Temporal Properties of Conditional Independence in Dynamic Bayesian Networks

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Abstract

Dynamic Bayesian networks (DBNs) are compact graphical representations used to model probabilistic systems where interdependent random variables and their distributions evolve over time. In this paper, we study the verification of the evolution of conditional-independence (CI) propositions against temporal logic specifications. To this end, we consider two specification formalisms over CI propositions: linear temporal logic (LTL), and non-deterministic Büchi automata (NBAs). This problem has two variants. Stochastic CI properties take the given concrete probability distributions into account, while structural CI properties are viewed purely in terms of the graphical structure of the DBN. We show that deciding if a stochastic CI proposition eventually holds is at least as hard as the Skolem problem for linear recurrence sequences, a long-standing open problem in number theory. On the other hand, we show that verifying the evolution of structural CI propositions against LTL and NBA specifications is in PSPACE, and is NP- and coNP-hard. We also identify natural restrictions on the graphical structure of DBNs that make the verification of structural CI properties tractable.

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1 Introduction

Bayesian networks (BNs) (Pearl 1985, 1988; Neapolitan 1990) are prominent tools in both data science and artificial intelligence that enable modeling and reasoning under uncertainty. BNs succinctly represent a full joint probability distribution by using a directed acyclic graph (DAG) as a template to capture dependencies between variables and prescribe the probability distribution of each variable conditioned on its parents. BNs have successfully been applied in medical AI (Lucas, van der Gaag, and Abu-Hanna 2004), natural language processing (Manning and Schütze 1999), robotics (Thrun, Burgard, and Fox 2005), bioinformatics (Friedman 2000), and risk assessment (Fenton and Neil 2012).

Dynamic Bayesian Networks (DBNs) extend BNs to describe systems where the outcomes modeled by random variables evolve with time (Murphy 2002; Koller and Friedman 2009). DBNs succinctly represent a *sequence* of full joint probability distributions of a set of random variables, i.e., a DBN prescribes an initial joint probability distribution for the variables $\mathbf{V}^0$, and also prescribes the joint distribution of $\mathbf{V}^{t+1}$, the variables at time step (or time slice) $t+1$, conditioned on the variables $\mathbf{V}^t$ at time step t. These are respectively given by an *initial* BN and a *step* BN, and their corresponding DAGs are collectively referred to as the *DBN template*. To make a concrete DBN, the template is instantiated with *conditional probability distributions* (CPDs).

The temporal dimension of DBNs has motivated applications in robotics (Thrun, Burgard, and Fox 2005) and systems biology (Palaniappan and Thiagarajan 2012). Today, DBNs find applications in various areas, and the following examples illustrate the continued relevance of DBNs in modern AI. Integrating DBNs can make computer-vision algorithms more adaptive and efficient (Piedade and Miraldo 2023), and furthermore, DBNs and LLMs have been combined to build multimodal AI systems that interact with users in a context-aware manner (Han et al. 2025). In healthcare, DBNs support early sepsis prediction in the ICU while remaining interpretable and robust to missing data (Agard et al. 2025). Recent neuroscience work uses multi-timescale DBNs to infer directed, behavior-dependent interactions between brain regions, demonstrating utility on high-quality datasets (Das et al. 2024). Beyond medicine, DBNs are applied, e.g., to solar power generation forecasting (Zhang, Yan, and Liu 2024) and resilience analysis of dynamic engineering systems such as transportation networks (Kammouh, Gardoni, and Cimellaro 2020). Given their relevance, algorithms for learning DBN structures from data are an active area of research (see, e.g., Meng et al. (2024)).

Example 1.1. To illustrate DBNs and DBN-templates, consider a system coordinating different probabilistic components either having access only to *low-security* information or also to *high-security* information. In each time step t, the low-security components provide an input L^t and the high-security components provide an input H^t. The system then produces a low-security output O^t and a high-security output S^t (S for secret). The dependencies between these variables are depicted in the DBN-template depicted in Fig. 1a: The *initial template* marked with 0 expresses that initially O^0 depends only on L^0 and S^0 depends only on H^0. All other pairs of variables are independent. The *step template*

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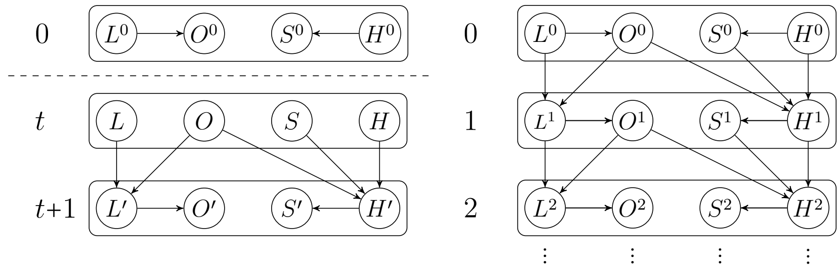

(a) Example of a DBN-template. (b) Unfolding of the DBN-template.

		L'	
L	O	0	1
0	0	0.2	0.8
0	1	0.5	0.5
1	0	0.3	0.7
1	1	0	1

(c) Example of a CPD.

Figure 1: The DBN-template described in Ex. 1.1 and its unfolding as well as an example CPD.

depicted below uses the variables L, O, S, and H representing the variables at the current time step as well as copies L', O', S', and H' representing the variables at the next time step. For example, this template captures that the next low-security input L^{t+1} always depends directly only on the previous low-security input L^t and output O^t. To represent all (direct) dependencies between variables in the timed sequence of variables, the template can be unfolded into one infinite directed acyclic graph (DAG), called the *unfolding of the DBN-template*, as depicted in Fig. 1b.

