

Generalized Distance Bribery

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Abstract

The bribery problem in elections asks whether an external agent can make some distinguished candidate win or prevent her from winning, by bribing some of the voters. This problem was studied with respect to the weighted swap distance between two votes by Elkind et al. (2009). We generalize this definition by introducing a bound on the distance between the original and the bribed votes. The distance measures we consider include a restriction of the weighted swap distance and variants of the footrule distance, which capture some real-world models of influence an external agent may have on the voters. We study constructive and destructive variants of distance bribery for scoring rules and obtain polynomial-time algorithms as well as NP-hardness results. For the case of element-weighted swap and element-weighted footrule distances, we give a complete dichotomy result for the class of pure scoring rules.

1 Introduction

In an election in which voters influence the election's result by casting their votes they might not be the only agents interested in its outcome. A well known example of strategic influence on elections is bribery, formally introduced by Faliszewski et al. (2009) (see also the book chapter by Faliszewski and Rothe (2016)). By paying voters with a resource, e. g. money, to change their vote, an external agent tries to change the election outcome to her advantage. In the constructive case the aim is to make a specific candidate a winner of the election, while in the destructive case the briber tries to prevent a candidate from winning. Originally the number of voters that can be bribed is limited. However, there is no limitation to the amount of change within a vote. It is natural to assume that a voter's cost for changing her vote to a large extent is higher than the cost for making small adjustments. This intuition is modeled in the microbribery problem introduced by Faliszewski et al. (2009), and in the swap-bribery problem introduced by Elkind et al. (2009). In both models the briber has to pay certain prices for swaps of adjacent candidates. In order to obtain a more realistic model, we propose to imply a bound on the distance between the truthful and the bribed votes.

The idea of limiting the distance between originally cast votes and their altered versions in strategic influences is not restricted to bribery. Obratsova and Elkind (2012) have introduced distances to manipulation, where a voter changes her vote in order to subjectively improve the election outcome. They focus on the unweighted swap distance and footrule distance, and the maximum displacement distance between the truthful and the strategic vote in the constructive case. We propose to extend the study of distances to both, the constructive and destructive bribery problem and use unweighted and weighted variants of the swap and footrule distance. One motivation for introducing a bound on the overall distance of the bribed profile follows from the work by Obratsova and Elkind (2012) on optimal manipulation. The distance between a truthful and a manipulative vote should be bounded since otherwise a manipulation may be detected more easily. In a more positive sense bribery can be seen as a form of campaign management where an external actor tries to make her desired candidate win by running specific campaigns. While the swap distance counts the inversions of pairs of candidates, it is a more realistic approach to assign a weight to a specific swap, as, in a natural election, voters are often very sure about the positioning of some candidates, while being less concerned with others. We show that different assumptions on how to calculate the weights have a crucial influence on the complexity. The restriction of element-weighted distance functions (studied by Kumar and Vassilvitskii (2010) for linear orders) assigns a weight to every candidate. The weight of a swap is the product of the corresponding candidate weights. This can be seen as an indicator of how sure a voter is about the position of a candidate in the vote. The same weight functions can also be applied to the footrule distance. Instead of inversions, here the positions a candidate is shifted by are counted. The idea is to measure the cumulative change in the candidate's standing in the vote, based on its position. In the weighted version, the influence of one candidate on the standing of another can be set by the weight function. Hence, the footrule distance is more realistic in terms of robustness or comparison of profiles.

We study the computational complexity of distance bribery for pure scoring rules with swap and footrule distances in the unweighted, element-weighted, and weighted case. This includes showing how to exploit the relation-

ships between the distances in order to transfer complexity results but also demonstrating the limits by giving examples for cases where the complexity differs. Weights only apply to the distances, votes remain unweighted. In particular, for the considered voting systems the destructive case is never harder than the constructive one. We present polynomial-time algorithms obtained through dynamic programming and NP-completeness results, solving a number of open questions from the literature. A detailed overview of our results is given in Section 2.2.

2 Preliminaries

An election is a pair (C, V) with a set of candidates C and a list of voters $V = (v_1, \dots, v_n)$ (also called profile), where a vote v_i is associated with a preference order $>_{v_i}$ over the candidates. If the vote v_i is clear from the context we omit the index or simply write $>_i$. The set of all linear orders over the candidates in C will be denoted by $\mathcal{L}(C)$. The position of candidate c in vote v_i will be denoted by $\text{pos}(v_i, c)$, and the candidate on position k in vote v_i will be denoted by $\text{cand}(v_i, k)$. For two lists $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_k)$ of votes let $A \cup B = (a_1, \dots, a_n, b_1, \dots, b_k)$ be the concatenation of both lists. A voting rule is a function \mathcal{E} that maps an election (C, V) to an element of 2^C . In this paper tie-breaking is not an issue since we focus on the so-called *non-unique* winner model, where a bribery action is successful if the designated candidate is one of the winning candidates in the constructive case and if she is none of the winners in the destructive case. However, our results can be adapted to the unique winner case by slight changes.

We will consider different voting rules, all belonging to the class of scoring rules. Every scoring rule is characterized by a scoring vector $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$ and value $\alpha_j \in \mathbb{N}_0$, for $1 \leq j \leq m$, where $m \geq 2$ is the number of candidates, and will be denoted by $\mathcal{E}_{\vec{\alpha}}$. Every candidate gets α_j points according to her position j in each vote. The winners are the candidates with the maximum number of points. By $\text{score}_{(C, V)}(c)$ we denote the points that candidate c gets in the election (C, V) . We assume that at least two different entries in the scoring vector exist. A scoring rule for an arbitrary number of candidates is defined as a series of scoring vectors for which we can determine the scoring vector for m candidates in polynomial-time depending on m . We will focus on so called pure scoring rules as defined by Betzler and Dorn (2010). A *pure scoring rule* is a scoring rule, for which the scoring vector for m candidates can be obtained from inserting a legal entry in the vector for $m - 1$ candidates. We call a pure scoring rule *polynomially bounded*, if α_1 is bounded by a polynomial depending on m . This includes *plurality*, which gives only one point for a first position and *k-approval*, where one point is given to every candidate placed in the top k positions of a vote. Accordingly this includes *veto* and *k-veto*, which give one point to all candidates but the candidate(s) in the last/last k positions who get zero points, and also *Borda*, characterized by the scoring vector $(m - 1, m - 2, \dots, 1, 0)$ for an m -candidate election.

