Random Walk Decay Centrality

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Abstract

We propose a new centrality measure, called the Random Walk Decay centrality. While most centralities in the literature are based on the notion of shortest paths, this new centrality measure stems from the random walk on the network. We provide an axiomatic characterization and show that the new centrality is closely related to PageRank. More in detail, we show that replacing only one axiom, called Lack of Self-Impact, with another one, called Edge Swap, results in the new axiomatization of PageRank. Finally, we argue that Lack of Self-Impact is desirable in various settings and explain why violating Edge Swap may be beneficial and may contribute to promoting diversity in the centrality measure.

Introduction

Centrality measures—methods for identifying the most important nodes in a network based solely on its topology—have been extensively studied in the literature on graph theory and network analysis for over 50 years (Newman 2010). Fueled by the ever-growing availability of relational data, as well as the access to unprecedented computational power, centrality measures have become an essential part of every network analysis toolkit over the past two decades (Freeman 2008). These measures are increasingly being applied in numerous subareas of computer science, including the world-wide web (Page et al. 1999), viral marketing (Hinz et al. 2011), and energy saving in communication networks (Bianzino et al. 2011), just to name a few.

Most standard centrality measures are based on the notion of shortest paths (Freeman 1979). One of the most fundamental such measures is Closeness centrality, whereby the importance of a node is determined based on the inverse of the sum of distances from all other nodes in the graph (Sabidussi 1966). Since this centrality measure is well-defined only for strongly connected graphs, Jackson (2008) proposed an alternative, named Decay centrality, which works also for disconnected graphs.

These standard centrality measures are based on two simplifying assumptions. Firstly, they assume that the nodes of the network do not have any weights—an assumption that does not always hold in practical applications. Secondly, they assume that information always travels in a network through the fastest route(s)—an assumption that requires all nodes to know the entire topology of the network, which is rarely the case in real-world networks. In fact, whether one is modeling the spread of gossip through a social group, the propagation of viruses through a computer network, or the way users surf the Internet, such processes tend to spread more chaotically in practice, e.g., through somewhat random paths as opposed to shortest paths (Lerman and Ghosh 2010; Borgatti 2005; Huberman et al. 1998).

To address these limitations, a number of centrality measures have been developed based on random walks in the graph. According to this approach, the source of information is a node chosen randomly according to a given distribution of node weights, and the information then propagates through the network by moving along random outgoing edges (Lovász 1993). Perhaps the first and most influential such centrality measure is PageRank (Page et al. 1999). Its success has lead to the development of random-walk versions of various centrality measures, such as Random Walk Closeness centrality (White and Smyth 2003) and Random Walk Betweenness centrality (Newman 2005).

Against this background, in this paper we propose a new centrality measure, called Random Walk Decay, which is a random-walk version of Decay centrality. To highlight the similarities and differences between both centralities, we use an axiomatic approach. Specifically, we show that Random Walk Decay centrality and Decay centrality both satisfy five basic properties (called axioms): Locality, Sink Merging, Directed Leaf Proportionality, One-Node Graph and Lack of Self-Impact. In addition to those five axioms, if the centrality measure is based on shortest paths (i.e., satisfies the Shortest Paths Property), we obtain Decay centrality; on the other hand, if it is based on random walks (i.e., satisfies the Random Walk Property), we obtain Random Walk Decay.

Furthermore, using the axiomatic approach, we compare the Random Walk Decay centrality to PageRank. We show that replacing only one axiom (Lack of Self-Impact) with another one (Edge Swap) gives us a new axiomatization of PageRank. The former axiom, Lack of Self-Impact, states that “removing your own outgoing edges does not affect your centrality”; thus, it corresponds to a certain strategy-proof property satisfied by Random Walk Decay centrality. The latter axiom, Edge Swap, states that “if two nodes
have the same centralities and the same number of outgoing edges, then we can swap one of their edges without affecting their centralit"; thus, violating this axiom allows Random Walk Decay centrality to promote diversity of incoming edges. In result, we argue that Random Walk Decay centrality has properties violated by PageRank which can be found desirable in various setting.

**Preliminaries**

In this section, we introduce basic notations and definitions.

**Graphs:** In this paper, we consider directed multigraphs. A (multi)graph is an ordered pair, \( G = (V, E) \), where \( V \) is the set of nodes and \( E \subseteq V \times V \) is the multiset of directed edges. We will use \( \cup \) and \( - \) to denote multiset union and difference, respectively. For graph \( G \), the sets of nodes and edges are denoted by \( V(G) \) and \( E(G) \), respectively.

Furthermore, we associate with each node \( v \) a nonnegative weight, denoted by \( \beta(v) \). For \( v \in V(G) \) by \( \delta \), we denote the particular vector of node weights \( \beta \) such that \( \delta(v) = 1 \) and \( \delta(u) = 0 \) for every \( u \in V(G) \setminus \{v\} \). The sum of weights of all nodes in a graph \( G \) will be denoted by \( \beta(G) \).