A DBN based on this DBN-template additionally consists of the CPDs for each variable in the initial template and for each primed variable in the step template given its parents. Assuming that all variables take only values 0 and 1, an example CPD for the variable L' in the step template is depicted in Fig. 1c. For each combination of values of the parent variables L and O, it specifies the probability with which L' takes value 0 and 1. This CPD is applied at each time step in the unfolding of the DBN. For variables without parents, such as L^0 and H^0 in the initial template, the CPDs specify the probabilities with which they take value 0 and 1.

A fundamental concern (see, e.g., Heckerman, Geiger, and Chickering (1995)) in reasoning about probabilistic models is the characterization and/or deduction of stochastic conditional independence (CI) of sets $\mathbf{X}$ and $\mathbf{Y}$ of random variables given the values of a set $\mathbf{Z}$, denoted $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$.

In seminal work, (Geiger, Verma, and Pearl 1990) show that the truth of these CI propositions, which satisfy the so-called graphoid axioms, can be deduced from the underlying DAG template in the case of BNs. Specifically, they define structural conditional independence through the efficiently testable graphical notion of *d-separation*, denoted as $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$ when $\mathbf{Z}$ d-separates $\mathbf{X}$ and $\mathbf{Y}$. They then show soundness: if $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$, then $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ for all conditional probability distributions. Subsequently, Meek (1995) showed a form of completeness: i.e., if $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$ fails then $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ also fails for all but a (Lebesgue) measure-0 set of conditional probability distribution parameters.

DBNs can naturally be associated with an infinite sequence of BNs, with the t-th term being obtained by unfolding up to time slice t. We call a statement of the form $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$ (respectively, $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$) an atomic proposition of structural (respectively, stochastic) CI, and say that it holds at time t if $(\mathbf{X}^t \perp\!\!\!\perp \mathbf{Y}^t \mid \mathbf{Z}^t)$ (respectively, $(\mathbf{X}^t \perp \mathbf{Y}^t \mid \mathbf{Z}^t)$) holds in the unfolding of the DBN up to time t. Given a collection A of structural (respectively, stochastic) CI propositions, the DBN defines a *trace*, i.e., an infinite word over the alphabet 2^A whose t-th position records which of the propositions hold at time t.

In this paper, we concern ourselves with checking the properties of the trace, such as: is $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$ ever false? In Ex. 1.1, is the output always independent of the secret given the low-security input? In a system, is it always the case that if inputs I_1, I_2 are independent, then so are outputs O_1, O_2?

To express temporal properties of systems, the use of temporal logics such as linear temporal logic (LTL) and of non-deterministic Büchi automata (NBAs), capturing all ω-regular languages, has emerged as a success story over the past decades (see, e.g., (Baier and Katoen 2008)). We aim to employ these formalisms to talk about the temporal aspects of CIs.

Example 1.2. The three properties mentioned above are expressed in LTL as: (i) $\Diamond \neg(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$, where $\Diamond$ is the temporal modality for "eventually"; (ii) $\Box(O \perp\!\!\!\perp S \mid L)$, where $\Box$ is the temporal modality for "globally", and dual to $\Diamond$; (iii) $\Box((I_1 \perp\!\!\!\perp I_2) \rightarrow (O_1 \perp\!\!\!\perp O_2))$.

Given an LTL formula φ over the set of atomic propositions A or an NBA $\mathcal{B}$ over the alphabet 2^A, the *structural CI model-checking problem for DBN-templates* now asks whether the trace of a DBN-template satisfies φ or is accepted by $\mathcal{B}$, respectively. The *stochastic CI model-checking problem for DBNs* asks the analogous question for the trace of a DBN with respect to a set of stochastic CI propositions.

1.1 Contributions

1. In Sec. 3, we introduce temporal specification mechanisms for the evolution of structural or stochastic CI propositions in DBN-templates and DBNs, respectively, using LTL and NBAs. We formulate the resulting structural and stochastic CI model-checking problems.
2. In Sec. 4, we show that the structural CI model-checking problems of DBN-templates against LTL formulas and against NBAs are both in PSPACE and NP-hard as well as coNP-hard. Under the natural restriction that the initial template of a DBN-template only contains edges that also appear as intra-slice edges in the step template, we prove that the problems are in P.

3. Given full DBNs with CPDs, we show in Sec. 5 that checking eventual stochastic CI is as hard as the Skolem problem for linear recurrence sequences, a famous number-theoretic problem whose decidability status has been open for many decades. This implies that a decidability result for the stochastic CI model-checking problems is out of reach without a breakthrough in analytic number theory.

1.2 Related Work

The question how to detect structural CIs in BNs has been answered in the 1980s and 1990s by showing that d-separation characterizes all structural CIs that follow from the structure of a BN, that this is equivalent to stochastic CI under almost all choices of CPDs, and by showing that the d-separation can compute these structural CIs in polynomial time (see (Pearl 1988; Geiger, Verma, and Pearl 1990; Meek 1995)). Exactly determining whether a stochastic CI holds requires exact computation of the necessary conditional probability distributions. Methods for approximate testing of conditional independence of discrete random variables, however, are an active area of research (see, e.g., (Canonne et al. 2018; Teymur and Filippi 2020)). Orthogonally, seminal work by Boutilier et al. (1996) studies so-called context-sensitive independence expressing that variables might only be independent under specific assignments of values to other variables. Like stochastic CI, this kind of independence depends on the concrete CPDs.

We are not aware of thorough studies of d-separation and the detection of CIs in DBNs, let alone the formal verification of temporal properties of CIs in DBNs. Regarding other extension of BNs, Shen et al. (2019) study CIs in testing BNs, an extension of BNs representing a set of probability distributions instead of a singly distributions, and show that d-separation can still be used to detect structural CIs.

