining the last two mentioned inequalities, we get:

$$r(c_j) \leq r(c_{j+1}) < r(t_0). \tag{16}$$

This cannot be true because it implies that t_0 is not covered by c_j, which is a contradiction. Consequently,

$$l(t_0) < l(c_{j+1}) \tag{17}$$

must be true. Then for any c_k where $j+1 < k$ we have either $l(c_k) < l(c_{j+1})$ or $l(c_{j+1}) \leq l(c_k)$. The first inequality cannot be true because when combined with the interval order, we get:

$$l(c_k) < l(c_{j+1}) < r(c_{j+1}) \leq r(c_k), \tag{18}$$

implying that c_{j+1} is a subset of c_k, which is in contradiction with our initial statement. Therefore, the second inequality holds and can be combined with Equation (17), resulting in $l(t) < l(c_{j+1}) \leq l(c_k)$; hence c_k cannot cover t_0. Consequently, $(c_i, \ldots, c_j)$ is a consecutive subsequence of all covers from c containing t_0. □

Theorem 11. *Problem* $(\mathbb{I}_1, \mathbb{N} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0}, \textit{proper set} \mid \mathbb{N})$ *can be solved in polynomial time.*

Proof. The proof of this theorem is almost identical to the proof of Theorem 6. We can formulate the problem in a matrix form and then, as stated in Lemma 10, we can sort the covers by their right endpoints so that the matrix A consists of rows with the consecutive-ones property. Therefore, A is TU and, according to our definition of the problem, demand vector d is integral. These facts mean the LP relaxation of the problem has an integral optimum, and it can be solved in polynomial time. □

Corollary 12. *Problem* $(\mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0}, \textit{proper set} \mid \mathbb{N})$ *can be solved in polynomial time.*

Proof. This follows from Lemma 4, which proves that the target demands can be rounded, and Theorem 11. □

Now, we will show that the problem with "proper set" covers is equivalent the the problem in which all covers have the same weight.

Lemma 13. *Problem* $(\mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_1, \{v\} \mid \mathbb{N})$*, in which the weights are all equal to* $v \in \mathbb{Q}_{\geq 0}$*, can be transformed into problem* $(\mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_1, \{v\}, \textit{proper set} \mid \mathbb{N})$ *without loss of generality.*

Proof. The transformation algorithm takes instance of the original problem $(t, d, c, (v)_{i=1}^{n})$ and removes covers that are covered by any other cover:

$$c' = \{c_i \mid i \in \{1, \dots, n\}, (\nexists k \in \{1, \dots, n\})[c_i \subset c_k]\}, \tag{19}$$

producing instance $(t, d, c', (v)_{i=1}^{|c'|})$.

First, we will show that feasible solution $x^{\mathcal{A}}$ of instance $\mathcal{A}$ can be transformed into feasible solution $x^{\mathcal{B}}$ of instance $\mathcal{B}$. We can construct $x^{\mathcal{B}}$ as follows. First, initialize $x^{\mathcal{B}}$ by assigning the same usages as in $x^{\mathcal{A}}$ to the covers that are in c'. It is important to note that, according to Equation (19), for each cover c''_i not in c', there is cover c'_k from c' that covers at least the same targets as c''_i, if not more. Therefore, c''_i can be substituted with c'_k, which is done by increasing $x_k^{\mathcal{B}}$ by the value of $x_i^{\mathcal{A}}$. If we repeat this substitution for all covers not in c', the resulting $x^{\mathcal{B}}$ will be a feasible solution of instance $\mathcal{B}$.

Now, we will show that feasible solution $x^{\mathcal{B}}$ of instance $\mathcal{B}$ can be transformed into a feasible solution of instance $\mathcal{A}$. We know that due to the feasibility of $x^{\mathcal{B}}$, it satisfies all the targets t. We also have $c' \subseteq c$, because of the definition of c' described in Equation (19). Therefore, we can build feasible solution $x^{\mathcal{A}}$ of instance $\mathcal{A}$ by assigning zeros to the covers that are not in c' and assigning the same usages as in $x^{\mathcal{B}}$ to the covers that are in c'.