An edge \( (u, v) \) is an outgoing edge for the node \( u \) and an incoming edge for node \( v \). If \( u = v \), this edge is called a loop. The multiset of outgoing edges for \( v \) is denoted by \( \Gamma^+_G(v) \). Analogously, the multiset of incoming edges for \( v \) is denoted by \( \Gamma^-_G(v) \). Moreover, we define \( \Gamma^+_G(v) = \Gamma^+_G(v) \cup \Gamma^-_G(v) \). A node without outgoing edges is called a sink. A node without incoming and outgoing edges is isolated.

A walk, \( p = (v_1, \ldots, v_k) \) is an ordered sequence of nodes such that \( (v_i, v_{i+1}) \in E \) for every \( i \in \{1, \ldots, k - 1\} \). A (simple) path is a walk in which all nodes (except possibly the first and the last one) are distinct. A (simple) cycle is a path such that \( v_1 = v_k \). The length of a walk is the length of the sequence minus one. If there exists a walk that starts in \( u \) and ends in \( v \), then \( u \) is called a predecessor of node \( v \). The set of all predecessors of node \( v \) is denoted by \( P_G(v) \).

A graph is strongly connected if there exists at least one path between any two nodes. A graph is (weakly) connected if there exists at least one path between any two nodes if we treat it as an undirected graph.

The graph obtained from \( G \) with node weights \( \beta \) by merging node \( u \) with node \( v \) is denoted by \( M_{u \rightarrow v}(G, \beta) \). Formally: \[
M_{u \rightarrow v}(G, \beta) = (V(G) \setminus \{u\}, E(G) - \Gamma^+_G(u) \cup E', \beta'),
\]
where \( E' = \bigcup_{(w, v) \in E(G)}\{w, v\} \cup \bigcup_{(w, v) \in E(G)}\{w, v\} \) and \( \beta'(v) = \beta(v) + \beta(u) \), and \( \beta'(w) = \beta(w) \) for \( w \in V \setminus \{u, v\} \). Also, for graph \( G \) and multiset of edges \( E' \), we define \( G + E' = (V(G), E(G) \cup E') \) and \( G - E' = (V(G), E(G) - E') \).

We say that two graphs, \( G, G' \), overlap on \( S \) if \( V(G) \cap V(G') = S \). If \( V(G) \cap V(G') = \emptyset \), then \( G \) and \( G' \) do not overlap, and are said to be disjoint. The sum of two graphs along with their node weights is defined as: \[
(G, \beta) + (G', \beta') = ((V(G) \cup V(G'), E(G) \cup E(G')), \beta''),
\]
where \( \beta''(v) = \beta(v) + \beta'(v) \) for \( v \in V(G) \cup V(G') \), \( \beta''(v) = \beta(v) \) for \( v \in V(G) \setminus V(G') \) and \( \beta''(v) = \beta'(v) \) for \( v \in V(G') \setminus V(G) \).

**Centrality measures:** A centrality measure, \( F \), is a function that assigns to every node, \( v \), in every graph, \( G \), a real value reflecting the importance of \( v \) in \( G \).

Freeman (1979) in his seminal work identified three centrality measures that capture different aspects of a node in the graph. The most basic one, called the Degree centrality, assesses a node by the number of its edges. For directed graphs, both In- and Out-Degree centralities are considered. The other two centrality measures focus on the shortest paths in the graph. Specifically, the Betweenness centrality evaluates a node, \( v \), based on the proportion of shortest paths (between any two other nodes) to which \( v \) belongs. In contrast, the Closeness centrality, originally proposed by Sabidussi (1966), identifies the nodes that are closest to all other nodes, and that is by computing the inverse of the sum of distances to other nodes in the graph:

\[
C_v(G) = \frac{1}{\sum_{u \in V(G) \setminus \{v\}} \text{dist}(u,v)},
\]
where \( \text{dist}(u,v) \) is the distance from \( u \) to \( v \) defined as the length of a shortest path from \( u \) to \( v \).

The Closeness centrality is well-defined only for strongly connected graphs. To address this shortcoming, Jackson (2008) proposed an alternative, called Decay centrality:

\[
Y_v(G) = \sum_{u \in V(G) \setminus \{v\}} a^{\text{dist}(u,v)},
\]
for a decay parameter \( a \in (0, 1) \). Here, if we treat \( a \) as the probability of a successful move from one node to another via an edge, then Decay centrality can be interpreted as the expected number of nodes that can reach \( v \) via shortest paths.