2 Preliminaries

Probability spaces and conditional independence. We assume knowledge of the basics of probability theory (Klenke 2007). In this paper, we work with discrete random variables. Disjoint tuples of random variables $\mathbf{X}, \mathbf{Y}$ are considered conditionally independent given $\mathbf{Z}$ (denoted as $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$), if for any values $\mathbf{x}, \mathbf{y}, \mathbf{z}$ (provided $\Pr[\mathbf{Z} = \mathbf{z})] > 0$, the following holds: $\Pr(\mathbf{X} = \mathbf{x}, \mathbf{Y} = \mathbf{y} \mid \mathbf{Z} = \mathbf{z}) = \Pr(\mathbf{X} = \mathbf{x} \mid \mathbf{Z} = \mathbf{z}) \cdot \Pr(\mathbf{Y} = \mathbf{y} \mid \mathbf{Z} = \mathbf{z})$.

Bayesian networks A Bayesian network (BN) is a type of probabilistic graphical model that expresses a set of variables and their conditional dependencies using a directed acyclic graph (DAG), where each node represents a variable, and the edges indicate direct probabilistic dependencies between the variables.

Definition 2.1 (Bayesian Network)**.** A Bayesian network (BN) over a finite set $\mathbf{V}$ of discrete random variables is a tuple $B = \langle \mathbf{V}, \mathcal{E}, \mathcal{P} \rangle$, where:

- Each element of $\mathbf{V}$ is represented as a vertex of a DAG;
- The set of directed edges is $\mathcal{E} \subseteq \mathbf{V} \times \mathbf{V}$; we call the DAG $\langle \mathbf{V}, \mathcal{E} \rangle$ the *template* of the BN.
- The probability distribution of $\mathbf{V}$ is expressed in terms of a collection $\mathcal{P}$ of conditional probability distributions (CPDs), i.e., for each variable X with parents $\mathsf{pa}(X) = \mathbf{Y}$, $\mathcal{P}$ prescribes $\Pr[X = x \mid \mathbf{Y} = \mathbf{y}]$ for all possible $x, \mathbf{y}$.

We refer to the set of all BNs with a given BN-template $\mathcal{T}$ as the family $\mathsf{Fam}(\mathcal{T})$.

As an illustrative scenario, consider a BN where all variables are binary. Then $\mathcal{P}$ consists of $\sum_{X \in \mathbf{V}} 2^{|\mathsf{pa}(X)|}$ parameters, each term of the summation counting the number of parameters required to prescribe the probability of X being 1, depending on the values taken by its parents.

Definition 2.2 (d-paths and d-separation)**.** Given a BN-template (i.e., DAG) $\mathcal{T} = (\mathbf{V}, \mathcal{E})$ and three pairwise disjoint sets $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \subseteq \mathbf{V}$ of nodes, a d-path from $\mathbf{X}$ to $\mathbf{Y}$ with respect to $\mathbf{Z}$ is a sequence $W_0, \ldots, W_k$ of nodes with the following properties:

- $W_0 \in \mathbf{X}$ and $W_k \in \mathbf{Y}$,
- for each $i < k$, either $(W_i, W_{i+1}) \in \mathcal{E}$ or $(W_{i+1}, W_i) \in \mathcal{E}$
- for all $0 < i < k$, if the node W_i has an outgoing edge to W_{i-1} or W_{i+1} (or both), then $W_i \notin \mathbf{Z}$,
- for all $0 < i < k$, if the node W_i has incoming edges from both W_{i-1} and W_{i+1}, then one of the descendants of W_i is in $\mathbf{Z}$ (we consider a node to be its own descendant and ancestor). We call such a node W_i a *collider* and say that the collision is *attributed* to the descendants of W_i in $\mathbf{Z}$.

If there is no such path, we say that $\mathbf{Z}$ d-separates $\mathbf{X}$ and $\mathbf{Y}$. In this case, we write $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$ and say that $\mathbf{X}$ and $\mathbf{Y}$ are *structurally independent* given $\mathbf{Z}$.

We remark that both structural and stochastic conditional independence (CI) satisfy the *graphoid* axioms (Geiger, Verma, and Pearl 1990, p. 511, (4a)-(4d)) (see also (Spohn 1980) for a proof of the stochastic case). Structural CI via d-separation in a BN-template $\mathcal{T}$ is known to be equivalent to stochastic CI in all members of the family $\mathsf{Fam}(\mathcal{T})$.

Theorem 2.3 (Soundness and completeness of d-separation, (Pearl 1988), (Meek 1995))**.** *Given a BN-template $\mathcal{T} = \langle \mathbf{V}, \mathcal{E} \rangle$ and pairwise disjoint sets of random variables $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \subseteq \mathbf{V}$, the following two statements are equivalent and can be checked in polynomial time:*

- $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$,
- *for all BNs in $\mathsf{Fam}(\mathcal{T})$, we have $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$, i.e., $\mathbf{X}$ and $\mathbf{Y}$ are (stochastically) independent given $\mathbf{Z}$.*

Dynamic Bayesian networks (DBNs). We use Dynamic Bayesian networks (Murphy 2002; Koller and Friedman 2009) to model probabilistic systems where the entities captured by random variables evolve over time. Formally, consider a finite set $\mathbf{V}$ of random variables. We track the evolution of $\mathbf{V}$ with time through a countably infinite sequence $(\mathbf{V}^t)_{t=0}^{\infty}$ of copies of the random variables in $\mathbf{V}$.

The evolution itself is modeled as a dynamical system whose state at time t is a BN involving the variables $\bigcup_{i=0}^{t} \mathbf{V}^t$. The initial BN is given by $\langle \mathbf{V}^0, \mathcal{E}^0, \mathcal{P}^0 \rangle$.