Since every feasible solution for one instance can be transformed into a feasible solution for the other instance without changing the total cover usage and the weights of the covers are the same, the optimal solutions must also be the same.

□

Corollary 14. *Problem* $(\mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_1, \{v\} \mid \mathbb{N})$ *can be solved in polynomial time.*

Proof. This follows from Lemma 4, Lemma 13, and Theorem 11. □

We have shown that some single-interval MIC problems can be solved in polynomial time. Unfortunately, the MIC problem can easily become NP-hard. This occurs simply by extending the cover domain to include 2-intervals, as proven by the following theorem.

Problem	Complexity
$(\mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{Q}_{\geq 0})$	Polynomial
$(\mathbb{I}_1, \mathbb{Q}_{\geq 0}, \text{proper set} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$	Polynomial
$(\mathbb{I}_1, \{v\} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{N}), v \in \mathbb{Q}_{\geq 0}$	Polynomial
$(\mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_1, \mathbb{Q}_{\geq 0}, \text{proper set} \mid \mathbb{N})$	Polynomial
$(\mathbb{I}_1, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_1, \{v\} \mid \mathbb{N}), v \in \mathbb{Q}_{\geq 0}$	Polynomial
$(\mathbb{I}_1, \{1\} \mid \mathbb{I}_{\leq 2}, \{1, 2\} \mid \mathbb{N})$	NP-hard
$(\mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$	NP-hard

Table 1: The summary of key complexity results for MIC.

Theorem 15. *Problem* $(\mathbb{I}_1, \{1\} \mid \mathbb{I}_{\leq 2}, \{1, 2\} \mid \mathbb{N})$ *is NP-hard.*

Proof. Hochbaum and Levin (2006) proved that the MC problem, as defined in Theorem 2, in which each column of the matrix a has at most two blocks of consecutive ones is NP-hard. Moreover, they prove this by a reduction that takes a variant of the SAT problem and constructs an instance of MC such that $\forall i \in \{1, \dots, n\} : \gamma_i \in \{1, 2\}$ and $\forall j \in \{1, \dots, m\} : b_j = 1$. We denote these instances of MC as 2MC.

We can use the same reduction as in Theorem 2 on any 2MC instance. This will result in an MIC instance comprising single-interval targets with unit demands and at most 2-interval covers with costs from $\{1, 2\}$. Consequently, $(\mathbb{I}_1, \{1\} \mid \mathbb{I}_{\leq 2}, \{1, 2\} \mid \mathbb{N})$ is NP-hard. □

Corollary 16. *For an arbitrary* $k \in \{1, 2, \dots\}$ *and* $l \in \{2, 3, \dots\}$ *the problem* $(\mathbb{I}_{\leq k}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_{\leq l}, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$ *is NP-hard.*

Proof. This follows from Theorem 15, because $\forall k \in \{1, 2, \dots\}, \forall l \in \{2, 3 \dots\} : (\mathbb{I}_1, \{1\} \mid \mathbb{I}_{\leq 2}, \{1, 2\} \mid \mathbb{N}) \subset (\mathbb{I}_{\leq k}, \mathbb{Q}_{\geq 0} \mid \mathbb{I}_{\leq l}, \mathbb{Q}_{\geq 0} \mid \mathbb{N})$. □

7 Conclusion

In this paper, we investigated the theoretical properties of tasks that are connected to PSSRM. We proved the completeness of LRA, meaning LRA constructs covers that cover all possible emitter combinations, and no additional covers are necessary. Then, we defined the MIC problem and introduced a notation system to distinguish between its variants. We showed that some of the discrete single-interval MIC problems can be solved in polynomial time. Additionally, we proved that any discrete MIC problem with a cover domain that includes 2-intervals is NP-hard. A summary of these complexity results can be found in Table 1. In the future, it would be interesting to study the complexity of discrete single-interval MIC without the "proper set" constraint, as well as MIC problems with multiple-interval targets and single-interval covers.

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