**Personalized centrality measures:** Most standard centrality measures were proposed for graphs without weights of nodes. However, they can usually be easily adapted to this richer setting (Koschützki et al. 2005). In this context, we define the personalized Decay centrality as follows:

\[
Y_v(G, \beta) = \sum_{u \in V(G)} \beta(u) \cdot a^{\text{dist}(u,v)}. \tag{1}
\]

The personalized Decay centrality introduces two modifications to the original definition. Firstly, the contribution of a node \( u \) to the centrality of \( v \) (i.e., \( a^{\text{dist}(u,v)} \)) is now multiplied by the weight of \( u \). Secondly, we now sum over all nodes \( (\sum_u \text{dist}(u,v)) \), rather than over all nodes other than \( v \) \( (\sum_{v \neq u} \text{dist}(u,v)) \). To understand the rationale behind this latter modification, consider an extreme scenario in which only a single node, say \( v \), has a positive weight. Here, if we sum over all nodes other than \( v \), then any node with a connection to \( v \) would have a positive centrality, whereas \( v \) itself would have a centrality equal to zero, as all nodes not connected to \( v \) would have a rather unintuitive outcome in most interpretations of node weights.

An important personalized centrality measure is PageRank (Page et al. 1999). This measure is defined by the following recursive formula:

\[
PR_v(G, \beta) = a \cdot \sum_{(u,v) \in E(G)} \frac{PR_u(G, \beta)}{|\Gamma^+_G(u)|} + \beta(v), \tag{2}
\]
for a parameter \( a \in (0, 1) \). Since we assume multiple edges between two nodes, then node \( u \) may appear multiple times on the right-hand side of the equation. It has been proven that, for a fixed \( a \), this formula uniquely characterizes a centrality measure (see, e.g., Bianchini, Gori, and Scarselli 2005).

**Random Walk Decay Centrality**

In this paper, we propose a new centrality measure that is based on the notion of a random walk on a graph. The random walk is defined in the following way (Lovász 1993):

- at the beginning (at moment \( t = 0 \)), we choose one node according to the distribution of node weights;
- in the \( k \)-th step (at moment \( t = k \), for \( k \geq 1 \)), while in node \( u \), we choose one of the outgoing edges of \( u \), say \((u, v) \in \Gamma_G(u)\), uniformly at random, and move along this edge to node \( v \).

Formally, the random walk is a sequence of random nodes \( w = (w(0), w(1), \ldots) \) that is a Markov chain, defined through its initial distribution, i.e.,

\[
\mathbb{P}_{G,\beta}(w(0) = v) = \beta(v) / \beta(G),
\]

and a transition matrix \( M = (p_G(u, v))_{u, v \in V} \) where the probability of moving from node \( u \) to node \( v \), denoted by \( p_G(u, v) \), is the number of edges from \( u \) to \( v \) divided by the number of all outgoing edges from \( u \):

\[
p_G(u, v) = \frac{|(u, v) \in \Gamma_G(u)|}{|\Gamma_G(u)|} \quad \text{for } v \in \Gamma_G(u).
\]

To deal with the fact that sinks would break the infinite walk, we assume that—besides the nodes of the graph—there exists one additional “terminal” absorbing state \( e \) to which we move from all sinks in the graph. Formally, we have \( p_G(u, e) = 1 \) if \( u \) is a sink, \( p_G(u, e) = 0 \) otherwise, and \( p_G(e, e) = 1 \). In result, we can think of the random walk as the set of all possible infinite walks on the graph, each associated with its probability.

**Example 1.** Consider the random walk on the graph from Figure 1. The random walk starts in node \( u \) or node \( w \), because these are the only nodes with non-zero weights. From node \( u \), the walk moves to \( v \) with probability \( 2 / 3 \), and stays in \( u \) with probability \( 1 / 3 \). From node \( w \), the walk moves either to \( v \) or to \( t \), both with probability \( 1 / 2 \). From node \( v \), the walk always moves to \( w \). Node \( t \) is a sink, so from \( t \) the walk moves to the absorbing state \( e \) and loops therein. Consequently, the probabilities of the different combinations of the first four nodes in the walk are as follows:

\[
\begin{array}{c|c|c}
1 / 54 & (u, u, u, u, \ldots) & 1 / 8 \\
1 / 27 & (u, u, u, v, \ldots) & 1 / 8 \\
1 / 9 & (u, u, v, w, \ldots) & 1 / 4 \\
1 / 6 & (u, v, w, v, \ldots) & \\
1 / 6 & (u, v, w, t, \ldots) & \\
\end{array}
\]

Let us introduce some additional terminology. For node \( v \in V(G) \), we will consider the probability that it will be visited for the \( k \)-th time in moment \( t \). We will call it \( k \)-th visiting probability in moment \( t \) and denote it by \( VP_{G,\beta}(t, k) \):

\[
VP_{G,\beta}(t, k) = \mathbb{P}_{G,\beta}(w(t) = v, |s \leq t : w(s) = v| = k).
\]

Now, we say that a centrality measure is a random walk centrality if it depends solely on the node’s visiting probabilities.

**Random Walk Property (RWP):** For every two graphs \( G, G' \) with node weights \( \beta, \beta' \) such that \( \beta(G) = \beta'(G') \) and node \( v \in V(G) \cap V(G') \), if \( VP_{G,\beta}(t, k) = VP_{G',\beta'}(t, k) \) for every \( t, k \), then \( F_v(G, \beta) = F_v(G', \beta') \).

