Facility Location Games with Optional Preferences: A Revisit

Xingchen Sha[1], Shuyu Bao[1], Hau Chan[2], Vincent Chau[3*], Ken C. K. Fong[4*], Minming Li[1]

[1]Department of Computer Science, City University of Hong Kong, HKSAR China
[2]Department of Computer Science and Engineering, University of Nebraska-Lincoln, USA
[3]School of Computer Science and Engineering, Southeast University, China
[4]Division of Artificial Intelligence, School of Data Science, Lingnan University, HKSAR China
xingchsha2-c@my.cityu.edu.hk, sybao3-c@my.cityu.edu.hk, hchan3@unl.edu, vincentchau@seu.edu.cn, kenfong@ln.edu.hk, minming.li@cityu.edu.hk

Abstract

We study the k-facility location games with optional preferences on the line. In the games, each strategic agent has a public location preference on the k facility locations and a private optional preference on the preferred/acceptable set of facilities out of the k facilities. Our goal is to design strategyproof mechanisms to elicit agents' optional preferences and locate k facilities to minimize the social or maximum cost of agents based on their facility preferences and public agent locations. We consider two variants of the facility location games with optional preferences: the Min variant and the Max variant where the agent's cost is defined as their distance to the closest acceptable facility and the farthest acceptable facility, respectively. For the Min variant, we present two deterministic strategyproof mechanisms to minimize the maximum cost and social cost with $k \geq 3$ facilities and well-separated n agents, achieving approximation ratios of 3 and $2n + 1$ respectively. We complement the results by establishing lower bounds of $\frac{3}{2}$ and $\frac{n}{4}$ for the approximation ratios achievable by any deterministic strategyproof mechanisms for the maximum cost and social cost, respectively. We then improve our results in a special setting of the Min variant where there are exactly three facilities and present two deterministic strategyproof mechanisms to minimize the maximum cost and social cost. For the Max variant, we present an optimal deterministic strategyproof mechanism for the maximum cost and a k-approximation deterministic strategyproof mechanism for the social cost.

Introduction

In recent years, extensive research has been conducted on facility location games (Chan et al. 2021) due to their potential application for modeling various real-world preference aggregation scenarios, such as voting and representative selection (Black 1948; Blin and Satterthwaite 1976; Moulin 1980), and theoretical interests involving strategic agents.

In the classical setting of facility location games, strategic agents have location preferences on the ideal locations or positions of the facilities in some metric space (e.g., line space). Depending on the facility location context, agent location preferences of the facilities can have a wide range of interpretations. For instance, facilities can be viewed as physical entities (e.g., schools, parks, libraries, or bus stops) in geographical domains or non-physical entities (e.g., political viewpoints and room temperatures) in non-geographical domains (Chan et al. 2021). Because the agents' preferences are private, the typical goal of the social planner is to design mechanisms that elicit agents' preferences and determine the facilities' positions on the metric space that (approximately) minimize the social/total or maximum costs (or distances) of the agents to the facilities. As agents often have incentives to misreport preferences, existing studies have primarily sought to design mechanisms that are strategyproof in which no agents have incentives to misreport preferences.

Moulin (1980) first characterized the set of strategyproof mechanisms for the classical single facility location games (i.e., or 1-facility location games) for locating a single facility on a line space, where agents have single-peaked preferences. Procaccia and Tennenholtz (2013) extended the study to k-facility location games for locating $k \leq 2$ facilities focusing on optimizing the total and maximum costs and initiated approximate mechanism design paradigm proposing to approximately optimize these costs when optimality and strategyproofness cannot be achieved simultaneously.

Facility Location Games with Optional Preferences While the majority of the studies on facility location games have assumed that agent location preferences are private and agents are indifferent between facilities (where agents are only interested in the closest facilities), there is an emerging line of facility location game studies that examine agents having preferences on the facilities and aim to design strategyproof mechanisms to elicit agent facility preferences while locating facilities to minimize social/total or maximum agent costs to the facilities based on the facility preferences and public agent locations.

Serafino and Ventre (2014) first considered k-facility location games with facility preferences for locating $k = 2$ facilities where agents have private preferences on the set of acceptable facilities and the cost of each agent is the sum of their distances to the acceptable facilities. Later, Chen et al. (2020) proposed the k-facility location games with optional (facility) preferences under two variants for $k \leq 2$ facilities: the Min variant and the Max variant where the agent's cost is defined as their distance to the closest acceptable facility and the agent's cost is defined as their distance to the farthest ac-

*Corresponding author

Objective	$k=2$	$k=3$	$k>3$
Maximum Cost	UB: $2^{[2]}$	UB: 2	UB: 3^*
	LB: $\frac{4}{3}^{[2]}$	LB: $\frac{3}{2}$	LB: $\frac{3}{2}$
Social Cost	UB: $2.75^{[1]}$	UB: $n+2$	UB: $2n+1^*$
	LB: $2-\frac{1}{n-2}^{[2]}$	LB: $\frac{n}{4}$	LB: $\frac{n}{4}$

* = well-separated, UB = upper bound, LB = lower bound, $^{[1]}$ = (Li et al. 2020), $^{[2]}$ = (Chen et al. 2020).

Table 1: Summary of results for k-facility location games in the Min variant. It is trivial to find the optimal strategyproof mechanism for each cost objective with single facility.

