Understanding EFX Allocations: Counting and Variants

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Abstract

Envy-freeness up to any good (EFX) is a popular and important fairness property in the fair allocation of indivisible goods, of which its existence in general is still an open question. In this work, we investigate the problem of determining the minimum number of EFX allocations for a given instance, arguing that this approach may yield valuable insights into the existence and computation of EFX allocations. We focus on restricted instances where the number of goods slightly exceeds the number of agents, and extend our analysis to weighted EFX (WEFX) and a novel variant of EFX for general monotone valuations, termed EFX+. In doing so, we identify the transition threshold for the existence of allocations satisfying these fairness notions. Notably, we resolve open problems regarding WEFX by proving polynomial-time computability under binary additive valuations and establishing the first constant-factor approximation for two agents.

1 Introduction

The field of *fair division* models many real-world problems such as course allocation (Budish and Cantillon 2012), inheritance division (Pratt and Zeckhauser 1990), and divorce settlements (Brams and Taylor 1996) (also refer to survey by Amanatidis et al. (2023)). Among the numerous fairness properties that have been proposed, *envy-freeness (EF)* (and its approximations or variants) remains one of the most widely studied in the field. Intuitively, an allocation is said to be envy-free if every agent values his bundle of goods at least as much as he values any other agent's bundle of goods. However, in the setting with indivisible goods, a complete envy-free allocation may fail to exist. Hence, many works consider relaxations of envy-freeness.

Perhaps the most prominent of these relaxations are *envy-freeness up to any good (EFX)* (Caragiannis et al. 2019) and *envy-freeness up to one good (EF1)* (Lipton et al. 2004). An allocation satisfies EFX if any envy that an agent has towards another agent can be eliminated by removing *any* good in the latter agent's bundle, and it satisfies EF1 if any envy that an agent has toward another agent can be eliminated by removing *some* good in the latter agent's bundle (refer to Section 2 for the formal definitions). The argument in favor of EFX as the preferred notion of fairness lies in the fact that EFX is considerably stronger compared to EF1. However, while EF1 allocations always exists and can be found in polynomial time (even when agents have arbitrary monotone valuations over the goods) (Budish 2011; Lipton et al. 2004), the existence of EFX remains one of the biggest open problems in the field of fair division (Caragiannis et al. 2019; Moulin 2019; Procaccia 2020).

Despite substantial efforts, the existence of EFX allocations have only been proven for several restricted instances. These include, but are not limited to (1) when $n = 2$ (Plaut and Roughgarden 2020), (2) when $n = 3$ with at least one additive agent (Akrami et al. 2023), (3) when $m = n + 3$ (Mahara 2023), and (4) under identical valuations (Plaut and Roughgarden 2020). Moreover, many previous algorithms have been shown to not extend to more general cases (Chaudhury et al. 2021b; Mahara 2023).

In this work, we ask a more general question: *How many EFX allocations are there?* If there is an instance where EFX fail to exist, the answer is clearly 0. However, for instances where EFX allocation(s) always exists, the answer to this question is substantially unclear. This question was first explored in Suksompong (2020) where they show that while there is always an exponential (with respect to the number of goods) number of EF1 allocations, there could be far fewer EFX allocations. In particular, when there are two agents, there can be as few as two EFX allocations, no matter the number of goods. In this paper, we study the problem of counting the minimum number of EFX allocations for several restricted instances, and for two variants of EFX in more general instances.

We posit that pursuing this line of inquiry may uncover valuable insights regarding (the existence/computation of) EFX allocations, and make some progress towards resolving "fair division's most enigmatic question" (Procaccia 2020):

- Establishing tighter upper bounds on the minimum number of EFX allocations allows us to identify 'hard' instances for EFX. These 'hard' instances can be used either as a first step to find an instance where an EFX allocation fails to exist, or to design algorithms tackling these 'hard' instances.
- Establishing tighter lower bounds on special cases allows us to better understand the kind of EFX allocations that exist in those special cases.

Moreover, in practice, it may be desirable to have differing EFX allocations. The psychological phenomena of *single-option aversion* suggests that humans may be uncomfortable with decision-making if they are only presented with one option, even if it is an option they really like (Mochon 2013). Beyond psychology, consider an instance with two agents and three goods, where each agent values every good identically. Clearly, a complete EF allocation is not possible.[1] The best that any *deterministic* mechanism can achieve in this instance is an EFX allocation of allocating any two goods to an agent and one good to the other.

However, consider a *randomized* mechanism that gives the first agent two goods with a probability of half, and one good otherwise. By having more than one EFX allocation, we can design a mechanism that only outputs EFX allocations and is EF in expectation! This idea is also captured in the *Best-of-Both-Worlds (BoBW)* framework where the goal is to design a mechanism that has stronger ex-ante guarantees despite relatively weaker ex-post guarantees. Under this framework, it was shown that there exist an algorithm that is ex-ante EF and ex-post EF1 (Aziz 2020; Freeman, Shah, and Vaish 2020). Such algorithms implicitly hinges on the fact there are numerous EF1 allocations that one can randomize over, so as to achieve a fair outcome in expectation.

1.1 Our Contributions

We consider the standard fair division model with indivisible goods. Our results are summarized in Table 1.