**Example 2.** Consider again the random walk on the graph from Figure 1. Here, \( VP_{G,\beta}(t, k) \) equals:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{ } & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline k \backslash t & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline 1 & 0 & 7/12 & 1/9 & 1/27 & 1/81 & 1/243 & 1/729 \\
2 & 0 & 0 & 0 & 7/24 & 1/18 & 1/54 & 1/162 \\
3 & 0 & 0 & 0 & 0 & 0 & 7/48 & 1/36 \\
\end{array}
\]

In more detail, we clearly have \( VP_{G,\beta}(t, k) = 0 \) if \( t < k \). Furthermore, \( VP_{G,\beta}(1, 1) = 1 / 2 \cdot 1 / 3 + 1 / 2 \cdot 1 / 2 = 7 / 24 + 1 / 18 \). Additionally, \( VP_{G,\beta}(t, 1) = 1 / 2 \cdot 1 / 3 \cdot t^{-1} \cdot 2 / 3 \) for \( t > 1 \) because we have to start at \( u \) and loop for \( t - 1 \) times there in order to enter node \( v \) for the first time at moment \( t > 1 \). Finally, we have \( VP_{G,\beta}(t, k) = 1 / 2 \cdot VP_{G,\beta}(t - 2, k - 1) \) for \( t, k > 1 \) because we may only return back to \( v \) in two steps, which happens with probability \( 1 / 2 \).

Random walk centrality measures evaluate the nodes in a given graph by analyzing different properties of the random walk. For instance, Random Walk Closeness centrality is defined as the inverse of the expected time needed for the random walk to reach a specific node for the first time (White and Smyth 2003). Formally, for every graph \( G \) and every node \( v \in V(G) \):

\[
RWC_v(G, \beta) = 1 / \left( \sum_{t=0}^{\infty} t \cdot VP_{G,\beta}(t, 1) \right).
\]

The Random Walk Closeness centrality suffers from several problems of its original—the Closeness centrality. In particular, if there exists a node with non-zero weight from which \( v \) cannot be reached, then the centrality of \( v \) equals zero.

In this paper, we propose the following centrality measure, which is a translation of the Decay centrality (Jackson 2008) to the random walk model.

**Definition 1.** Random Walk Decay centrality is a centrality measure defined for every graph \( G \), and every node, \( v \in V(G) \), as:

\[
RWD_v(G, \beta) = \beta(G) \cdot \sum_{t=0}^{\infty} a^t \cdot VP_{G,\beta}(t, 1),
\]

for a decay parameter \( a \in (0, 1) \).
Let us explain this formula. For node $v$ and moment $t$, $V^p_{G,v}(t,1)$ is the probability that node $v$ will be reached by the random walk at moment $t$ for the first time. If we assume, as in the interpretation of the Decay centrality, that each step succeeds with probability $a$, then $a^t \cdot V^p_{G,v}(t,1)$ is the probability of reaching the node $v$ successfully. This expression is summed over all possible moments, $t \geq 0$. Finally, the whole expression is multiplied by $\beta(G)$. This is because each node $u$, is a starting point with probability $\beta(u)/\beta(G)$. Thus, by multiplying the sum by $\beta(G)$, the random walk that starts from node $u$ is considered with the weight $\beta(u)$, just as in the personalized Decay centrality (Eq. (1)). To put it differently, the Random Walk Decay measures the probability of reaching a node by the random walk assuming a constant probability, $(1-a)$, of breaking the walk.

**Example 3.** Let us compute the Random Walk Decay centrality for node $v$ in the graph from Figure 1. Based on Example 2 we get:

$$RWD_v(G,\beta) = 2\left(\frac{7a}{12} + \frac{a^2}{9} + \frac{a^3}{27} + \ldots\right) = \frac{21a - 7a^2 + 18}{6(3-a)}.$$  

For $a = 1/2$ we get $RWD_v(G,\beta) = 214/105$. Similar calculations show that $RWD_v(G,\beta) = 1/2$, $RWD_w(G,\beta) = 3/5$ and $RWD_0(G,\beta) = 3/10$.

We end this section by discussing PageRank. Recall that Page et al. (1999) proposed PageRank along with a random walk interpretation namely the random surfer model. The Markov chain defined by the random surfer model differs from that of the random walk. More in details, in the random surfer model, at each step the surfer stops moving along edges with some probability, and instead jumps to a randomly selected node. Nevertheless, in the following proposition we show that PageRank also satisfies the Random Walk Property (RWP), i.e., it can be expressed in terms of the (standard) random walk on a graph.

**Proposition 1.** PageRank is equal to

$$PR_v(G,\beta) = \beta(G) \cdot \sum_{t=0}^{\infty} \left( a^t \cdot \sum_{k=1}^{\infty} V^p_{G,v}(t,k) \right).$$

(7)

In result, PageRank satisfies Random Walk Property (RWP).