Objective	$k=2$	$k\geq 3$
Maximum Cost	$1^{[1]}$	1
Social Cost	$2^{[1]}$	k

$^{[1]}$=(Chen et al. 2020).

Table 2: Known upper bounds for k-facility location games in the Max variant. It is trivial to find the optimal strategyproof mechanism for each cost objective with one facility.

ceptable facility, respectively. Notice that the optional preference of an agent refers to an approval set of the facilities rather than a ranking of facilities based on a utility function. These facility preferences enable each agent to express their preference over a set of facilities that are acceptable to the agents in various real-life situations (see e.g., (Li et al. 2020; Chen et al. 2020)). For instance, when determining the locations of different parks, the agents can specify the parks that they would prefer to use on a regular basis. When locating the bus stops for different bus routes, the agents can indicate the sets of bus stops that are able to help them commute to their targeted destinations. Furthermore, when determining the temperatures of several classrooms, the agents can specify the subsets of classrooms they would use throughout the academic year.

Our Contributions

Following the studies of facility location games with agent facility preferences, we examine the k-facility location games with optional preferences on a line metric space $\mathbb{R}$ for locating k facilities[1].

In the considered games, there are n agents that can have different optional preferences over these facilities (i.e., the subsets of facilities that they prefer/accept). Because the agents' locations are publicly known while their optional preferences are private information, our goal is to design strategyproof mechanisms to elicit their true optional preferences and optimize certain objectives by determining facilities' locations. Following the work of (Chen et al. 2020; Li et al. 2020), the cost of an agent is the distance to their closest (resp. farthest) acceptable facility in the Min (resp. Max) variant. We consider the objectives of minimizing maximum cost or social cost, which are defined to be the maximum and the sum of the agents' cost, respectively.

Notice that while (Chen et al. 2020; Li et al. 2020) considered $k\leq 2$ facilities, we focus on $k\geq 3$ facilities, which is an open problem that has not been examined previously. Tables 1 and 2 provide a summary of our results in terms of the lower bounds on the approximation ratios of any strategyproof mechanisms and the upper bounds on the approximation ratios of our designed strategyproof mechanisms. More specifically,

- For the Min variant, we first consider the setting with $k\geq 3$ facilities. We present a deterministic strategyproof mechanism for the maximum cost of well-separated agents with an approximation ratio of 3. We then provide a deterministic strategyproof mechanism for the social cost of well-separated agents with an approximation ratio of $2n+1$. We complement these results by establishing a lower bound of $\frac{3}{2}$ for the maximum cost and $\frac{n}{4}$ for the social cost by any deterministic strategyproof mechanism. We also show that our results can be extended to other metric spaces. Then, we further improve our results for a special setting where there are exactly three facilities. We present a deterministic strategyproof mechanism for the maximum cost which achieves an approximation ratio of 2 and a deterministic strategyproof mechanism for the social cost with an approximation ratio of $n+2$.
- For the Max variant, we present an optimal deterministic strategyproof mechanism for the maximum cost and a k-approximation deterministic strategyproof mechanism for the social cost.

Outline The remainder of this paper is organized as follows. We formally define the problem in Section 2. In Section 3, we study the Min variant with at least three facilities (i.e., $k\geq 3$). In Section 4, we discuss the Max variant. We conclude our work and discuss open questions in Section 5.

Related Work

In the realm of mechanism design for facility location problems, Moulin (1980) first characterized strategyproof mechanisms for the classic facility location problem on a linear structure, where each agent has a single-peaked preference.

One significant approach to study the facility location games is by Procaccia and Tennenholtz (2013), which initiated approximate mechanism design without money, leveraging facility location problems as a case study. They designed several approximately optimal (both deterministic and randomized) strategyproof mechanisms for the classic facility location games, focusing on the social and the maximum cost. Their analyses encompass scenarios with single facility, two homogeneous facilities and multiple locations per agent. The subsequent work by Lu et al. (2010) further improved their lower and upper bounds. Fotakis and Tzamos (2014) studied k-facility location games, showing a strong impossibility theorem for deterministic anonymous strategyproof mechanisms with a bounded approximation ratio. They provided a randomized strategyproof mechanism for

[1]Previous studies of facility location games with or without optional facilities have primarily studied and focused on the line metric space (Chen et al. 2020; Li et al. 2020; Chan et al. 2021).

any number of facilities, the $EqualCost$ mechanism, which achieves approximation ratios of n and 2 for the social cost and maximum cost respectively (Fotakis and Tzamos 2013).

Since then, numerous variants for the classic facility location games have been proposed. One major direction is to consider models where agents can have different preferences and decide their participation over heterogeneous facilities.

Regarding the preferences of agents in the context of a single facility, Cheng, Yu, and Zhang (2013) initiated studies on obnoxious facility location games, where agents strive to be far away from the facility. This was followed by much subsequent research (Ibara and Nagamochi 2012; Ye, Mei, and Zhang 2015; Oomine and Nagamochi 2016). Additionally, researchers also explored dual preferences where agents held different inclinations towards a single facility (Feigenbaum and Sethuraman 2015; Zou and Li 2015). Filos-Ratsikas et al. (2017) studied single facility on a real line where agents have double-peaked preferences.