We use a copy $\mathbf{V}'$ of the random variables to express the update dynamics, which at each step $t \geq 1$, introduce the variables $\mathbf{V}^t$ with parents in $\mathbf{V}^{t-1} \cup \mathbf{V}^t$, while keeping the rest of the network unchanged. Formally, we have a *step template* which is a BN-template with variables $\mathbf{V} \cup \mathbf{V}'$ and edge relation $\mathcal{E}^{\mathsf{step}} \subseteq (\mathbf{V} \cup \mathbf{V}') \times \mathbf{V}'$. Semantically, if X' has parents $Y'_{i_1}, \ldots, Y'_{i_\ell}, Y_{j_1}, \ldots, Y_{j_k}$ in the step template, then for each $t \geq 1$, X^t has parents $Y^t_{i_1}, \ldots, Y^t_{i_\ell}, Y^{t-1}_{j_1}, \ldots, Y^{t-1}_{j_k}$. Finally, we have the conditional probability distribution parameters $\mathcal{P}^{\mathsf{step}}$ which prescribe $\Pr[X' = x \mid \mathsf{pa}(X') = \mathbf{y}]$ for all $x, \mathbf{y}$. Semantically, we have that the distribution of X^t conditioned on its parents is the same for all $t \geq 1$.

We can thus specify a DBN via $\langle \mathbf{V}, \mathcal{E}^0, \mathcal{P}^0, \mathcal{E}^{\mathsf{step}}, \mathcal{P}^{\mathsf{step}} \rangle$. Its structural properties are given by the tuple $\mathcal{T}_{\mathsf{DBN}} = \langle \mathbf{V}, \mathcal{E}^0, \mathcal{E}^{\mathsf{step}} \rangle$, which we refer to as a *DBN-template*. Analogous to BNs, we refer to the set of all DBNs with a given template $\mathcal{T}_{\mathsf{DBN}}$ as the family $\mathsf{Fam}(\mathcal{T}_{\mathsf{DBN}})$. As explained earlier, in a DBN of binary variables, $\mathcal{P}^0$ consists of $\sum_{X \in \mathbf{V}^0} 2^{|\mathsf{pa}(X)|}$ parameters, and $\mathcal{P}^{\mathsf{step}}$ consists of $\sum_{X \in \mathbf{V}'} 2^{|\mathsf{pa}(X)|}$ parameters.

We say that a DBN-template is *restricted* if whenever $(X^0, Y^0) \in \mathcal{E}^0$, we also have that $(X', Y') \in \mathcal{E}^{\mathsf{step}}$. So, the dependencies that exist initially at time step 0 also have to be present at all later time steps, which is captured by their presence in the step template between the corresponding primed variables. The DBN-template depicted in Fig. 1 is an example of a restricted DBN template.

Equivalence of DBNs with Markov Chains. That the semantics of a DBN can be expressed in terms of the evolution of a Markov chain is folklore. For completeness, we state an elementary lemma expressing this fact formally.

Lemma 2.4. *Given a DBN with k binary random variables, we can construct an equivalent Markov chain with* 2^k *states.*

Given a Markov chain with K *states, we can construct an equivalent DBN with* $\lceil \log K \rceil$ *binary random variables.*

3 Specification Formalisms for Temporal Conditional Independence Properties

In this paper, we study the verification of DBNs and DBN-templates against linear-temporal properties regarding the evolution of conditional independencies (CIs) over time. For DBN-templates with variables $\mathbf{V}$ we use atomic propositions of the form $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$, which we call *structural CI propositions*. We say that $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$ holds in DBN-template $\mathcal{T}$ at time t if $(\mathbf{X}^t \perp\!\!\!\perp \mathbf{Y}^t \mid \mathbf{Z}^t)$ holds in the unfolding of $\mathcal{T}$ to a BN-template after t time steps. Similarly, for full DBNs with concrete CPDs, we use *stochastic CI propositions* $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ that hold at time t in a DBN $\mathcal{B}$ if $(\mathbf{X}^t \perp \mathbf{Y}^t \mid \mathbf{Z}^t)$ holds in the BN formed at time t. A first result connecting structural CI and stochastic CI is immediate by simply unfolding and applying Thm. 2.3.

Proposition 3.1. *If* $(\mathbf{X}^t \perp\!\!\!\perp \mathbf{Y}^t \mid \mathbf{Z}^t)$ *in a DBN-template* $\mathcal{T}$ *for some t, then,* $(\mathbf{X}^t \perp \mathbf{Y}^t \mid \mathbf{Z}^t)$ *in every DBN* $\mathcal{B} \in \mathrm{Fam}(\mathcal{T})$.

The converse direction, i.e., the completeness of d-separation for DBNs, however, turns out to be intricate. We discuss this issue in Sec. 6.

Trace of a DBN(-template). For any (finite) set A of such structural CI propositions, a DBN-template defines a *trace* $\tau \in (2^A)^\omega$, i.e., an infinite word over the alphabet 2^A, where the letter at position t indicates which propositions hold at time t. Likewise, a full DBN defines a trace $\tau \in (2^B)^\omega$ for any (finite) set B of stochastic CI propositions.

We seek to verify whether the trace τ of a DBN-template or of a DBN satisfies a logical specification. We use the notation $\tau(t)$, to refer to the t-th position of τ, and the notation $\tau[t : \infty]$ to refer to the suffix of τ starting at position t, e.g., $\tau[0 : \infty] = \tau$, $\tau[t : \infty][t' : \infty] = \tau[t + t' : \infty]$.