2.1 Distances

Now we will introduce different distance measures. Formally, a distance measure d on a space A is a mapping $d : A \times A \rightarrow \mathbb{R}$ that fulfills the following properties for all $a, b, c \in A$: (i) $d(a, b) \geq 0$ (non-negativity), (ii) $d(a, b) = 0$ if and only if $a = b$ (identity of indiscernibles), (iii) $d(a, b) = d(b, a)$ (symmetry), and (iv) $d(a, b) + d(b, c) \geq d(a, c)$ (triangle inequality). We will focus on distances between votes. For a fixed set of candidates this is a distance of the form $d : \mathcal{L}(C) \times \mathcal{L}(C) \rightarrow \mathbb{R}$. As we will study the complexity of problems involving such distances we actually consider a family of distances $(d^m)_{m>1}$ that contains one distance function for each possible candidate set size m .

The two basic distance measures we consider are the swap distance and the footrule distance. For two votes v and v' from $\mathcal{L}(C)$ the *swap distance* $\text{swap}(v, v')$ counts the number of inverted pairs and is formally defined as $\text{swap}(v, v') = |\{(x, y) \in C \times C \mid x >_v y \text{ and } y >_{v'} x\}|$. Beside the swap distance, we will also consider the footrule distance. Instead of counting swaps, this distance counts the positions that a candidate needs to be shifted by to obtain the target order. The *footrule distance* $\text{fr}(v, v')$ is defined as $\text{fr}(v, v') = \sum_{y \in C} |\text{pos}(v, y) - \text{pos}(v', y)|$.

In addition to the well-known unweighted versions of these distance measures we also consider weighted forms. The most general form of the swap distance is the weighted swap distance which is also used for the swap bribery problem (see Elkind et al. (2009)). In order to define the weighted distance we introduce the general notion of a weight function $\pi : C \times C \rightarrow \mathbb{N}_0$, that assigns a weight to each pair of candidates. We will always assume that $\pi(x, x) = 0$ for every $x \in C$ and $\pi(x, y) = \pi(y, x)$ for every $x, y \in C$. This function is analogous to the price function in the context of swap bribery.

Definition 1 (Weighted swap/footrule distance) Let π be a weight function, the weighted swap distance between two votes v, v' from $\mathcal{L}(C)$ is $\text{swap}_{\pi}^w(v, v') = \sum_{(x, y) \in C \times C: x >_v y \text{ and } y >_{v'} x} \pi(x, y)$, and the weighted footrule distance between v and v' is $\text{fr}_{\pi}^w(v, v') = \sum_{y \in C} \left| \sum_{x \in C: x >_v y} \pi(x, y) - \sum_{x \in C: x >_{v'} y} \pi(x, y) \right|$.

For matters of consistency we denote the unweighted distances by fr_{π} and swap_{π} , where the weight function is $\pi(x, y) = 1$ for all $x, y \in C, x \neq y$. The following example illustrates how these distances are calculated.

Example 2 Consider the votes $v : a > b > c > d > e$ and $v' : a > d > c > b > e$. The swap distance is $\text{swap}(v, v') = |\{(b, c), (b, d), (c, d)\}| = 3$. For a weight function π , the weighted swap distance is the sum of the weights of these pairs: $\text{swap}_{\pi}^w(v, v') = \pi(b, c) + \pi(b, d) + \pi(c, d)$. The unweighted footrule distance is $\text{fr}(v, v') = |2 - 4| + |4 - 2| = 4$, whereas the weighted footrule distance for a weight function π is $\text{fr}_{\pi}^w(v, v') = |-\pi(d, b) - \pi(c, b)| + |\pi(b, c) - \pi(d, c)| + |\pi(b, d) + \pi(c, d)|$.

In addition to these general weighted distances we will consider a special weighted variant that corresponds to the

element-weighted distances described by Kumar and Vassilvitskii (2010). In the *element-weighted swap/footrule distance* (denoted by $\text{swap}_\pi^{\text{ew}}$ and $\text{fr}_\pi^{\text{ew}}$ respectively), the function π is restricted to the form $\pi(x, y) = \varphi(x) \cdot \varphi(y)$ for $x \sim y$, where $\varphi : C \rightarrow \mathbb{N}_0$ is a function that assigns a weight to every candidate. While our distances are obviously heavily inspired by the weighted distances proposed by Kumar and Vassilvitskii (2010), they actually differ widely. Their distances are invariant, i. e., they are not dependent on the actual identity of each element but on the permutation between votes v and v' . Our distances are not invariant for a good reason. They should depend on the identity of each candidate and the measurable differences or similarities between them. If weights of 0 are not allowed for a pair of two distinct candidates, the metric properties are still fulfilled for our definition of the element-weighted swap distance since it is a special case of the weighted swap distance, for which it is not hard to show these properties. For the triangle inequality, note that for a swap to appear between two candidates comparing a vote v_a to a vote v_c this swap also has to appear comparing v_a to a third vote v_b or v_b to v_c . This argument can be modified to show that the weighted footrule distance fulfills the triangle inequality as well. Non-negativity, identity of indiscernibles, and symmetry are obviously also fulfilled by the weighted footrule distance.