Another approach of studying the heterogeneous facility location games was initiated by Serafino and Ventre (2014, 2015), where the cost for an agent favoring two facilities is the sum of the distances to both. Later Kanellopoulos, Voudouris, and Zhang (2023a) substantially improved the bounds. They also studied the problem where the two facilities are selected from a given finite set of candidate locations and provided deterministic strategyproof mechanisms with constant bounds for both social cost and maximum cost (Kanellopoulos, Voudouris, and Zhang 2023b). In cases where the cost is influenced by either the closer or farther facility, models of optional preference were initiated and studied by Chen et al. (2020). The upper bound for the social cost in the Min variant was later substantially improved by Li et al. (2020) using a similar approach. However, they also showed that their mechanism cannot be extended to the setting with $k \geq 3$ facilities. Xu, Zhang, and Xie (2023) studied optional preference model for the facility location games with a minimum distance requirement between the two facilities. Moreover, Fong et al. (2018) proposed the fractional preference model, where the cost is a weighted sum of the distances to both facilities. Anastasiadis and Deligkas (2020) worked on a scenario with k facilities, where each agent's preference for a facility falls into one of three categories: close to the facility, far from the facility, or indifferent about its presence. Their setup can be considered as a combination of the dual preferences and optional preferences models by extending them to multiple facilities.

To the best of our knowledge, our work stands as the first to consider k-facility location games with optional preferences. More work on facility location games can be found in a recent survey on mechanism design for facility location problems (Chan et al. 2021).

Preliminaries

We are given a collection of agents $N = \{1, \dots, n\}$. In this setting, there are k facilities (named as $F = \{F_1, \dots, F_k\}$), and each facility is given a unique label F_j with the index $j \in \{1, \dots, k\}$. We denote the facility location of F_j as y_j. Each agent i is at location x_i and has a preference $p_i \subseteq F$, which is an acceptable set of facilities that can serve agent i. In this paper, a preferred facility by agent i refers to a facility within their acceptable set of facilities. We denote the location profile of all agents as $X \coloneqq (x_1, \dots, x_n)$ and the preference profile as $P \coloneqq (p_1, \dots, p_n)$.

In this paper, we consider two variants of the facility location games with optional preferences, the Min variant and the Max variant. In the Min variant of the optional preference model, an agent i's cost is the distance between their location and the closest preferred (acceptable) facility, i.e., $cost_{X,P,i}(y_1, \dots, y_k) = \min_{F_j \in p_i} |y_j - x_i|$. While in the Max variant, an agent i's cost is the distance between their location and the farthest preferred (acceptable) facility, i.e., $cost_{X,P,i}(y_1, \dots, y_k) = \max_{F_j \in p_i} |y_j - x_i|$.

A deterministic mechanism M maps (X, P) to a profile of facility locations $Y \coloneqq (y_1, \dots, y_k)$. We aim to minimize maximum cost and social cost. Given a deterministic mechanism M and an instance (X, P), maximum cost is defined as $MC_{X,P}\ (M(X, P)) = \max_{i \in N} cost_{X,P,i}(M(X, P))$ and social cost is defined as $SC_{X,P}(M(X, P)) = \sum_{i \in N} cost_{X,P,i}(M(X, P))$. A mechanism M achieves an approximation ratio of ρ for the social cost (resp. maximum cost), if for any profile (X, P), $SC_{X,P}(M(X, P)) \leq \rho \cdot SC_{X,P}(OPT(X, P))$ (resp. $MC_{X,P}(M(X, P)) \leq \rho \cdot MC_{X,P}(OPT(X, P))$) where $OPT(X, P)$ is the optimal solution that minimizes the target cost.

A deterministic mechanism M is strategyproof if no agent can benefit by reporting a false preference. Denote (P_{-i}, p'_i) as the tuple P with p'_i in place of p_i. Below, we provide a formal definition of strategyproofness in this problem.

Definition 1. *A deterministic mechanism M is strategyproof if for all X, P, any agent i and preference p'_i, we have*

$$cost_{X,P,i}(M(X, P)) \leq cost_{X,P,i}(M(X, (P_{-i}, p'_i))).$$

Optional Preference (Min)

In this section, we consider a basic setting of k-facility location games with optional preference (Min), where there are at least three facilities. We first introduce two strategyproof mechanisms for well-separated agents under each cost objective and evaluate their performance by providing respective bounds of approximation ratios. It is interesting that when the number of facilities increases to three, the approximation ratio achievable by any strategyproof mechanism for the social cost grows linearly to the number of agents.

General Case of $k \geq 3$ Facilities

Maximum Cost. We first consider the maximum cost and provide a deterministic strategyproof mechanism for $k \geq 3$ facilities. Similar to the dynamic programming algorithm by Love (Love 1976) for optimizing the social cost, we design a dynamic programming $OPTMC(X)$ for Mechanism 1 to compute an optimal solution $M = \{m_1, m_2, \dots, m_k\}$ regardless of the preference profile, i.e., suppose every agent prefers all the facilities. Without loss of generality, we assume $m_1 \leq m_2 \leq \dots \leq m_k$. Denote $|M|$ as the number of distinct locations in M. We divide the agents into k groups, i.e., $G_1, G_2, \dots, G_k$, such that the closest point in M for each agent i in G_j is m_j, i.e., $m_j = \arg\min_{m \in M} |m - x_i|$. Denote $|G_j|$ as the number of agents in G_j. We say that

Algorithm 1: OPTMC(X)

Input: Agent location profile $X = (x_1, \ldots, x_n)$. Without loss of generality, assume $x_1 \leq \cdots \leq x_n$.