Linear temporal logic (LTL). We consider two common logical formalisms (for a comprehensive exposition of which the reader is referred to (Baier and Katoen 2008)). The first is linear temporal logic (LTL, introduced in (Pnueli 1977)), whose formulae φ over a set of atomic propositions A are syntactically given by the grammar $\varphi := a \mid \neg\varphi \mid \varphi \wedge \varphi \mid \bigcirc\varphi \mid \varphi\,\mathcal{U}\,\varphi$ where $a \in A$ is an atomic proposition. The operator $\bigcirc$ is called "next" and the operator $\mathcal{U}$ is called "until". Semantically, it is defined recursively whether an infinite word τ over 2^A satisfies an LTL formula (written $\tau \models \varphi$) as follows:

- $\tau \models a$ if and only if $a \in \tau(0)$,
- $\tau \models \neg\varphi$ if and only if $\tau \not\models \varphi$,
- $\tau \models \varphi_1 \wedge \varphi_2$ if and only if both $\tau \models \varphi_1$ and $\tau \models \varphi_2$,
- $\tau \models \bigcirc\varphi$ if and only if $\tau[1 : \infty] \models \varphi$,
- $\tau \models \varphi_1\,\mathcal{U}\,\varphi_2$ if and only if there exists t s.t. $\tau[t : \infty] \models \varphi_2$ and for all $t' < t$, $\tau[t' : \infty] \models \varphi_1$.

For notational convenience, we allow access to all the usual Boolean connectives, true, false, as well as the temporal modalities $\diamondsuit$ ("eventually"; $\diamondsuit\varphi$ is equivalent to true $\mathcal{U}\,\varphi$) and its dual $\square$ ("globally"; $\square\varphi$ is equivalent to $\neg\diamondsuit\neg\varphi$).

E.g., consider the DBN-template given in Example 1.1. We can use d-separation to argue that the structural formula $\square(O \perp\!\!\!\perp S \mid L)$ holds, i.e., $(O^t \perp\!\!\!\perp S^t \mid L^t)$ holds for all t. Using the temporal LTL-operators, also more involved properties can be expressed: the formula $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y})\,\mathcal{U}\,\neg(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z})$, e.g., expresses that the sets of variables $\mathbf{X}$ and $\mathbf{Y}$ are structurally independent at least until $\mathbf{Y}$ and $\mathbf{Z}$ are dependent.

For LTL, we investigate the following two problems:

- **Structural LTL model-checking of DBN-templates:** For a DBN-template $\mathcal{T}$ with variables $\mathbf{V}$ and an LTL formula φ over the set A of structural CI propositions using $\mathbf{V}$, decide whether the trace $\tau \in (2^A)^\omega$ of $\mathcal{T}$ satisfies φ.
- **Stochastic LTL model-checking of DBNs:** For a DBN $\mathcal{B}$ with variables $\mathbf{V}$ and an LTL formula φ over the set B of stochastic CI propositions using $\mathbf{V}$, decide whether the trace $\tau \in (2^B)^\omega$ of $\mathcal{B}$ satisfies φ.

We indeed employ d-separation as a tool to reason more generally about specifications involving only structural independence propositions in Sec. 4. On the other hand, we prove that evaluating stochastic formulae as simple as $\diamondsuit(X \perp Y)$ can be number-theoretically hard (Lem. 5.1) showing that a decidability result for stochastic LTL or NBA model-checking of DBNs is out of reach without a breakthrough in number theory.

Non-deterministic Büchi automata (NBAs). The second formalism we consider is that of nondeterministic Büchi automata (NBAs, introduced in (Richard Büchi 1966)), which express precisely the class of ω-regular temporal properties. An NBA is a tuple $\mathcal{A} = (Q, \Sigma, \Delta, Q_0, F)$ where Q is a finite set of states, Σ is the alphabet, $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation, $Q_0 \subseteq Q$ is the set of initial states and $F \subseteq Q$ is the set of accepting states. A run of $\mathcal{A}$ on an infinite word $\tau = w_0 w_1 w_2 \cdots \in \Sigma^\omega$ is a sequence $\rho = q_0 q_1 \dots$ of states such that $q_0 \in Q_0$ and $(q_i, w_i, q_{i+1}) \in \Delta$ for each $i \in \mathbb{N}$. The run ρ is accepting if $q_j \in F$ for infinitely many $j \in \mathbb{N}$. We say that $\mathcal{A}$ accepts τ if there exists an accepting run on τ.

We remark that it is known that NBAs are strictly more expressive than LTL formulae, however translating an LTL formula into an equivalent NBA can lead to an exponential increase in size. The resulting NBA model-checking problems we consider are:

- **Structural NBA model-checking of DBN-templates:** For a DBN-template $\mathcal{T}$ with variables $\mathbf{V}$ and an NBA $\mathcal{A}$ over the alphabet 2^A where A is the set of structural CI propositions using $\mathbf{V}$, decide whether the trace $\tau \in (2^A)^\omega$ of $\mathcal{T}$ is accepted by $\mathcal{A}$.
- **Stochastic NBA model-checking of DBNs:** For a DBN $\mathcal{B}$ with variables $\mathbf{V}$ and an NBA $\mathcal{A}$ over the alphabet 2^B where B is the set of stochastic CI propositions over $\mathbf{V}$, decide if the trace $\tau \in (2^B)^\omega$ of $\mathcal{B}$ is accepted by $\mathcal{A}$.

4 Structural Conditional Independence

In this section, we study the properties of the trace $\tau \in (2^A)^\omega$ of a DBN-template with variables $\mathbf{V}$ with respect to a set A of structural CI propositions and show how to check whether the trace satisfies a logical specification. The main result we will establish is the following:

Main result 4.1. *The structural LTL and NBA model-checking problems for DBN-templates are in PSPACE and NP-hard as well as coNP-hard.*

For restricted DBN-templates, the structural LTL and NBA model-checking problems are in PTIME.

To prove this result, we will show that the trace of a DBN-template with respect to structural CI propositions is ultimately periodic by virtue of being represented as the run of a deterministic transition system with $2^{O(|\mathbf{V}|^2)}$ states. We shall further demonstrate that in this transition system, a state can be represented in $O(|\mathbf{V}|^2)$ space, the successor can be computed in time polynomial in $|\mathbf{V}|$, and the labeling function (i.e., which propositions hold in a given state) can be computed in time polynomial in $|\mathbf{V}|, |A|$.