Note that if weights of 0 are allowed for a pair of two distinct candidates, all distance measures only fulfill the requirements for a pseudometric, i. e., the identity of indiscernibles is replaced by the property $d(a, a) = 0$. For the sake of readability we neglect identity of indiscernibles in some of our proofs. However, these proofs can be easily rewritten to fulfill this property.

Even though the swap and footrule distances focus on different aspects of the differences between two votes, it can be shown that they are somehow related, and we will make use of this in our proofs.

Lemma 3 *For a set of candidates C , a weight function π over C , and $v, v', v^* \in \mathcal{L}(C)$ the following statements hold:*

$$\text{swap}_\pi(v, v') \leq \text{fr}_\pi(v, v') \leq 2 \cdot \text{swap}_\pi(v, v'); \quad (1)$$

$$\text{swap}_\pi^{\text{ew}}(v, v') \leq \text{fr}_\pi^{\text{ew}}(v, v') \leq 2 \cdot \text{swap}_\pi^{\text{ew}}(v, v'); \quad (2)$$

$$2 \cdot \hat{\pi}(v, v') \leq \text{fr}_\pi^{\text{w}}(v, v') \leq 2 \cdot \text{swap}_\pi^{\text{w}}(v, v') \quad (3)$$

with $\hat{\pi}(v, v') = \max\{\pi(x, y) \mid x >_v y \text{ and } y >_{v'} x\}$.

If v and v^* are 3-inversion free¹, it holds that

$$\text{fr}_\pi^{\text{w}}(v, v^*) = 2 \cdot \text{swap}_\pi^{\text{w}}(v, v^*). \quad (4)$$

The first part is the well-known Diaconis–Graham inequality (see Diaconis and Graham (1977)). The remaining proofs can be shown by slight adaptations.

2.2 Distance Bribery

Now we will define decision problems in order to study their computational complexity. For details on computational complexity we refer to the textbooks by Papadimitriou (1994) and Arora and Barak (2009). The study of the

¹There is a 3-inversion between v and v^* if $a >_v b >_{v^*} c$ and $c >_{v^*} b >_v a$ for some candidates a, b , and c .

computational complexity of such manipulative actions is important, since in the context of voting in multi-agent systems it is crucial to know about the possibility of an external intervention. Of course, a worst-case complexity analysis is only the first step and should be followed by a more fine-grained analysis and the design of approximation algorithms and fast algorithms for the average case.

The original bribery problem asks if for a given election a designated candidate can be turned into a winner of the election by bribing a specific number of voters, whereas the swap bribery problem (introduced by Elkind et al. (2009)) requires that the sum of the swaps' weights may not exceed a budget. In the related shift bribery problem, also introduced by Elkind et al. (2009), only swaps involving the desired candidate are allowed. Note that there is a reduction of shift bribery to the element-weighted variant of our problem. However, it does not transfer any results to the problems studied in this paper.

For a given distance function \mathcal{D} and a voting rule \mathcal{E} we define a generalization of the swap bribery problem as follows:

$(\mathcal{D}, \mathcal{E})$ -DISTANCE BRIBERY	
Given:	An election $E = (C, V)$, with $V = (v_1, \dots, v_n)$, a weight function $\pi_i, 1 \leq i \leq n$, for every voter, a positive integer K , and a distinguished candidate $p \in C$.
Question:	Is there a list of votes $V' = (v'_1, \dots, v'_n)$ such that $\sum_{i \in \{1, \dots, n\}} \mathcal{D}_{\pi_i}(v_i, v'_i) \leq K$ and $p \in \mathcal{E}(C, V')$?

In the destructive variant $(\mathcal{D}, \mathcal{E})$ -DESTRUCTIVE DISTANCE BRIBERY we have the same input, but we ask whether there is an alternative profile $V' = (v'_1, \dots, v'_n)$ with $\sum_{i \in \{1, \dots, n\}} \mathcal{D}_{\pi_i}(v_i, v'_i) \leq K$ and $p \notin \mathcal{E}(C, V')$. Hence, we want to prevent p from being a winner of the election.

One restriction is that the weight function is identical for all voters. This follows the work on bribery where only a certain number of votes can be changed arbitrarily (see Faliszewski et al. (2009)). All our polynomial-time results obviously cover this restricted version, and we mention whether our NP-hardness results also hold in this special case. The most general problems are those for the weighted distances, while unweighted distances lead to the most restricted problems. Hence, polynomial-time algorithms for $\text{swap}_\pi^{\text{w}}$ and fr_π^{w} carry over to the corresponding problems for $\text{swap}_\pi^{\text{ew}}$ and $\text{fr}_\pi^{\text{ew}}$, and both to the corresponding problems for swap_π and fr_π . At the same time, NP-hardness for the unweighted distances (swap_π and fr_π) implies NP-hardness for the element-weighted distances ($\text{swap}_\pi^{\text{ew}}$ and $\text{fr}_\pi^{\text{ew}}$) which in turn imply NP-hardness for the weighted distances ($\text{swap}_\pi^{\text{w}}$ and fr_π^{w}). Obviously, all decision problems are contained in NP, since the winners for all voting rules considered here can be computed in polynomial time and as, given two votes, it is possible to compute all introduced distances in polynomial time.

Results for the constructive and destructive variants for the different distances are summarized in Table 1. Whenever results were previously known (or follow by the above

described relations), the respective source is given. For all other cases, the number of the corresponding theorem, for which the proof is provided in this paper, is given in brackets. The results show that the introduction of weighted distances may turn the problem NP-complete. We study the scoring vector $(2, 1, \dots, 1, 0)$, since this is a very simple scoring vector with only 3 different values. Interestingly this is an example for a scoring rule for which the result differs in the constructive case for the weighted swap distance and the weighted footrule distance. To the best of our knowledge, such a difference was not known before for other problems.