**Proof.** It suffices to prove that the centrality measure defined in (7) satisfies the recursive formula from (2). By considering the nodes from which the random walk can move to node $v$, we get:

$$\sum_{k=1}^{\infty} V^p_{G,v}(t,k) = \sum_{u \in V} \sum_{k=1}^{\infty} p_G(u,v) \cdot V^p_{G,v}(t-1,k), \quad \text{for } t > 0,$$

and $\sum_{k=1}^{\infty} V^p_{G,v}(t,k) = \beta(v)/\beta(G)$ for $t = 0$. Combining it with Eq. (7) leads to (2).

Formula (7) is similar to the formula for the Random Walk Decay. In a nutshell, the Random Walk Decay takes into account only the first time the random walk visits a node, while PageRank takes into account also further times. Thus, PageRank measures the expected number of times a node is reached by the random walk assuming a constant probability, $(1-a)$, of breaking the walk.

**Axiomatic Characterization**

In this section, we axiomatically characterize our new centrality measure—the Random Walk Decay centrality. The characterization is built in a close relation to the Decay centrality and PageRank.

We begin with axioms satisfied by all three centralities—the Decay centrality, the Random Walk Decay centrality and PageRank. Our first axiom, Locality, states that the centrality of a node depends solely on nodes connected to it. To put it differently, the centrality of a node does not change if we add to the graph a second, disjoint graph.

**Locality (LOC):** For every two disjoint graphs $G, G'$, node weights $\beta, \beta'$ and node $v \in V(G)$

$$F_v((G,\beta) + (G',\beta')) = F_v(G,\beta).$$

This basic axiom was proposed for graphs without weights by Skibski et al. (2016).

Our second axiom is called Sink Merging. It states that if we merge two nodes without outgoing edges, i.e., sinks, without joint predecessors, then the centrality of the resulting node will be the sum of the centralities of both sinks; moreover, the centralities of other nodes will not change.

**Sink Merging (SM):** For every graph $G$, node weights $\beta$ and sinks $u, v \in V(G)$ such that $P_G(v) \cap P_G(u) = \emptyset$

$$F_v(M_{u\rightarrow v}(G,\beta)) = F_u(G,\beta) + F_v(G,\beta),$$

and $F_w(M_{u\rightarrow v}(G,\beta)) = F_u(G,\beta)$ for any $w \in V(G) \setminus \{u,v\}$.

This axiom is a much weaker version of Merging, proposed for PageRank by Ws and Skibski (2018), that considered merging arbitrary nodes, possibly with outgoing edges.

The third axiom is called Directed Leaf Proportionality, which requires that, if we add an edge from a sink $u$ to an isolated node $v$, then the gain in the centrality of $v$ is proportional to the centrality of $u$.

**Directed Leaf Proportionality (DLP):** There exists a constant, $a > 0$, such that for every graph $G$, node weights $\beta$, sink $u \in V(G)$ and isolated node $v \in V(G)$

$$F_v((G,\beta),u,v) = a \cdot F_u(G,\beta).$$

This axiom is a directed and weighted version of Leaf Proportionality, proposed by Skibski and Sosnowska (2018).

Our fourth axiom, One-Node Graph, is a simple normalization property: if there is only one node in the graph and its weight equals 1, then its centrality also equals 1.

**One-Node Graph (1-NG):** For every node $v$

$$F_v((v,\emptyset),\delta_v) = 1.$$
Lack of Self-Impact (LSI): For every graph \( G \), node weights \( \beta \) and \( (v, u) \in E(G) \)
\[
F_v(G, \beta) = F_v(G - (v, u), \beta).
\]

In the following theorem, we show that the Random Walk Decay centrality is the only centrality measure that satisfies the Random Walk Property and the above five axioms: Locality, Sink Merging, Directed Leaf Proportionality, One-Node Graph and Lack of Self-Impact.

**Theorem 2.** The Random Walk Decay centrality is the unique centrality measure that satisfies LOC, SM, DLP, 1-NG, LSI and RWP.

**Proof (Sketch).** Due to space restrictions, we present only the sketches of the proofs in the paper.

The main idea of the proof relies on the class of broken cactus graphs. A graph \( G \) is called a (directed) cactus if it is strongly connected and each edge is part of exactly one cycle (Palbom 2005). A graph \( G \) is called a broken cactus if there exist two nodes, \( s, t \), called start and end nodes, such that \( t \) is a sink and \( M_{s-t}(G, \beta) \) is a cactus graph.

Assume that a centrality measure \( F \) satisfies LOC, SM, DLP and RWP. We begin our proof by showing that if the graph is a broken cactus such that only its start has non-zero weight, then the centrality \( F \) of its end equals the Random Walk Decay centrality (up to scalar multiplication).

**Claim 1:** If centrality measure \( F \) satisfies LOC, SM, DLP and RWP, then there exists \( \alpha \geq 0 \) such that for every broken cactus \( G \) that begins in \( s \) and ends in \( t \) it holds that \( F_v(G, \delta_s) = \alpha \text{RWD}_v(G, \beta) \).

This claim is proved by observing that every broken cactus can be obtained from a path by adding cactus graphs that overlap with the path on a single node.