We then rely on a classical argument whose key ingredients can be found in (Baier and Katoen 2008, Proof of Lem. 5.47) and (Vardi and Wolper 1986, Thm. 3.2).

Lemma 4.2. *Let TS be a deterministic labelled transition system of size N, represented such that a state's label and successor can be computed in space polynomial in* $\log N$, *and let τ be its trace. Given a LTL formula φ of size M, we can decide in space polynomial in* $M, \log N$ *whether τ satisfies φ. Given an NBA $\mathcal{A}$ of size K, we can decide in space polynomial in* $\log K, \log N$ *whether $\mathcal{A}$ accepts τ.*

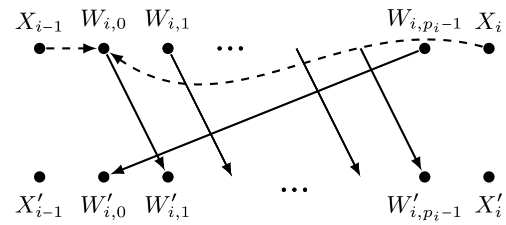

Figure 2: Bridge between islands X_{i-1} and X_i. Initial template in dashed edges.

The improved complexity for restricted DBN-templates follows from the insight that the trace of a restricted DBN-template on variables $\mathbf{V}$ is constant from time $|\mathbf{V}|^2$ onwards.

To convey a feeling for the hardness results, we first provide an example of a family of DBN-templates whose traces have periods exponential in the number of used variables:

Example 4.3. The k-th DBN-template of this family has variables $X_0, X_1, \dots, X_k$ as well as $W_{1,0}, W_{1,1}, W_{2,0}, \dots, W_{k,0}, W_{k,1}, \dots, W_{k,p_k-1}$, where p_i refers to the i-th prime. The initial template has edges of the form $(X^0_{i-1}, W^0_{i,0})$ and $(X^0_i, W^0_{i,0})$ for all i, and the step template has edges $(X_0, X'_0), (X_k, X'_k)$, and edges from $W_{i,r}$ to $W'_{i,r+1(\mathrm{mod}\, p_i)}$. By construction, we have $(X_0 \perp\!\!\!\perp X_k \mid \{W_{1,0}, \dots W_{k,0}\})$ if and only if the timestep t is divisible by all of $2, 3, \dots, p_k$ (see Fig. 2). Intuitively, we go from X_0 to X_k via "islands" $X_1, \dots, X_{k-1}$. The "bridge" between successive islands X_{i-1}, X_i uses the edges of the initial template, and is open precisely when t is divisible by p_i, i.e., a collision can be attributed to $W^t_{i,0}$. We need all bridges to be open simultaneously to make the journey: this gives us a period $2 \cdot 3 \cdot \dots \cdot p_k$, which is exponential in $|\mathbf{V}| = (k+1) + 2 + 3 + \dots + p_k$.

We establish NP-hardness of deciding whether $\Diamond(X \perp\!\!\!\perp Y \mid \mathbf{Z})$ holds: we reduce from the NP-complete intersection problem for unary DFA (Blondin, Krebs, and McKenzie 2016). Since we also have access to the negated formula, we can also reduce from the complementary problem. Further, both properties can be expressed by fixed NBAs, and we get the following result whose proof is an adaptation of the example above.

Lemma 4.4. *The LTL and NBA model-checking problems with structural-independence propositions are hard for NP as well as for coNP.*

4.1 DBN Traces Through Transition Systems

We shall now prove the PSPACE upper bounds for the structural model-checking problems by showing that any trace of a DBN-template with variables $\mathbf{V}$ and structural independence propositions A can be obtained as the run of a deterministic transition system with $2^{O(|\mathbf{V}|^2)}$ states, each of whose states can be represented in $O(|\mathbf{V}|^2)$ space, whose successor function can be computed in time polynomial in $|\mathbf{V}|$, and whose labelling function can be computed in time polynomial in $|\mathbf{V}|, |A|$.

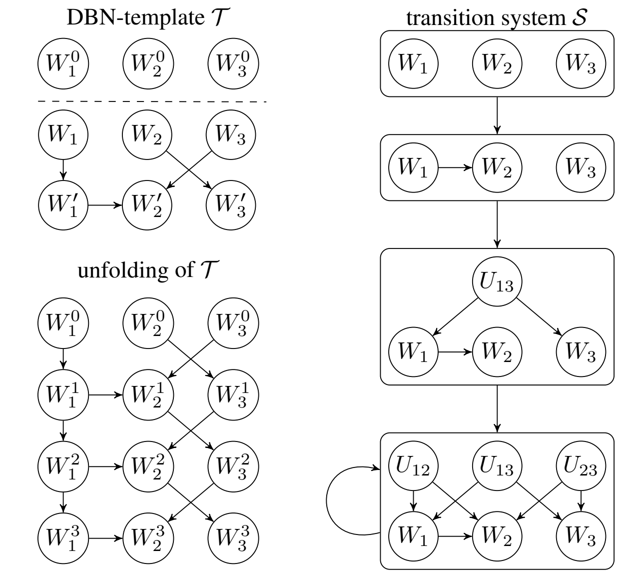

Figure 3: An example of a DBN-template $\mathcal{T}$ with its unfolding and the transition system $\mathcal{S}$ constructed from $\mathcal{T}$. Variables U_{ij} without outgoing edges are omitted. Note, e.g., that the variable U_{13} is connected to W_1 and W_2 after two time steps reflecting that there is a collision-free d-path $W_1^2, W_1^1, W_2^1, W_3^2$ connecting W_1^2 and W_3^2 in the unfolding through the previous time slices.

We start by making a key observation about d-paths. The proof is by contradiction: if there are more collisions, a pigeonhole argument forces two to be attributed to the same descendant, which can be used to "tunnel through" and obtain a path with fewer collisions.