Our results solve a number of open questions from the literature. Shiryaev et al. (2013) conjectured that swap_π^w DESTRUCTIVE DISTANCE BRIBERY is hard for many scoring rules, especially Borda and k -approval for any fixed $k \geq 2$. We confirmed this conjecture for Borda (Corollary 11) but disproved it for k -approval (Corollary 6). Obraztsova and Elkind (2012) posed the question whether the swap bribery problem, where only one voter can be bribed, remains tractable for scoring rules. We give a negative answer in the case of Borda elections. In our NP-hardness proof of $(\text{swap}_\pi^w, \text{Borda})$ -DESTRUCTIVE DISTANCE BRIBERY (Theorem 10) there is only one voter that may be bribed. Thereby, the proof also shows that the corresponding optimal manipulation problem for weighted swap distances is NP-complete. In addition, Obraztsova and Elkind (2012) suggested to extend the study of optimal manipulation to other distances. Since the optimal manipulation problem is a special case of bribery where only one voter can be bribed, all our P results carry over to the corresponding optimal manipulation problems. Furthermore, all our hardness results for the destructive cases carry over, since in our reductions, there is only one voter that may be bribed.

Related Work Distance bribery is a general framework for many problems. Based on the distance used, and whether we look at the destructive or constructive variant, we get various possible motivations such as robustness (Shiryaev et al. (2013)), the standing of a non-winning candidate (Brelsford et al. (2008)), optimal manipulation (Obraztsova and Elkind (2012)), possible winner (Konczak and Lang (2005)), coalitional manipulation (Conitzer and Sandholm (2002)), tie-breaking, and so on.

The use of individual prices for the change of each vote has been introduced by Faliszewski et al. (2009) for the priced bribery problem. There is however one price fixed irrespective of the change in the vote. Distance restrictions on constructive and destructive bribery problems have also been studied by Yang et al. (2016) for the swap distance and the Hamming distance. The related bribery problems with distance restrictions introduced by Yang et al. additionally include an upper bound on the number of votes that can be bribed. Obviously, our hardness results also hold in this case. Since we follow the work on swap bribery we relate the upper bound of allowed bribery actions to the distance at hand. Furthermore, we consider the distance bound to be part of the input, whereas Yang et al. fix it in the problem name. It is obvious that their distance restricted bribery problem

for an unlimited distance bound equals the ordinary bribery problem, as well as ours does, if one uses the discrete distance. Another difference is that Yang et al. bound the distance for each bribed voter, while we have a global bound on the distance, consistent with the swap bribery problem. Elkind et al. (2009) showed that the swap bribery problem is a generalization of the possible winner problem. In our notation the swap bribery problem corresponds to the distance bribery problem with the swap_π^w distance. However, there is no direct reduction for problems with other distance functions than the weighted swap distance. Even for the element-weighted swap distance their reduction no longer holds. On the one hand, our problem differs from the nonuniform bribery, which was introduced by Faliszewski (2008), since they use utility-based voting, while our definition applies to profiles where votes are linear orders over the candidates. On the other hand, it should be mentioned that our model can also be applied to various other preference representations, as well as to the approval vectors used by Faliszewski (2008), as long as meaningful distances can be defined on that form of representation.

Another related problem is the margin of victory in an election studied, e.g., by Magrino et al. (2011) and Xia (2012), and the robustness of an election studied by Shiryaev et al. (2013). They focus on the possibility to change the election winner through some changes in the votes. It is argued that the amount of change needed to obtain a different outcome can be seen as a measure for the robustness of a voting rule. Similar as for the optimal manipulation problem the motivation is again to detect fraud in elections. The latter problem is equivalent to destructive swap bribery, and the same arguments obviously apply to the bribery problem. Hence, the distance measures used in this paper also provide a way of measuring the robustness of a voting rule.

3 Results

To construct relative scores in the reductions below, we use the circular block votes defined by Betzler and Dorn (2010). This allows us to generate a set of votes with arbitrary scores that can be generated from the scoring vector, for an offset β and a dummy candidate d , which gets less than β points.

Destructive Distance Bribery In this section we will consistently denote the undesirable candidate as q and a possibly more preferable candidate as p . Before stating our results, we introduce a notion for the minimum necessary distance to improve the designated candidate p compared to some other candidate. For a given instance of $(\mathcal{D}, \mathcal{E})$ -DESTRUCTIVE DISTANCE BRIBERY with election (C, V) , and $p, q \in C$, let $\text{cost}_{(C, V)}(p, q, i, r)$ denote the minimum necessary distance to improve the relative score of p compared to q in vote v_i by at least r , with $0 \leq r \leq \alpha_1 - \alpha_m$. We set $\text{cost}_{(C, V)}(p, q, i, r) \rightarrow \infty$ if the relative score cannot be improved by at least r .