Parameter: Define $M(i, j)$ to be the facility location profile that minimizes maximum cost of the first i-th agents from left to right with j facilities. Define $MC(i, j)$ to be the maximum cost of the first i-th agents with $M(i, j)$. Initialize $M(i, 1) = \{\frac{x_i + x_1}{2}\}$. Initialize $MC(i, j) =$

$$\begin{cases} \frac{x_i - x_1}{2} & \textbf{if } j = 1, \\ \infty & \textbf{otherwise}. \end{cases}$$

Output: $M(n, k)$.

for $i \in \{2, \ldots, n\}$, $j \in \{2, \ldots, k\}$ **do**
  for $m \in \{1, \ldots, i-1\}$ **do**
    if $\max\{MC(m, j-1), \frac{x_i - x_{m+1}}{2}\} < MC(i, j)$ **then**
      $MC(i, j) = max\{MC(m, j-1), \frac{x_i - x_{m+1}}{2}\}$
      $M(i, j) = M(m, j-1).append(\frac{x_i + x_{m+1}}{2})$
    end if
  end for
end for
return $M(n, k)$

agents are well-separated if as a candidate in M is deleted, all agents closest to it have the same new closest candidate. Mechanism 1 considers several ways of placing the facilities, and outputs the facility location profile by cases. It is clear that the running time of Mechanism 1 is $O(2^k n^2)$ where k is publicly known and constant in real life scenarios.

Mechanism 1.

$M \leftarrow OPTMC(X)$
if $|M| = 1$ ***then***
  return M
end if
if $|M| = 2$ ***then***
  return $\text{argmin}_{y_1, \ldots, y_k \in M} MC_{X,P}(y_1, \ldots, y_k)$[2]
end if
$M' \leftarrow M$
for *ascending* $d = |m_a - m_b|$ *where* $m_a, m_b \in M$ ***do***
  if $|M'| = 2$ ***then***
    return $\text{argmin}_{y_1, \ldots, y_k \in M'} MC_{X,P}(y_1, \ldots, y_k)$[2]
  end if
  for $y_1, \ldots, y_k \in M'$ ***do***
    if *each agent is served by a facility at the closest candidate location in* M' ***then***
      return Y[2]
    end if
  end for
  $m_c \leftarrow \arg\min_{m_c \in M'} \left|\frac{m_1 + m_k}{2} - m_c\right|$
  Call $Delete(m_1, m_k, m_c, d)$
end for
$Delete(m_l, m_r, m_c, d)$:
if $\min(m_c - m_l, m_r - m_c) \leq d$ ***then***
  $M'.delete(m_c)$
else
  $m_{c_1} \leftarrow \arg\min_{m_{c_1} \in M} \left|\frac{m_l + m_c}{2} - m_{c_1}\right|$
  $m_{c_2} \leftarrow \arg\min_{m_{c_2} \in M} \left|\frac{m_c + m_r}{2} - m_{c_2}\right|$
  if $m_l < m_{c_1} < m_c$ ***then***
    Call $Delete(m_l, m_c, m_{c_1}, d)$
  end if
  if $m_c < m_{c_2} < m_r$ ***then***
    Call $Delete(m_c, m_r, m_{c_2}, d)$
  end if
end if

[2]Put the facility with a smaller index on the left to break the tie.

Theorem 1. *Mechanism 1 is strategyproof, and has an approximation ratio of 3 for the maximum cost of well-separated agents.*

Lemma 1. *For the maximum cost, there exists no α-approximation deterministic strategyproof mechanism for $k \geq 3$ facilities with $\alpha < \frac{3}{2}$.*

Proof. Consider two agents who only prefer $\{F_1\}$ locating at 1 and 2, $k-1$ agents who prefer $\{F_1, F_2, \ldots, F_{j-1}, F_{j+1}, \ldots, F_k\}$ locating at $3\,(j-2)$ respectively, with $2 \leq j \leq k$, and one agent who prefers $\{F_2, F_3, \ldots, F_k\}$ locating at $3\,(k-1)$. One optimal solution for this instance is to place F_1 at 1, F_j at $3\,(j-1)$, $2 \leq j \leq k$, which has maximum cost of 1. Now suppose there exists an α-approximation deterministic strategyproof mechanism with $\alpha < \frac{3}{2}$. Agents except for the two agents at 1 and 2 must be served by k different facilities according to the mechanism. Otherwise, maximum cost is at least $\frac{3}{2}$, contradicting the assumption of the approximation ratio. Denote the location of F_1 by the mechanism as y_1. Because of the approximation ratio, F_1 must be placed at either $\left(\frac{1}{2}, \frac{3}{2}\right)$, if the agent at 0 is served by F_1, or $\left(\frac{3}{2}, \frac{5}{2}\right)$, if the agent at 3 is served by F_1. Suppose $y_1 \in \left(\frac{3}{2}, \frac{5}{2}\right)$, i.e., the agent at 3 is served by F_1 according to the mechanism. Now consider the agent at 1 misreporting his preference from $\{F_1\}$ to $\{F_2\}$. The optimal maximum cost of the new instance is still 1 where F_1 is placed at 1, F_2 is placed at 2, F_j is placed at $3\,(j-1)$, $3 \leq j \leq k$. The agent at 0 must be served by F_1, and the agent at 3 must be served by F_2 according to the mechanism. Denote the new location of F_1 as y_1'. Due to the approximation ratio, F_1 should be placed at $\left(\frac{1}{2}, \frac{3}{2}\right)$. To prevent the agent at 1 from misreporting, we have $y_1' \geq y_1 > \frac{3}{2}$, which is a contradiction. When $y_1 \in \left(\frac{1}{2}, \frac{3}{2}\right)$, consider the agent at 2 misreporting his preference from $\{F_1\}$ to $\{F_3\}$ and the proof is similar.