Lemma 4.5. *If there exists a d-path from X to Y with respect to a set $\mathbf{Z}$ of size k in an arbitrary BN, there is one with at most k collisions, all of which occur inside $\mathbf{Z}$.*

The above claim motivates us to compute the pairs in $\{X^t, Y^t, Z_1^t, \dots, Z_k^t\}$ that are connected by collision-free paths in order to determine whether there is a d-path relative to $\mathbf{Z}^t$ from X^t to Y^t. However, we must be slightly careful: a path that concatenates collision-free paths via edges (Z^t, X^t) and (Z^t, Y^t) is actually blocked by Z^t.

Construction of the transition system. We construct a transition system $\mathcal{S} = (Q, q_0, \to, A, L)$, where Q will be the state space with an initial state q_0, $\to$ a deterministic successor relation, A the set of structural CI propositions, and $L\colon Q \to 2^A$ the labeling function. The states in S will be BN-templates that represent the connections via collision-free d-paths in the DBN-template at some time t. For an illustration of all steps of the construction, see Fig. 3.

State space and labeling function: The representative BN-template for time $t = 0$ is simply the initial BN-template. The representative BN-templates for times $t > 0$ use variables $\mathbf{V} = \{W_1, \dots, W_n\}$, and auxiliary variables $\mathbf{U} = \{U_{ij} : 1 \le i, j \le n\}$, corresponding to unordered pairs of distinct i, j. The edge (W_i, W_j) is always present if and only if (W_i', W_j') is an edge in the step-template BN. In the representative at time t, we additionally draw edges (U_{ij}, W_i) and (U_{ij}, W_j) if there is a collision-free d-path from W_i^t to W_j^t in the original DBN-template, and the intermediate vertices along this path do not belong to $\mathbf{V}^t$. Note how the latter requirement rules out the possibility of a path being blocked by an observed variable. So, by definition, $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$ holds at time t in the DBN if and only if $(\mathbf{X}^t \perp\!\!\!\perp \mathbf{Y}^t \mid \mathbf{Z}^t)$ holds in the representative BN-template; running d-separation queries on the latter settles the issue of the labelling function, which hence can be computed in time polynomial in $|\mathbf{V}|$ and A.

We observe that for $t > 0$, there are $2^{\binom{|\mathbf{V}|}{2}}$ possible representatives, depending on the choice of which U_{ij} are made parents, each representable as a graph with $O(|\mathbf{V}|^2)$ vertices. This establishes the size requirements described at the beginning of Sec. 4.

Transition relation: Above, the transition relation of the reachable part of the state space is implicitly given by following the time steps. Now, we describe how to compute the successor of a representative BN-template $\mathcal{B}$ on $\mathbf{V} \cup \mathbf{U}$ directly in polynomial time. For this, consider the graph with vertices $\mathbf{U} \cup \mathbf{V} \cup \mathbf{V}' \cup \mathbf{U}'$. Draw edges in the subgraph induced by $\mathbf{U} \cup \mathbf{V}$ as prescribed by $\mathcal{B}$, and edges in $(\mathbf{V} \cup \mathbf{V}') \times (\mathbf{V}')$ as prescribed by the step template. Now, for each $i \neq j$, we add edges from U_{ij}' to W_i' and W_j' if there is an $A \in \mathbf{U} \cup \mathbf{V}$ that can reach W_i' and W_j' in the graph constructed so far without using edges inside $\mathbf{V}'$.

The correctness of this construction can be seen as follows: If A is in $\mathbf{V}$, this is a collision-free d-path from W_i' to W_j'. If $A \in \mathbf{U}$ it has edges to some $B_i \in \mathbf{V}$ reaching W_i' and $B_j \in \mathbf{V}$ reaching W_j'. The edges to B_i and B_j represent a collision-free d-path which can then be extended to a collision-free d-path from W_i' to W_j'. Finally, we restrict the graph to $\mathbf{V}' \cup \mathbf{U}'$ to obtain the successor of $\mathcal{B}$. Clearly, this successor can be computed in time polynomial in $|\mathbf{V}|$.

PSPACE upper bound: So, we have constructed the deterministic transition system satisfying the requirements described at the beginning of Sec. 4 and conclude:

Theorem 4.6. *The structural LTL and NBA model-checking problems for DBN-templates are in PSPACE.*

4.2 The Special Case of Restricted DBNs

The difficulty of exponentially long periods is circumvented when we consider restricted DBNs: in this case, the trace is constant from position $t = |\mathbf{V}|^2$ onwards. To show this, we argue that for restricted DBNs, all times $t > |\mathbf{V}|^2$ have the same representative. Since going all the way to $\mathbf{V}^0$ does not give access to any "new" connecting edges in this setting. Formally, we obtain via a pigeonhole argument:

Lemma 4.7. *In a restricted DBN-template, if there is a collision-free path from X^t to Y^t, there is one that goes back at most $|\mathbf{V}|^2$ time slices.*

Thus, in the case of restricted DBNs, the trace is ultimately constant after at most $|\mathbf{V}|^2$ time steps, and the entire transition system can be computed in time polynomial in $|\mathbf{V}|, |A|$. This reduced complexity allows us to use standard techniques (graph-based automaton emptiness, unrolling LTL semantics in dynamic-programming fashion) to show:

Theorem 4.8. *The LTL and NBA model-checking problems for restricted DBNs with structural-independence propositions can be solved in polynomial time.*

5 Stochastic Conditional Independence

In this section, we establish the number-theoretic hardness of reasoning about stochastic CIs when concrete conditional probability distributions are given. Specifically, we shall show that deciding whether formulae of the form $\Diamond(X \perp Y)$ hold is at least as hard as the Skolem problem for rational linear recurrence sequences (LRS).