Theorem 4 $(\mathcal{D}, \mathcal{E})$ -DESTRUCTIVE DISTANCE BRIBERY is in P for each distance function \mathcal{D} and each polynomially bounded pure scoring rule \mathcal{E} for which $\text{cost}_{(C, V)}$ can be determined efficiently.

		swap_π	$\text{swap}_\pi^{\text{ew}}$	$\text{swap}_\pi^{\text{w}}$	fr_π	$\text{fr}_\pi^{\text{ew}}$	fr_π^{w}
plurality/veto	C	P [▲]	P [▲]	P [▲]	P, (12)	P, (12)	P, (12)
	D	P [♣]	P [♣]	P [♣]	P, (6)	P, (6)	P, (7)
k -approval	C	P [★]	NP-c., (16)	NP-c. [▲]	P, (14)	NP-c., (16)	NP-c., (16)
	D	P [♣]	P, (6)	P, (6)	P, (6)	P, (6)	P, (7)
$(2, 1, \dots, 1, 0)$	C	NP-c., (13)	NP-c., (13)	NP-c., (13)		NP-c., (16)	NP-c., (16)
	D	P [♣]	P, (6)	P, (6)	P, (6)	P, (6)	NP-c., (8)
Borda	C	NP-c., (15)	NP-c., (15)	NP-c. [▲]	NP-c., (15)	NP-c., (15)	NP-c., (15)
	D	P [♣]		NP-c., (11)	P, (9)		NP-c., (11)

Table 1: Overview of complexity results for the constructive and destructive DISTANCE BRIBERY problems for various distances, where “NP-c.” stands for NP-complete. Results that were previously known (or directly follow from them) are marked by their corresponding source: [▲] Elkind et al. (2009), [★] Dorn and Schlotter (2012), and [♣] Shiryayev et al. (2013). For all other results the reference to the corresponding theorem is given in brackets. The NP-completeness of constructive distance bribery for $\text{swap}_\pi^{\text{ew}}$, $\text{fr}_\pi^{\text{ew}}$, $\text{swap}_\pi^{\text{w}}$, and fr_π^{w} for k -approval holds for a fixed $k \geq 2$.

Proof. We will use the following method for each candidate $p \in C \setminus \{q\}$, to compute an overall distance minimal bribery making p better than q . To solve the decision problem, we then have to compare the minimum overall distance of the destructive bribery with the distance limit K . Given a set of candidates C with $\{p, q\} \subseteq C$ and a list of votes $V = (v_1, \dots, v_n)$, such that $\text{score}_{(C,V)}(q) \geq \text{score}_{(C,V)}(p)$ (otherwise q is not a winner). The weight function for voter v_i will be denoted by π_i . Let $\delta = \text{score}_{(C,V)}(q) - \text{score}_{(C,V)}(p) + 1$ be the deficit between q and p .

The following formula is a straightforward generalization of the formula presented by Kaczmarczyk and Faliszewski (2016) for destructive shift bribery. By $f(i, r)$ with $i \in \{1, 2, \dots, n\}$ we denote the minimum overall distance for reducing the deficit by at least r in the votes v_1, v_2, \dots, v_i . The value of $f(0, \cdot)$ is only defined as an initial value for technical reasons. For f it holds that

$$f(0, 0) = 0, \quad f(0, g) = \infty \text{ for } g > 0, \text{ and}$$

$$f(i, r) = \min_{r' \leq r} [f(i-1, r-r') + \text{cost}_{(C,V)}(p, q, i, r')].$$

Therefore, the minimum necessary overall distance to eliminate the deficit between p and q is given by $f(n, \delta)$ which can be determined in polynomial-time if $\text{cost}_{(C,V)}$ can be determined efficiently. \square

We apply this result and show that $\text{cost}_{(C,V)}$ can be computed in polynomial time for a large class of rules.

Theorem 5 *Let \mathcal{E} be a pure scoring rule with a limited number of different values, and additionally only one value has an unlimited number of entries, then $(\mathcal{D}, \mathcal{E})$ -DESTRUCTIVE DISTANCE BRIBERY is in P for $\mathcal{D} \in \{\text{swap}_\pi, \text{swap}_\pi^{\text{ew}}, \text{swap}_\pi^{\text{w}}, \text{fr}_\pi, \text{fr}_\pi^{\text{ew}}\}$.*

Proof (sketch). First note that the above restriction on the pure scoring rule implies that it is polynomially bounded. Hence, due to Theorem 4 we only show that $\text{cost}_{(C,V)}$ can be computed in polynomial time. This can indeed be done for all listed distances by a brute force approach.

For swap distances, the idea is to calculate the minimum necessary distance $\text{cost}_{(C,V)}$ by looking at each possible distribution of candidates to the values.

The minimum distance is obtained by placing the candidates of some value in their original order. Since there is only one value with an unlimited number of entries, the number of distributions is polynomially bounded by the number of candidates. By placing the candidates assigned to a value in the original order the minimum element-weighted footrule distance is produced. Therefore, it suffices to show that every corrective swap in the unweighted footrule distance reduces or maintains the distance. This can be shown via a connection between the unweighted and element-weighted distance. \square

Corollary 6 *For plurality, veto, k -approval, and $(2, 1, 1, \dots, 1, 0)$ elections $(\mathcal{D}, \mathcal{E})$ -DESTRUCTIVE DISTANCE BRIBERY is in P for $\mathcal{D} \in \{\text{swap}_\pi, \text{swap}_\pi^{\text{ew}}, \text{swap}_\pi^{\text{w}}, \text{fr}_\pi, \text{fr}_\pi^{\text{ew}}\}$.*

This corollary follows directly from Theorem 5. The approach cannot be extended to the weighted footrule distance since additional inversions can actually lower the distance. Consider the votes $v : q > a > b > p$, $v' : p > a > b > q$, and $v^* : p > b > a > q$ and the weight function given by $\pi(a, b) = 4$, $\pi(a, p) = 1$, $\pi(a, q) = 4$, $\pi(b, p) = 4$, $\pi(b, q) = 1$, and $\pi(p, q) = 1$. We receive the following distances: $\text{fr}_\pi^{\text{w}}(v, v') = 18$ and $\text{fr}_\pi^{\text{w}}(v, v^*) = 14$. Next, we show that for k -approval elections it is indeed possible to extend the approach to the weighted footrule distance. Note that for $k = 1$ this proof covers plurality, and a very similar approach can be used to show polynomial-time solvability for k -veto.