Next, we show that for every sink we can construct a collection of broken cactus graphs such that the weighted sum of visiting probabilities of the ends of these graphs equals the visiting probability of a sink in the original graph.

**Claim 2:** For every graph \( G \), node weights \( \beta \) and sink \( v \in V(G) \), there exists a collection of broken cactus graphs \( G_1, \ldots, G_n \) that start in \( s_1, \ldots, s_n \), respectively, end in \( v \) and pairwise overlap on \( |v| \) such that
\[
VP_{G, \beta}(t, k) = \sum_{i=1}^{n} c_i \cdot VP_{G_i, \delta_{s_i}}(t, k)
\]
for some constants \( c_1, \ldots, c_n \geq 0 \), for every \( t \geq 0 \) and \( k \geq 1 \).

We prove this claim by induction on the number of incoming edges of nodes that are not sinks and considering graphs with only one node having a non-zero weight.

Note that Claim 2 is a general property of the random walk and visiting probabilities, and does not depend on the axiom nor the centrality measure definitions. However, by combining it with RWP we get that the centrality of a sink can be determined from the centralities of the ends of broken cactus graphs. Hence, Claim 1 and Claim 2 imply that the centrality of a sink is equal to the Random Walk Decay centrality (up to scalar multiplication).

Now, consider an arbitrary graph, \( G \), and a node, \( v \in V(G) \). From LSI we know that the centrality of node \( v \) in graph \( G \) is the same as in the graph \( G - \Gamma^+_G(v) \) obtained by removing all outgoing edges of \( v \). In the latter graph, node \( v \) is a sink, hence from Claim 3 and LSI we know that there exists a constant \( \alpha \) such that \( F_v(G, \beta) = F_v(G - \Gamma^+_G(v), \beta) = \alpha \text{RWD}_v(G, \beta) \). Finally, 1-NG implies that \( \alpha = 1 \); this concludes the proof of Theorem 2.

The personalized Decay centrality based on the shortest paths also satisfies the five axioms stated above—LOC, SM, DLP, 1-NG and LSI—but violates the Random Walk Property. To obtain a unique axiomatic characterization, we introduce the Shortest Paths Property—an axiom that captures the fact that the centrality is based on distance, i.e., shortest paths, from other nodes in the graph. Our axiom is a direct translation of the definition of the class of distance based centralities by Skibski and Sosnowska (2018) to weighted and directed graphs.

**Shortest Paths Property (SPP):** For every two graphs \( G, G' \), node weights \( \beta, \beta' \) such that \( \beta(G) = \beta'(G') \) and node \( v \in V(G) \cap V(G') \), if
\[
|\{ u \in V(G) : \text{dist}(u, v) = k, \beta(u) = \alpha \}| = |\{ u \in V(G') : \text{dist}(u, v) = k, \beta'(u) = \alpha \}|
\]
for every \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \), then \( F_v(G, \beta) = F_v(G', \beta') \).

The following theorem shows that replacing the Random Walk Property with the Shortest Paths Property in the axiomatization of the Random Walk Decay centrality results in an axiomatization of the personalized Decay centrality. It is easy to observe that Lack of Self-Impact is implied by the Shortest Paths Property, so we omit the former axiom from the axiomatic characterization.

**Theorem 3.** The personalized Decay centrality is the unique centrality measure that satisfies LOC, SM, DLP, 1-NG and SPP.

**Proof (Sketch).** We will use induction on the number of edges in graph \( G \). Based on LOC, it suffices to consider only connected graphs—if graph is not connected, then the centrality of every node is the same as in a connected graph with the same or less number of edges.

If a connected graph, \( G \), has no edges, then it must have only one node, and from 1-NG, LOC and SM it can be shown that \( F_v(G, \beta) = \beta(v) = Y_v(G, \beta) \). Now assume that \( G \) has at least one edge and for every graph \( G' \) with less edges it holds that \( F_v(G', \beta) = Y_v(G', \beta) \) for every \( v \in V(G') \) and weights \( \beta \). Fix node \( v \in V(G) \). We will show that the centrality of \( v \) in \( G \) can be computed based on centralities in graphs with a smaller number of edges; hence it is unique.

First, observe that if there exists a node, \( u \), with more than one outgoing edge, then at least one of these edges, say \((u, w)\), can be removed without changing the distance from \( u \)
Table 1: Summary of our axiomatic characterizations. The plus sign (+) indicates that the centrality measure satisfies the axiom, whereas the minus sign (-) indicates that the centrality measure violates it.

<table>
<thead>
<tr>
<th>Decay</th>
<th>LOC</th>
<th>SM</th>
<th>DLP</th>
<th>1-NG</th>
<th>SPP</th>
<th>RWP</th>
<th>LSI</th>
<th>ES</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPP</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>RWP</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>PageRank</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Theorem 2 and the fact that for every sink the Random Walk Decay centrality and PageRank are equal.