A rational LRS of order k is a sequence $(u_n)_{n=0}^{\infty}$ of rational numbers satisfying the recurrence relation $u_{n+k} = a_{k-1}u_{n+k-1} + \cdots + a_0 u_n$, where $a_{k-1}, \ldots, a_0$ are rational numbers with $a_0 \neq 0$. It is given by the coefficients $a_0, \ldots, a_{k-1}$, and the initial terms $u_0, \ldots, u_{k-1}$. The Skolem problem takes as input an LRS, where the recurrence relation and initial terms are respectively encoded as a companion matrix $A \in \mathbb{Q}^{k \times k}$ and a vector $u \in \mathbb{Q}^k$, and asks whether there exists an n such that $u_n = 0$, i.e., $A^n u$ contains a 0-entry.

The Skolem problem has been open for nearly a century (Everest et al. 2003; Tao 2008). It is open even if we restrict the LRS to have order five (Ouaknine and Worrell 2012). It is known to be decidable at orders four and below (Tijdeman, Mignotte, and Shorey 1984; Vereshchagin 1985).

Lemma 5.1. *Consider a rational LRS of order k, given by its companion matrix $A \in \mathbb{Q}^{k \times k}$, and vector $u \in \mathbb{Q}^k$ of initial values. We can compute a DBN with $\lceil \log k \rceil + 2$ binary variables $X, Z_1, \ldots, Z_\ell, Y$ where $\ell = \lceil \log k \rceil$ and rational conditional probabilities, such that $\Diamond(X \perp Y)$ holds if and only if the LRS has a zero term.*

The reduction uses (Aghamov et al. 2025, Cor. 1) to "embed" the given LRS into a Markov chain (M, v), and then Lem. 2.4 to convert the Markov chain into a DBN. We remark that as a corollary, our construction can also be used to reduce the closely related Positivity problem for LRS (see, e.g., (Ouaknine and Worrell 2014) for arguments of number-theoretic hardness) to the problem of deciding whether Y always "positively influences" X.

6 Discussion: Faithfulness in DBNs

Prop. 3.1 demonstrates an analog of Thm. 2.3 for DBNs, albeit in one direction. However, in future work, we aim to formally prove that the concept of structural independence is faithful to stochastic independence in DBNs, establishing a complete analog of Thm. 2.3 for DBNs. The distinction from the known result is that, when transitioning to DBNs, we impose constraints on the parameters by identifying the distributions of the same variables across different time slices. This reduces the dimensionality of the parameter space, leading to a strictly smaller family of admissible Bayesian networks at any given time t. This is the key obstacle in generalising the proof of Thm. 2.3 in (Meek 1995, Section 6.4).

We say that the parameters $\langle \mathcal{P}^0, \mathcal{P}^{\mathsf{step}} \rangle$ are *t-unfaithful* if $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ holds at time t but $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z})$ does not. They are called *unfaithful* if this occurs for some t. In other words, unfaithful parameters are those for which the structural and stochastic conditional independencies diverge. We aim to prove that only a measure-0 set of parameters is unfaithful.

Recall that by definition, if $(\mathbf{X}^t \perp \mathbf{Y}^t \mid \mathbf{Z}^t)$, then for every $\mathbf{x}, \mathbf{y}, \mathbf{z}$, we must have that the expression

$$\Pr[(\mathbf{X}^t, \mathbf{Y}^t, \mathbf{Z}^t) = (\mathbf{x}, \mathbf{y}, \mathbf{z})] \cdot \Pr[\mathbf{Z}^t = \mathbf{z}] \\ - \Pr[(\mathbf{X}^t, \mathbf{Z}^t) = (\mathbf{x}, \mathbf{z})] \cdot \Pr[(\mathbf{Y}^t, \mathbf{Z}^t) = (\mathbf{y}, \mathbf{z})]$$

is equal to 0. We observe that in the DBN setting, we can use Lem. 2.4 to argue that the above expression is a (degree $O(t)$) polynomial in the parameters $\langle \mathcal{P}^0, \mathcal{P}^{\mathsf{step}} \rangle$. In particular, if the polynomial is not identically 0, then for all but a measure-0 set of parameters, it returns a nonzero value. Our strategy is thus to prove that if structural dependence holds, the corresponding polynomial cannot be identically 0. Using that zero-sets of polynomials have measure 0 and are closed under countable unions, we would deduce that unfaithful parameters form a measure-0 set.

A promising direction comes from algebraic statistics (Sullivant 2018), which would apply tools from algebraic geometry and combinatorics to analyze the algebraic varieties resulting from the polynomials encoding conditional independence. Another line of attack could also possibly involve observing that the sequence of polynomials characterizing conditional (in)dependence at time t forms a linear recurrence over the field of multivariate rational functions, and judiciously appealing to the Skolem-Mahler-Lech theorem (the set of zeroes of a linear recurrence over a field of characteristic 0 is the union of a finite set and finitely many effective arithmetic progressions).

7 Conclusion

We introduced LTL-based and NBA-based specification formalisms to express temporal properties of CIs in DBNs. These formalisms can express desirable system properties such as non-interference in security applications and open the possibility to verify systems against all kinds of desirable specifications regarding the temporal evolution of CIs.

We restricted here to CI propositions that state CIs between variables at the same time slice. Our techniques, however, offer the possibility to introduce CI propositions talking about variables at different time slices, simply by augmenting the DBN by creating an appropriate number of copies of the variables to "look ahead". A syntax for such propositions could, e.g., be $(\mathbf{X}^{+2} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z}^{+1})$, which holds at time point t in a DBN-template if $(\mathbf{X}^{t+2} \perp\!\!\!\perp \mathbf{Y}^t \mid \mathbf{Z}^{t+1})$.

Regarding stochastic CIs in DBNs, our Skolem-hardness result is sobering regarding the potential of verifying systems against temporal specifications with respect to stochastic CI statements—which might come as a surprise. The key to obtain this hardness result was establishing the intricate connection between LRSs and DBNs.

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