Theorem 7 *($\text{fr}_\pi^{\text{w}}, k$ -approval)-DESTRUCTIVE DISTANCE BRIBERY is in P.*

Proof (sketch). Again, we will make use of Theorem 4. To adapt the proof for k -approval, it suffices to show that the vote in which the candidates assigned to 1 and 0 maintain their original order respectively inflicts the minimum

weighted footrule distance. This holds, since one can show that correcting the left- or rightmost inversion of two adjacency candidates always lowers or maintains the weighted footrule distance. \square

For $(2, 1, \dots, 1, 0)$ elections it is indeed not possible to extend the previous result.

Theorem 8 $(\text{fr}_\pi^w, \mathcal{E}_{(2,1,1,\dots,1,0)})$ -DESTRUCTIVE DISTANCE BRIBERY is NP-complete.

Proof. NP-hardness will be shown via a reduction from the NP-complete problem SUBSET SUM (see Garey and Johnson (1979)). A SUBSET SUM instance consists of a set of integers $A = \{a_1, a_2, \dots, a_h\}$ and a positive integer T . The question is whether there exists a subset $A' \subseteq A$ with $\sum_{a_i \in A'} a_i = T$. We construct an instance of the destructive bribery problem as follows. The set of candidates is $C = \{p, q, s, u_1, u_2, \dots, u_h\}$. The profile contains the vote $v_1 : q > s > u_1 > u_2 > \dots > u_h > p$ and a list of votes V_d . The latter is constructed by using the circular block votes with some u_i as the dummy candidate to ensure the following scores for some score offset β . The current winner of the election should be q with $\beta + 3$ points, followed by p with β points. Each other candidate should receive at most $\beta - 4$ points. The weight function π_1 is as follows: $\pi_1(p, u_i) = a_i$, $\pi_1(p, s) = 0$, $\pi_1(p, q) = 0$, $\pi_1(q, u_i) = 0$, $\pi_1(q, s) = 2T$, $\pi_1(s, u_i) = 2a_i$, $\pi_1(u_i, u_j) = 0$, for $1 \leq i, j \leq h$. For every other voter we set the weights for every swap to $K + 1$, that means only v_1 can be bribed. The distance limit is $K = 2 \cdot \sum_{a_i \in A} a_i + 2T$. Since only v_1 is bribable and only p can beat q , the only way of defeating q is to move p to the first and q to the last position in v_1 . The resulting vote $v'_1 = p > s > u_1 > u_2 > \dots > u_h > q$ inflicts the distance $\text{fr}_{\pi_1}^w(v_1, v'_1) = 2 \sum_{a_i \in A} a_i + 4T$. Due to the weight function, the only way of lowering the distance, is to swap u_i s in front of s . Lowering the distance by $2T$ is possible if and only if there exists a subset $A' \subseteq A$, with $\sum_{a_i \in A'} a_i = T$ such that swapping each u_i , where $a_i \in A'$, in front of s , lowers the term of s by $2T$. Note that the term of each u_i is not lowered by swapping with s . \square

One can extend this proof to all pure scoring rules, which have two distinct values separated by entries of at least one other value, with the number of entries increasing at least linearly with a polynomial number of candidates.

Next, we will consider the unweighted footrule distance.

Theorem 9 $(\text{fr}_\pi, \mathcal{E})$ -DESTRUCTIVE DISTANCE BRIBERY is in P for all pure scoring rules.

Proof. The approach used by Shiryaev et al. (2013) for the $(\text{swap}_\pi, \mathcal{E})$ -DESTRUCTIVE DISTANCE BRIBERY regarding scoring rules, can be easily generalized to apply to all distances which are polynomially bounded depending on the number of candidates, for which one can efficiently determine the maximum deficit reduction in a vote for a given bound on the distance. This is indeed possible for the unweighted footrule distance by testing every possible combination of positions of p and q and placing the remaining candidates according to their original order. \square

We continue, by considering the destructive variant of distance bribery for Borda elections. It is known that destructive shift bribery and destructive unweighted swap bribery are in P for Borda (see, e. g., Kaczmarczyk and Faliszewski (2016) and Shiryaev et al. (2013)). In contrast we will show NP-completeness for DESTRUCTIVE DISTANCE BRIBERY for a class of pure scoring rules, including Borda, when using the weighted swap distance. We will call a pure scoring rule \mathcal{E} separable with polynomial effort, if there exists a polynomially bounded function $f : \mathbb{N} \rightarrow \mathbb{N}$, so that for each integer r with $m = f(r)$ and $\vec{\alpha}$ as the scoring vector for m candidates the following holds: $\exists k \in \{1, \dots, m-1\} : \alpha_k > \alpha_{k+1}$ and $r \leq k \leq (m-r)$. Intuitively this condition means that the candidates may be divided into two “large enough” groups that do not get the same number of points. The integer r determines the minimal size of these groups.

Theorem 10 $(\mathcal{D}, \mathcal{E})$ -DESTRUCTIVE DISTANCE BRIBERY is NP-complete for $\mathcal{D} \in \{\text{swap}_\pi^w, \text{fr}_\pi^w\}$ for every pure scoring rule \mathcal{E} which is separable with polynomial effort, even if the weight functions are identical for all voters.

The proof is a reduction from the NP-complete problem BALANCED BICLIQUE. The above theorem includes scoring rules like Borda, $\lfloor \frac{m}{2} \rfloor$ -approval and many more rules.

Corollary 11 $(\mathcal{D}, \text{Borda})$ -DESTRUCTIVE DISTANCE BRIBERY is NP-complete for $\mathcal{D} \in \{\text{swap}_\pi^w, \text{fr}_\pi^w\}$, even if the weight functions are identical for all voters.