Therefore, there exists no α-approximation deterministic strategyproof mechanism with $\alpha < \frac{3}{2}$ for the maximum cost. □

Social Cost. The approximation ratio of Mechanism 1 for the social cost is bounded by $3n$. Below, we provide a deterministic strategyproof mechanism for $k \geq 3$ facilities that has a better upper bound of $2n + 1$ for minimizing the social cost, and prove a corresponding lower bound of $\frac{n}{4}$.

An optimal solution $S = \{s_1, \ldots, s_k\}$ can also be calculated by similar algorithms in polynomial time regardless of the preference profile, i.e., suppose every agent prefers all the facilities. For instance, Love solved the facility location problem on the real line by a dynamic programming algorithm with the observation that agents must be assigned contiguously to the new facilities in an optimal solution (Love 1976). Without loss of generality, we assume $s_1 \leq \cdots \leq s_k$. Again, we divide the agents into k groups, i.e., $G_1, \ldots, G_k$, such that the closest point in S for each agent i in G_j is s_j, i.e., $s_j = \arg\min_{s \in S} |s - x_i|$. Denote $|G_j|$ as the number of agents in G_j. Similarly, agents are well-separated if as a candidate in S is deleted, all agents closest to it have the same new closest candidate. Mechanism 2 considers several ways of placing the facilities, and outputs the facility location profile by cases.

Mechanism 2.

Let $OPTSC(X)$ be an optimal algorithm for the social cost of k-facility location problems.

$S \leftarrow OPTSC(X)$
if $|S| = 1$ ***then***
  return S
end if
if $|S| = 2$ ***then***
  return $\text{argmin}_{y_1,\ldots,y_k \in S} SC_{X,P}(y_1, \ldots, y_k)$[3]
end if
$S' \leftarrow S$
for *ascending* $d = |s_a - s_b|$ *where* $s_a, s_b \in S$ ***do***
  if $|S'| = 2$ ***then***
    return $\text{argmin}_{y_1,\ldots,y_k \in S'} SC_{X,P}(y_1, \ldots, y_k)$[3]
  end if
  for $y_1, \ldots, y_k \in S'$ ***do***
    if *each agent is served by a facility at the closest candidate location in* S' ***then***
      return Y[3]
    end if
  end for
  $s_c \leftarrow \text{argmin}_{s_c \in S'} |\frac{s_1 + s_k}{2} - s_c|$
  Call $Delete(s_1, s_k, s_c, d)$
end for
$Delete(s_l, s_r, s_c, d)$:
if $\min(s_c - s_l,\, s_r - s_c) \leq d$ ***then***
  $S'.delete(m_c)$
else
  $s_{c_1} \leftarrow \arg\min_{s_{c_1} \in S} |\frac{s_l + s_c}{2} - s_{c_1}|$
  $s_{c_2} \leftarrow \arg\min_{s_{c_2} \in S} |\frac{s_c + s_r}{2} - s_{c_2}|$
  if $s_l < s_{c_1} < s_c$ ***then***
    Call $Delete(s_l, s_c, s_{c_1}, d)$
  end if
  if $s_c < s_{c_2} < s_r$ ***then***
    Call $Delete(s_c, s_r, s_{c_2}, d)$.
  end if
end if

Theorem 2. *Mechanism 2 is strategyproof, and has an approximation ratio of $2n + 1$ for the social cost of well-separated agents.*

Lemma 2. *For the social cost, there exists no α-approximation deterministic strategyproof mechanism for $k \geq 3$ facilities with $\alpha < \frac{n}{4}$.*

[3]Put the facility with a smaller index on the left to break the tie.

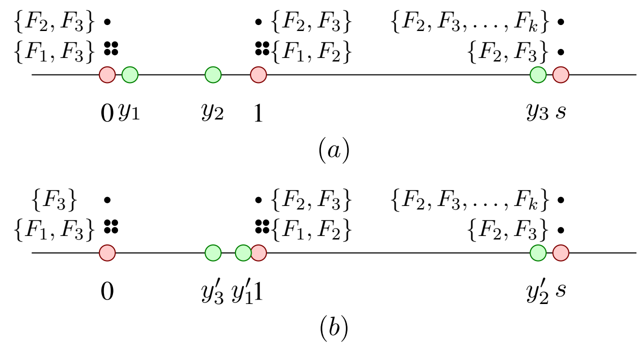

Figure 1: An example depicting the proof of the lower bound for the social cost.