Now, consider an arbitrary graph $G$ with $k$ cycles and node weights $\beta$. Fix $v \in V(G)$ that belongs to some cycle. Let us construct a new graph $G' = (\{v', w\}, \{(v', w), \ldots, (v, w)\})$, disjoint with $G$, where $\beta(v') = F_v(G)$ and $|E(G')| = |\Gamma_v^+(v)|$. From LOC we know that $F_v(G) = F_v(G + G')$ for every $u \in V(G)$. Moreover, it can be shown from 1-NG, LOC and SM that $F_v(G') = \beta(v) = F_v(G)$. Thus, nodes $v$ and $v'$ have the same centralities in $G + G'$ and the same number of outgoing edges. Now, from ES replacing all outgoing edges of $v$ with all outgoing edges of $v'$ does not affect the centralities in the graph. Observe that this operation breaks the cycles that $v$ belongs to in $G$. Hence, the obtained graph has less cycles than $G$ and the thesis follows from the inductive assumption.

Table 1 summarizes our axiomatic results. Since SPP implies LSI, based on Theorem 3 we know that there exists no centrality that satisfies LOC, SM, DLP, 1-NG, SPP and ES.

Comparison with PageRank

Our axiomatic characterizations highlight two differences between the Random Walk Decay centrality and PageRank. In this section, we focus on these two differences and show how they affect the behaviour of these centrality measures.

Strategy-proofness (with respect to outgoing edges)

In many settings, outgoing edges are subject to the node’s decision or manipulations. Examples include the Twitter social network (where outgoing edges represent the accounts that are followed by a user) and the World Wide Web (where outgoing edges represent the links to other websites). Consequently, Lack of Self-Impact can be considered a property of strategy-proofness for centrality measures—if outgoing edges do not affect the centrality of a node, then the node has no incentive to manipulate its outgoing connections.

Interestingly, PageRank does not satisfy Lack of Self-Impact. In the following example we show how, by adding outgoing edges, a node can increase its centrality and position in the ranking according to PageRank, but not according to the Random Walk Decay centrality.

Example 4. Consider graph $G$ from Figure 2. Graph $G$ consists of two 4-cycles, $(u_1, u_2, u_3, u_4, u_1)$, $(v_1, v_2, v_3, v_4, v_1)$. The two cycles are connected via 3 edges: $(v_4, u_4)$, $(u_3, v_3)$, and $(u_2, v_2)$. Due to the edges connecting both cycles, the nodes $v_2$, $v_3$, and $v_4$ are visited more often (and earlier) by the random walk, and are thus ranked first by both PageRank and the Random Walk Decay centrality. Node $u_1$, that will be of our interest, is ranked 5th according to both measures.

Figure 2 also depicts $G'$, which is obtained from $G$ by adding the edge $(u_1, u_4)$. Since this is an outgoing edge for $u_1$, adding it does not affect the Random Walk Decay centrality of $u_1$. In contrast, this edge has a significant impact on PageRank of $u_1$. The reason lies in the fact that the random walk will now visit $u_1$ much more often—whenever the random walk reaches node $u_1$, with probability $1/2$ it will go back to $u_4$ from which the only outgoing edge goes to $u_1$. In
result, both u1 and u4 top the ranking according to PageRank. The centralities of all nodes for α = 0.8 are as follows:

<table>
<thead>
<tr>
<th>node v</th>
<th>PR(G, β)</th>
<th>PR(G', β)</th>
<th>RWD(G, β)</th>
<th>RWD(G', β)</th>
</tr>
</thead>
<tbody>
<tr>
<td>v3</td>
<td>6.82 (1st)</td>
<td>6.08 (3rd)</td>
<td>4.67 (1st)</td>
<td>4.36 (1st)</td>
</tr>
<tr>
<td>v4</td>
<td>6.45 (2nd)</td>
<td>5.86 (4th)</td>
<td>4.43 (2nd)</td>
<td>4.21 (2nd)</td>
</tr>
<tr>
<td>v2</td>
<td>5.80 (3rd)</td>
<td>5.13 (5th)</td>
<td>4.13 (3rd)</td>
<td>3.77 (4th)</td>
</tr>
<tr>
<td>u2</td>
<td>4.84 (4th)</td>
<td>3.62</td>
<td>3.75 (4th)</td>
<td>3.02</td>
</tr>
<tr>
<td>u1</td>
<td>4.81 (5th)</td>
<td>6.56 (2nd)</td>
<td>3.72 (5th)</td>
<td>3.72 (5th)</td>
</tr>
<tr>
<td>v1</td>
<td>3.58</td>
<td>3.35</td>
<td>2.76</td>
<td>2.60</td>
</tr>
<tr>
<td>u3</td>
<td>2.94</td>
<td>2.45</td>
<td>2.48</td>
<td>2.17</td>
</tr>
</tbody>
</table>

In Example 4, a node improved its PageRank by adding an edge to its direct predecessor. In the next section, we will discuss how incoming edges affect both centrality measures.