Constructive Distance Bribery We now turn to the constructive case. For the sake of readability, candidate p will always be the candidate who should be made a winner of the election through the bribery action.

Theorem 12 For plurality and veto $(\mathcal{D}, \mathcal{E})$ -DISTANCE BRIBERY is in P for $\mathcal{D} \in \{\text{fr}_\pi, \text{fr}_\pi^{\text{ew}}, \text{fr}_\pi^w\}$.

Proof. We show polynomial-time solvability for the weighted variant, which directly implies the results for the other two variants. Elkind et al. (2009) argue that for the bribery problem with the weighted swap distance for plurality and veto elections it suffices to calculate the minimum distance for replacing the top or rearmost candidate with another specific candidate and use this value as input for the polynomial-time nonuniform-bribery algorithm by Faliszewski (2008). Hence, it is sufficient to show that these values can be computed in polynomial time for the weighted footrule distance.

The more general proof of Theorem 7 implies that for plurality the most efficient way of transferring the points from candidate $\text{cand}(v_i, 1)$ to candidate $\text{cand}(v_i, j)$ with $2 \leq j \leq m$ is to shift $\text{cand}(v_i, j)$ to the top position while moving each candidate in between one position downwards. For veto the most efficient way of transferring one point from $\text{cand}(v_i, j)$ with $1 \leq j \leq m-1$ to $\text{cand}(v_i, m)$ is to shift $\text{cand}(v_i, j)$ downwards to position m while moving each candidate in between one position upwards. \square

Theorem 13 $(\mathcal{D}, \mathcal{E}_{(2,1,1,\dots,1,0)})$ -DISTANCE BRIBERY is NP-complete for $\mathcal{D} \in \{\text{swap}_\pi, \text{swap}_\pi^{\text{ew}}, \text{swap}_\pi^w\}$, even if the weight functions are identical for all voters.

Proof. We show the result for the unweighted variant, which directly implies the results for the other two variants. NP-hardness will be shown via a reduction from the NP-complete problem X3C (see Garey and Johnson (1979)). An X3C instance consists of a set $U = \{u_1, \dots, u_{3q}\}$ of items and a family $S = \{S_1, \dots, S_r\}$ of 3-element subsets of U . The question is, whether there is an exact cover of U , i. e., a subset $S' \subseteq S$ such that $\bigcup_{S_i \in S'} S_i = U$. For the reduction, the set of candidates is given by $C = U \cup \{p\} \cup \{s_i, c_i, d_i \mid 1 \leq i \leq r\} \cup \{x_1, x_2, \dots, x_{12r}\}$. The number of candidates is $m = 3r + 12r + 3q + 1$ and we assume $q < r$ and $r \geq 5$. The list of votes is $V = V_s \cup V_d$. V_s contains the following three votes for each $S_i = \{u_{i_1}, u_{i_2}, u_{i_3}\}$:

$$\begin{aligned} v_{i,1} : & u_{i_1} > X > \dots > D_{-r} > s_i > p > d_r; \\ v_{i,2} : & u_{i_2} > X > \dots > D_{-r} > d_r > c_i > p > s_i; \\ v_{i,3} : & u_{i_3} > c_i > X > \dots > D_{-r} > p > d_r. \end{aligned}$$

Here X is a shorthand for the order $x_1 > \dots > x_{12r}$, D_{-r} for the order $d_1 > \dots > d_{r-1}$, and the ellipsis in the votes contains all the candidates that are not specified in the vote itself in arbitrary order. Here, β is the number of points, candidate p receives. The current winners of the election should be each u_j with $\beta + 2q + 1$ points, followed by each x_j , c_i , and s_i with $\beta + 2q$ points. Each d_j should receive at most β points. We can construct the scores through the following votes in V_d , increasing the relative score of the selected candidate by 1 point compared to each other candidate except d_r with 2 points. For p we use $p > X > \dots > D_{-r} > d_r$, for $c \in \{u_j, c_i, s_i\}$ we use $c > X > \dots > D_{-r} > p > d_r$, and for x_j we use $x_j > x_1 > \dots > x_{12r} > \dots > D_{-r} > p > d_r$, where $x_1 > \dots > x_{12r}$ does not contain x_j . The distance limit is set to be $K = q(2m - 1)$. Note that decreasing the score of each x_j by one point alone would increase the overall distance by at least $12r \cdot 3r > K$ for $q < r$. Therefore, the briber has to increase the score of p by at least $2q$ points. Since p is either already ranked at the top position or on position $m - 1$, increasing the score of p by $2q$ makes $2q(m - 2)$ additional swaps, and therefore an increase in the overall distance by at least $2q(m - 2)$, necessary. Note that as $2q(m - 2) < K < (2q + 1)(m - 2)$, $\beta + 2q$ is the maximum score that p can reach. Thus, the briber has to increase the score of p by exactly $2q$ while lowering the score of each u_j by at least 1.

We will show that p can only be made the winner while staying inside the distance limit if and only if there exists a solution to the X3C instance with the corresponding votes arranged such that

$$\begin{aligned} v'_{i,1} : & p > u_{i_1} > X > \dots > D_{-r} > d_r > s_i; \\ v'_{i,2} : & p > u_{i_2} > X > \dots > D_{-r} > d_r > s_i > c_i; \\ v'_{i,3} : & c_i > u_{i_3} > X > \dots > D_{-r} > p > d_r. \end{aligned}$$

This increases the distance by $2m - 1$. We will show that swapping p forward to the first position in votes beside $v_{i,1}$ and $v_{i,2}$ makes it impossible to make p the winner while staying inside the distance limit. Note that beside swapping p upwards for an additional distance of $2q(m - 2)$ while

possibly already reducing the score of $2q$ items, there is a leftover distance of $3q$ for reducing the score of the other u_j by 1 with the cheapest way of reducing the score of an u_j being 3.