Proof. As illustrated in Figure 1, consider $\frac{n-4}{2}$ agents located at 0 whose preference profiles are $\{F_1, F_3\}$, $\frac{n-4}{2}$ agents located at 1 whose preference profiles are $\{F_1, F_2\}$, two agents located at 0 and 1 respectively who prefer $\{F_2, F_3\}$, and two agents located at $s \gg n \geq 5$ who prefer $\{F_2, F_3\}$ and $\{F_2, F_3, \ldots, F_k\}$ respectively. One optimal solution for this instance is to place F_1 at 0, F_2 at 1, and F_3 at s, which has social cost of 1. Now suppose there exists an α-approximation deterministic strategyproof mechanism with $\alpha < \frac{n}{4}$. Agents who have the preference profiles of $\{F_1, F_3\}$, $\{F_1, F_2\}$, and the agent at s who prefers $\{F_2, F_3\}$ must be served by three different facilities according to the mechanism. Otherwise, for example, if agents who prefer $\{F_1, F_3\}$ and $\{F_1, F_2\}$ are all served by F_1, social cost will be at least $\frac{n-4}{2} + 1 = \frac{n}{2} - 1 > \frac{n}{4}$, which contradicts the assumption of approximation ratio. Hence, there are only two possible ways of placing the facilities. If the agent at s who prefers $\{F_2, F_3\}$ is served by F_3, F_1 will be placed at [-$\frac{1}{2}$,$\frac{1}{2}$] and F_2 will be placed at [$\frac{1}{2}$,$\frac{3}{2}$]. Otherwise if the agent at s is served by F_2, F_1 will be placed at [$\frac{1}{2}$,$\frac{3}{2}$] and F_3 will be placed at [-$\frac{1}{2}$,$\frac{1}{2}$]. Without loss of generality, we consider the former case. Denote the location of F_j by the mechanism as y_j. Now consider the agent at 0 misreporting their preference from $\{F_2, F_3\}$ to $\{F_3\}$. The optimal social cost of the new instance is still 1 where F_3 is placed at 0, F_1 is placed at 1, and F_2 is placed at s. The agent at s who prefers $\{F_2, F_3\}$ must be served by F_2 according to the mechanism. Otherwise, social cost will be at least s, which also contradicts the assumption of the approximation ratio. Denote the new location of F_j by the mechanism as y'_j. Due to the strategyproofness, F_3 should be placed at either $[-\frac{1}{2}, -y_2]$ or $[y_2, \frac{3}{2}]$. Considering these two instances, we have

$$\alpha = \max\{\frac{n-4}{2} \cdot (1 - y_2) + 1, \frac{n-4}{2} \cdot y'_3 + 1\}$$
$$\geq \max\{\frac{n-4}{2} \cdot (1 - y_2) + 1, \frac{n-4}{2} \cdot y_2 + 1\}$$
$$\geq \frac{n-4}{4} + 1 = \frac{n}{4},$$

which contradicts the assumption. □

Next, we will discuss how to extend our mechanisms and results to other metric spaces.

Remark 1. *For any Euclidean space R^d, our strategyproof mechanisms for $k \geq 3$ facilities can be extended with the same approximation ratios. The key idea is that we can still use an optimal algorithm for the k-center (k-median) problems to compute a facility location profile M (resp. S) assuming that each agent prefers all facilities.*

The approach of determining the final facility location profile is similar to those in the line setting and involves iteratively deleting candidate facility locations from the computed profile M (resp. S) via a binary search until every agent can be served by a preferred facility at the closest possible candidate, or their closest candidate has been deleted in M' (resp. S'). In the binary search, we first identify two candidates that are farthest apart, then order the remaining candidates based on their distances to these two. Additionally, all lower bounds would still hold in this metric space.

Using a similar method, our results can also be extended to other metrics spaces studied as long as optimal algorithms exist for the k-center (k-median) problems (e.g., L_1 and L_2^2 spaces).

Special Case of Three Facilities

We also note that through a careful case analysis, our mechanisms can be further improved for the maximum cost and the social cost in the case of exactly three facilities.

Denote L_j (resp. R_j) as the location of the leftmost (resp. rightmost) agent in G_j. Notice that the optimal solution without considering agents' preferences will place each facility at the middle point of each group, which satisfies $m_j = \frac{L_j+R_j}{2}$ for $j \in \{1,2,3\}$.

Mechanism 3.
Denote $d_{max} = \max(R_1-L_1, R_2-L_2, R_3-L_3)$. Consider the corresponding $\hat{G}_1$, $\hat{G}_2$, $\hat{G}_3$ of $\hat{M} = \{\hat{m}_1, \hat{m}_2, \hat{m}_3\}$ such that $\hat{M} =$

$$\begin{cases} \{R_1, \max(R_1, R_2 - d_{max}), L_3\} & \text{if } L_2 - L_1 \leq R_3 - R_2, \\ \{R_1, \min(L_3, L_2 + d_{max}), L_3\} & \text{if } L_2 - L_1 > R_3 - R_2. \end{cases}$$

Let

$$\bar{G} = \begin{cases} \hat{G}_3 - G_3 & \text{if } L_2 - L_1 \leq R_3 - R_2, \\ \hat{G}_1 - G_1 & \text{if } L_2 - L_1 > R_3 - R_2. \end{cases}$$

Case 1. *If there exists a facility location profile of F over the candidate locations $\hat{M}$ such that agents in $\hat{G}_j$ are all served by one facility at $\hat{m}_j$ except that some agents in $\bar{G}$ are served by one facility at $\hat{m}_2$, output it as the facilities' locations.*

If $L_2 - L_1 \leq R_3 - R_2$, break the tie in this order: $(\hat{m}_2, \hat{m}_1, \hat{m}_3)$, $(\hat{m}_1, \hat{m}_2, \hat{m}_3)$, $(\hat{m}_3, \hat{m}_1, \hat{m}_2)$, $(\hat{m}_1, \hat{m}_3, \hat{m}_2)$, $(\hat{m}_3, \hat{m}_2, \hat{m}_1)$, $(\hat{m}_2, \hat{m}_3, \hat{m}_1)$.