Diversity (of incoming edges)

One of the characteristic properties of PageRank is its recursive formula (see (2)), which states that PageRank of a node depends solely on PageRank of its direct predecessors (or, more precisely, nodes incident to its incoming edges). Intuitively speaking, PageRank of a node does not depend on the position in the network of its predecessors, but only on their centrality. In our axiomatic characterization, this property is captured by Edge Swap, which implies that an incoming edge from a node with the lowest centrality in a densely connected part of the graph could be as profitable as an incoming edge from a node with the highest centrality in a different, less densely connected part.

The Random Walk Decay centrality does not satisfy Edge Swap. In fact, the node can achieve higher centrality if it has incoming edges from a diverse set of nodes, i.e., coming from different parts of the network. We demonstrate this point with the following example.

Example 5. Consider graph G from Figure 3. This graph consists of three more densely connected parts, so called communities: \{u1, u2, u3\}, \{v1, v2, v3\} and \{w1, w2, w3, w4\}. These communities are connected through nodes u1, v1, w1 which form a 3-clique. Since w1 belongs to the biggest community, both its Random Walk Decay centrality and its PageRank are the highest. The nodes u1 and v1 have the second highest values, with symmetrical positions in the graph.

Figure 3 also depicts the graph G', which is obtained from G by rewiring the two highlighted (red) edges. Specifically, the edges (u2, u1) and (v2, v1) are replaced by (u2, v1) and (v2, u1); in result, the two new edges connect two communities. Since u2 and v2 both have two edges and clearly the same centralities in graph G, from Edge Swap we know that PageRank of every node in G' is the same as in G. In contrast, the centralities of both nodes u1 and v1 increase according to the Random Walk Decay centrality. This is because, according to this centrality, an edge from a different community is more profitable than an edge from your own community. In our example, the random walk that starts from nodes v2 and v3 reaches node u1 faster in graph G'; the same holds for node v1. In result, in G', the Random Walk Decay centralities of u1 and v1 are higher than the Random Walk Decay centrality of node w1. The centralities of all nodes for α = 0.8 are:

<table>
<thead>
<tr>
<th>node v</th>
<th>PR(G, β)</th>
<th>PR(G', β)</th>
<th>RWD(G, β)</th>
<th>RWD(G', β)</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>7.15 (1st)</td>
<td>7.15 (1st)</td>
<td>4.40 (1st)</td>
<td>4.40 (1st)</td>
</tr>
<tr>
<td>u1</td>
<td>7.06 (2nd)</td>
<td>7.06 (2nd)</td>
<td>4.15 (2nd)</td>
<td>4.46 (1st)</td>
</tr>
<tr>
<td>v1</td>
<td>7.06 (2nd)</td>
<td>7.06 (2nd)</td>
<td>4.15 (2nd)</td>
<td>4.46 (1st)</td>
</tr>
<tr>
<td>w3</td>
<td>4.33 (4th)</td>
<td>4.33 (4th)</td>
<td>2.62</td>
<td>2.62</td>
</tr>
<tr>
<td>w2, w4</td>
<td>4.16 (5th)</td>
<td>4.16 (5th)</td>
<td>2.74 (4th)</td>
<td>2.74</td>
</tr>
<tr>
<td>u2, v2</td>
<td>4.02</td>
<td>4.02</td>
<td>2.56</td>
<td>2.85 (4th)</td>
</tr>
<tr>
<td>u3, v3</td>
<td>4.02</td>
<td>4.02</td>
<td>2.56</td>
<td>2.70</td>
</tr>
</tbody>
</table>

Example 5 shows that the Random Walk Decay centrality increases when incoming edges become more diverse. As such, it avoids putting at the top of the ranking several nodes from the same community, which often happens in PageRank (Avrachenkov and Litvak 2006; Zhirov, Zhirov, and Shepelyansky 2010).

Related Work

Our paper belongs to a line of papers that study the axiomatic properties of centrality measures (Boldi and Vigna 2014; Bloch, Jackson, and Tebaldi 2016; Skibski, Michalak, and Rahwan 2018). In particular, our axiomatization of the Decay centrality relies on a recent axiomatization for undirected graphs proposed by Skibski and Sosnowska (2018).

To date, the only axiomatized centrality measure based on the random walk was PageRank. Palacios-Huerta and Volij (2004) proposed an axiomatization of the simplified version of PageRank. Altmann and Tennenholtz (2005) also focused on a simplified version of PageRank, but axiomatized the ranking, rather than the numerical values. Recently, Wąs and Skibski (2018) proposed the first axiomatization of
PageRank in its general form. Our new axiomatization significantly differs from all of these axiomatizations.

Conclusions
In this paper, we proposed Random Walk Decay centrality—a new centrality measure based on the random walk. We provided an axiomatic characterization using six axioms, and proved that replacing only one property leads to an axiomatization of the Decay centrality. Furthermore, we showed that Random Walk Decay works similarly to PageRank, but has certain properties that can be more desirable in various settings. In our future work, we plan to perform a comparative experimental analysis of Random Walk Decay and PageRank using real-world networks.

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References


Bloch, F.; Jackson, M. O.; and Tebaldi, P. 2016. Centrality measures in networks. Available at SSRN 2749124.