Reducing the score of u_j by 1 by swapping it away from the last position without swapping it with p is not possible since it increases the distance by at least $15 \leq 3r$ ($r \geq 5$) (for reducing the score of the x again). Reducing the score of u_j (or any other relevant candidate) by swapping it down past d_1, \dots, d_r is also not an option since it increases the distance by at least 5 ($r \geq 5$). In both cases we could not reduce the score of all the at least $(q - 1)$ leftover items.

We can use the same argumentation to show that swapping down u_{i_3} in $v_{i,3}$ without swapping p forward to the first position in $v_{i,1}$ and $v_{i,2}$ is not an option. In both cases we have to swap s_i down in $v_{i,1}$ to swap c_i down in $v_{i,2}$ to swap u_{i_3} with c_i . But doing so without swapping p forward in one of the votes increases the distance by at least 4. Thereby we could not reduce the score of all the at least $(q - 1)$ leftover items. Therefore, the briber has to swap p forward in exactly q pairs of $v_{i,1}$ and $v_{i,2}$ exactly as shown above with swapping down u_{i_3} in $v_{i,3}$. Since it is necessary to reduce the score of each u_j by one, making p the winner with respect to the distance limit is only possible if and only if there exists a solution to the X3C instance. \square

Theorem 14 (fr_π, k -approval)-DISTANCE BRIBERY is in P, even if k is part of the input.

Proof. This proof makes use of an algorithm proposed by Dorn and Schlotter (2012, Theorem 1) to compute a swap_π distance optimal bribery for k -approval. Let $V^* = (v_1^*, \dots, v_n^*)$ be the swap_π distance optimal bribery computed by the above mentioned algorithm. Since V^* is swap_π distance optimal each vote v_i^* fulfills the order proposed in the proof of Theorem 5 with the candidates on the top k and latter $m - k$ positions in their original order. Since each v_i^* is 3-inversion free, it holds due to Diaconis and Graham (1977) that $\text{fr}_\pi(v_i, v_i^*) = 2 \cdot \text{swap}_\pi(v_i, v_i^*)$, which extends to the overall distance: $\sum_{i=1}^n \text{fr}_\pi(v_i, v_i^*) = 2 \cdot \sum_{i=1}^n \text{swap}_\pi(v_i, v_i^*)$. Assume there is a different fr_π distance optimal bribery $V' = (v'_1, \dots, v'_n)$ with $\sum_{i=1}^n \text{fr}_\pi(v_i, v'_i) < \sum_{i=1}^n \text{fr}_\pi(v_i, v_i^*)$. Again, since V' is distance optimal, by the proof of Theorem 7 we know that each v'_i fulfills the 3-inversion free order proposed in the proof, so $\sum_{i=1}^n \text{fr}_\pi(v_i, v'_i) = 2 \cdot \sum_{i=1}^n \text{swap}_\pi(v_i, v'_i)$ holds. Substituting fr_π with swap_π in the above inequality leads to $\sum_{i=1}^n \text{swap}_\pi(v_i, v'_i) < \sum_{i=1}^n \text{swap}_\pi(v_i, v_i^*)$ which is a contradiction to the swap_π distance optimality of V^* . \square

Theorem 15 ($\mathcal{D}, \text{Borda}$)-DISTANCE BRIBERY is NP-complete for $\mathcal{D} \in \{\text{swap}_\pi, \text{swap}_\pi^{\text{ew}}, \text{swap}_\pi^{\text{w}}, \text{fr}_\pi, \text{fr}_\pi^{\text{ew}}, \text{fr}_\pi^{\text{w}}\}$, even if the weight functions are identical for all voters.

Theorem 16 (\mathcal{D}, \mathcal{E})-DISTANCE BRIBERY is NP-complete for $\mathcal{D} \in \{\text{swap}_\pi^{\text{ew}}, \text{fr}_\pi^{\text{ew}}\}$ and for all pure scoring rules except plurality and veto, and for $\text{swap}_\pi^{\text{ew}}$ even if the weight functions are identical for all voters.

Since $(\mathcal{D}, \mathcal{E})$ -DISTANCE BRIBERY for $\mathcal{D} \in \{\text{swap}_{\pi}^{\text{ew}}, \text{fr}_{\pi}^{\text{ew}}\}$ is solvable in polynomial-time for plurality and veto this completes a dichotomy result for pure scoring rules.

4 Conclusions

We introduced various forms of distances, namely weighted and unweighted versions of the swap and footrule distance, to the bribery problem in elections. We showed how the relation between the distances may be used in the proofs, but we also showed that there are limits. For $(2, 1, \dots, 1, 0)$ we have a P result for the weighted swap distance, but NP-completeness for the weighted footrule distance in the destructive case. On the one hand, we have shown for pure scoring rules that in the constructive case the complexities of the element-weighted and weighted problems coincide. On the other hand, for $(2, 1, \dots, 1, 0)$ in the destructive case they differ. Regarding the open cases from Table 1, we conjecture NP-completeness for all of them. For many voting systems, which, like Borda, distinguish each position, the problem is already hard in the destructive (as well as constructive) case, even if only one voter is suggestible. Thus, the design of voting systems which are sensitive regarding the position of the candidates, combined with certain political integrity of the voters make the influence through bribery hard from a complexity theoretic point of view. This effects today's political agendas as distances can be considered in terms of campaigning and its limits. It is an interesting task to explore exactly which distance properties and which election properties are causing the jump from P to NP-completeness. As a next step the parameterized complexity of the presented problems and approximation algorithms for the hard problems should be considered in detail.

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