If $L_2 - L_1 > R_3 - R_2$, break the tie in this order: $(\hat{m}_3, \hat{m}_1, \hat{m}_2)$, $(\hat{m}_3, \hat{m}_2, \hat{m}_1)$, $(\hat{m}_2, \hat{m}_1, \hat{m}_3)$, $(\hat{m}_2, \hat{m}_3, \hat{m}_1)$, $(\hat{m}_1, \hat{m}_2, \hat{m}_3)$, $(\hat{m}_1, \hat{m}_3, \hat{m}_2)$.

Case 2. *If there does not exist a facility location profile that satisfies Case 1, let*

$$M' = \begin{cases} \{\hat{m}_2, \hat{m}_3\} & \text{if } L_2 - L_1 \leq R_3 - R_2, \\ \{\hat{m}_1, \hat{m}_2\} & \text{if } L_2 - L_1 > R_3 - R_2. \end{cases}$$

Output the facility location profile over M' with the minimum max cost among the agents in $N - \bar{G}$, i.e., $Y = (y_1, y_2, y_3) = \mathrm{argmin}_{y_1,y_2,y_3 \in M'} \max_{i \in N, i \notin \bar{G}} cost_{X,P,i}(y_1, y_2, y_3)$.
We break the tie by considering two sub-cases.

Case 2.1 *If each agent in $N - \bar{G}$ is served by a facility at the closest candidate location in M', choose the profile with a minimum number of agents from $\hat{G}_1$ (resp. $\hat{G}_3$) and $\hat{G}_2$ who have to be served by the same facility. If there are still multiple suitable profiles, we further break the tie by placing the facilities at $\hat{m}_2$ as many as possible and giving a higher priority to the facility F_j with a smaller index j, i.e., $F_1 \succ F_2 \succ F_3$.*

Case 2.2 *Otherwise, place the facility F_j with a smaller index j on the left first.*

Theorem 3. *Mechanism 3 is strategyproof, and has an approximation ratio of 2 for the maximum cost..*

Similarly, our mechanism can be further improved for the social cost in the case of three facilities as well.

Mechanism 4.
Case 1. *If there exists a facility location profile of F over the candidate locations S such that agents in G_j are all served by a facility at s_j, output it as the facilities' locations, and break the tie in this order: (s_1, s_2, s_3), (s_1, s_3, s_2), (s_2, s_1, s_3), (s_2, s_3, s_1), (s_3, s_1, s_2), (s_3, s_2, s_1).*

Case 2. *If there does not exist a facility location profile that satisfies Case 1, we remove one candidate location from S. Assuming, w.l.o.g., $s_2 - s_1 \leq s_3 - s_2$,*

$$S' = \begin{cases} \{s_1, s_3\} & \text{if } |G_1| \geq |G_2|, \\ \{s_2, s_3\} & \text{if } |G_1| < |G_2|. \end{cases}$$

Output the facility location profile over S' with the minimum social cost, i.e., $Y = (y_1, y_2, y_3) = \arg\min_{y_1,y_2,y_3 \in S'} \sum_{i \in N} cost_{X,P,i}(y_1, y_2, y_3)$. We break the tie by placing the facility F_j with a smaller index j on the left first.

Lemma 3. *Mechanism 4 is strategyproof.*

Proof. Firstly, since the candidate facility locations S and S' only depend on agents' location information, each agent i cannot change S or S' by misreporting their preference.

Secondly, by the definition of Mechanism 4, each agent is served by a preferred facility at the closest candidate location in S if Case 1 is satisfied. Therefore, no agent will benefit by misreporting from Case 2 to Case 1, and vice versa.

It remains to show that agents cannot benefit by misreporting within Case 2. Given agents' location profile X and preference profile P, denote $SC_{X,P}(Y)$ as social cost of agents with the facility location profile Y determined by Mechanism 4. After agent i misreports, the agents' preference profile becomes (P_{-i}, p'_i) where p_i is replaced by p'_i. In Case 2, a solution with the minimum achievable social cost by the facility location profile over S' is returned. The agent can only benefit if the output facility location profile becomes $Y' \neq Y$ after misreporting. Therefore, we have

$$SC_{X,P}(Y) \leq SC_{X,P}(Y'),$$
$$SC_{X,(P_{-i},p'_i)}(Y') \leq SC_{X,(P_{-i},p'_i)}(Y). \quad (1)$$

Suppose agent i will benefit by misreporting, which indicates that their distance cost with Y' is smaller than the distance cost with the original output Y, i.e.,

$$\min_{F_j \in p_i, y'_j \in Y'} |y'_j - x_i| < \min_{F_j \in p_i, y_j \in Y} |y_j - x_i|. \tag{2}$$

For all the other agents, the sum of their distance cost remains the same given the same facility location profile. By eq. (1), we have

$$\begin{aligned} & SC_{X,P}(Y) - SC_{X,(P_{-i},p'_i)}(Y) \\ = & \min_{F_j \in p_i, y_j \in Y} |y_j - x_i| - \min_{F_j \in p'_i, y_j \in Y} |y_j - x_i| \\ & \leq SC_{X,P}(Y') - SC_{X,(P_{-i},p'_i)}(Y') \\ = & \min_{F_j \in p_i, y'_j \in Y'} |y'_j - x_i| - \min_{F_j \in p'_i, y'_j \in Y'} |y'_j - x_i|. \end{aligned} \tag{3}$$

Notice that Y and Y' can only place the facilities at two candidate locations in S'. By eq. (2), we have

$$\begin{aligned} \min_{F_j \in p'_i, y_j \in Y} |y_j - x_i| &\leq \min_{F_j \in p_i, y_j \in Y} |y_j - x_i|, \\ \min_{F_j \in p_i, y'_j \in Y'} |y'_j - x_i| &\leq \min_{F_j \in p'_i, y'_j \in Y'} |y'_j - x_i|. \end{aligned} \tag{4}$$

It is clear that eq. (3) and eq. (4) can only hold when

$$\begin{aligned} \min_{F_j \in p_i, y_j \in Y} |y_j - x_i| &= \min_{F_j \in p'_i, y_j \in Y} |y_j - x_i|, \\ \min_{F_j \in p_i, y'_j \in Y'} |y'_j - x_i| &= \min_{F_j \in p'_i, y'_j \in Y'} |y'_j - x_i|, \end{aligned}$$

which indicates that

$$\begin{aligned} & SC_{X,P}(Y) = SC_{X,(P_{-i},p'_i)}(Y) \\ = & SC_{X,P}(Y') = SC_{X,(P_{-i},p'_i)}(Y'). \end{aligned}$$

Since we adopt a fixed tie-breaking rule to place the facilities, agents will not benefit by misreporting. □

Lemma 4. *Mechanism 4 has an approximation ratio of $n+2$ for the social cost.*

Proof Sketch. Given agents' location profile X and preference profile P, denote the facility location profile over S that achieves the minimum social cost as Y_S. Let the optimal facility location profile and the facility location profile output by Mechanism 4 be $OPT(X, P)$ and $M_4(X, P)$. We show that $SC_{X,P}(M_4(X, P)) \leq (n-1) \cdot SC_{X,P}(OPT(X, P)) + SC_{X,P}(Y_S) \leq (n + 2) \cdot SC_{X,P}(OPT(X, P))$. □

Optional Preference (Max)

In this section, we explore the Max variant of the optional preference model, where the cost of each agent is defined as their distance to the farthest preferred (acceptable) facility. Let $N_j = \{i | F_j \in p_i, i \in N\}$ be the set of agents whose preference contains F_j.

Mechanism 5. *Place F_j at the middle point of N_j.*

Theorem 4. *Mechanism 5 is strategyproof, and is optimal for the maximum cost.*

Proof Sketch. Given agents' location profile X and preference profile P, denote $MC_{X,P}(Y)$ as the maximum cost of agents with the facility location profile Y determined by Mechanism 5. For any two agents that both prefer F_j, it is clear that their maximum cost is at least half of their distance. Therefore, we have

$$\begin{aligned} MC_{X,P}(Y) = & \max_{F_j \in F} \frac{\max_{i \in N_j} x_i - \min_{i \in N_j} x_i}{2} \\ \leq & MC_{X,P}(OPT). \end{aligned} \tag{5}$$

□

Mechanism 6. *Place F_j at the median of N_j.*

Theorem 5. *Mechanism 6 is strategyproof, and has an approximation ratio of k for the social cost.*

Mechanism 6 has a parameterized approximation ratio related to the number of facilities. Since k is publicly known and constant in real life scenarios, Mechanism 6 achieves a constant approximation ratio in most cases. The following example shows that the bound given in Theorem 5 is tight.

Example 1. *As illustrated in Figure 2, consider the instance with $n = (k + 1) \cdot m$ agents: m agents who prefer $\{F_1, F_2, \ldots, F_k\}$ at 0, $n - m$ agents at 1, where each m of them prefer only one facility. Mechanism 6 places all the facilities at 0, which incurs social cost of $n - m$. However, one optimal solution can place all the facilities at 1, which incurs social cost of m. Hence, the approximation ratio of Mechanism 6 for the social cost is at least k.*

Figure 2: An example depicting the tightness of the upper bound for the social cost.

Conclusion

We study the k-facility location games with optional preferences from a mechanism design perspective. For the Min variant, we provide deterministic strategyproof mechanisms with approximation guarantees as well as lower bounds for the corresponding cost objectives. It is interesting that the approximation ratio by any deterministic strategyproof mechanism for the social cost grows from a constant to $O(n)$ as the number of facilities increases to three. For the Max variant, we provide optimal and parameterized strategyproof mechanisms for the social cost and maximum cost.

There are many directions for future work. The most important question is to match the bounds for deterministic strategyproof mechanisms in the Min variant. For the Max variant, a characterization on the optimal solution for the social cost will help establish meaningful lower bounds. Finally, it is intriguing to explore the potential of randomization to enhance the design of better mechanisms for k-facility location games with optional preferences.

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