In Section 3, we first show that given an instance, counting the number of EFX allocations is $\#P$-complete. We then show that the minimum number of EFX allocations is a function of both the number of agents (n) and number of goods (m) by investigating the restricted instance with few goods. We elucidate that this setting is non-trivial by showing that if an EFX allocation always exists when $m = n + \omega(1)$, then an EFX allocation always exists in general. For counting the minimum number of solutions, we show that there is a minimum of $\frac{n!}{(n-m)!}$ EFX allocations when $m \leq n$ and a minimum of n EFX allocations when $m = n + 1$. When $m = n + 2$, we provide a lower bound of n and an upper bound of n^2 for the minimum number of EFX allocations. Lastly, we show that there can always be as few as $n!$ EFX allocation for any number of goods.

In Section 4, we investigate weighted envy-freeness up to any good (WEFX), a generalization of EFX to the setting where agents have possibly differing entitlements. We show that under binary additive valuations, there always exists a WEFX allocation; but when $m \geq 3$, there could be as few as a single WEFX allocation. Moreover, we resolve an open question by providing a polynomial-time algorithm for computing WEFX and PO allocations for any number of agents under binary valuations. However, for the more general binary submodular or restricted additive valuation classes, we show that even after relaxing WEFX to a weaker variant—weak WEFX (WWEFX), a WWEFX allocation may still fail to exist in these settings. Lastly, under additive valuations, we show that there always exists a $\frac{1}{4}$-WEFX allocation when $n = 2$, improving upon the approximation factor previously shown by Hajiaghayi, Springer, and Yami (2023).

In Section 5, we investigate EFX+, a novel variant that is identical to EFX under additive valuations, but incomparable to EFX under general monotone valuations.

1.2 Further Related Work

Numerous variants (relaxations) of EFX (which is stronger than EF1) have been proposed and studied, such as a multiplicative relaxation of EFX (α-EFX) (Amanatidis, Markakis, and Ntokos 2020), envy-freeness up to one less-preferred good (EFL) (Barman et al. 2018), envy-freeness up to a random good (EFR) (Farhadi et al. 2021), and epistemic EFX (Caragiannis et al. 2022). Several other works consider the notion of EFX with charity, a relaxation of EFX where not all goods need to be allocated (Caragiannis, Gravin, and Huang 2019; Chaudhury et al. 2021b).

EFX has also been studied in the context of indivisible chores division. As the chores setting is known to be more complex in general, the existence of EFX is also an open problem there. There has been work showing that EFX for chores exists in restricted instances (Kobayashi, Mahara, and Sakamoto 2023; Miao et al. 2023), in constrained settings (Elkind, Igarashi, and Teh 2024), as well as upper bounds on approximate EFX (Bhaskar, Sricharan, and Vaish 2020; Zhou and Wu 2024).

2 Preliminaries

For each positive integer z, let $[z] := \{1, \ldots, z\}$. An instance consist of a set $N = [n]$ of n *agents* and a set $G = \{g_1, \ldots g_m\}$ of m *goods*. Each agent $i \in N$ has a *valuation function* $v_i : 2^G \to \mathbb{R}_{\geq 0}$. For each $i \in N$, we assume (as in standard in the literature) that the valuation function v_i is *monotone* (i.e., $S \subseteq T \implies v_i(S) \leq v_i(T)$) and $v_i(\emptyset) = 0$; v_i is said to be *additive* if for any subset of goods $S \subseteq G$, $v_i(S) = \sum_{g \in S} v_i(g)$.[2] For notational simplicity, we sometimes write $v_i(g)$ instead of $v_i(\{g\})$. Let the *valuation profile* of agents be $\mathbf{v} = (v_1, \ldots, v_n)$. An *instance* $I = (N', G', \mathbf{v}')$ is defined by a set of agents $N' \subseteq N$, a set of goods $G' \subseteq G$, and valuation profile $\mathbf{v}'$ corresponding to the set of agents and goods. An *allocation* $\mathcal{A} = (A_1, \ldots, A_n)$ is a partition of G such that each agent $i \in N$ receives the bundle A_i. Throughout this work, we assume the eventual goal is to obtain a *complete* allocation, i.e., all goods are eventually allocated.

Next, we formally define EF and EFX.

Definition 2.1 (EF). An allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is *envy-free (EF)* if for all agents $i, j \in N$, $v_i(A_i) \geq v_i(A_j)$.

Definition 2.2 (EFX). An allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is *envy-free up to any item (EFX)* if for all agents $i, j \in N$ and all $g \in A_j$, $v_i(A_i) \geq v_i(A_j \setminus \{g\})$.

[1] A *complete* allocation is one where all goods are allocated. This assumption is common in the literature; otherwise, one could leave all goods unallocated and it is trivially 'fair'.

[2] In general, we do not assume additivity for agents' valuation functions, but will use this in our counterexamples and upper bounds, which strengthens our negative results.

Property	Instance	$m \leq n$	$m = n+1$	$m = n+2$	$m \geq n+3$
EFX	General	$\frac{n!}{(m-n)!}$	n	UB: n^2, LB: n	UB: $n!$
	Identical Valuations	$\frac{n!}{(m-n)!}$	$n!$		
WEFX	$n = 2$, Binary Additive	2	1		
	$n = 2$, Binary Submodular	2	1	0	
	$n = 2$, Restricted Additive	2	1	0	
EFX+	$n = 2$	2			
	$n \geq 3$	$\frac{n!}{(m-n)!}$	0		

Table 1: Summary of our results: counting the minimum number of allocations satisfying the fairness property in various special cases. The upper bounds (UB) and lower bounds (LB) are provided in cases where the bound is not tight. Note that the lower bounds apply for general monotone valuations, but the examples constructed for the upper bounds only assume additive valuations, thereby strengthening our results.

It is instructive to note that the above definition of EFX is sometimes referred to as EFX_0 (introduced by Kyropoulou, Suksompong, and Voudouris (2020)), with EFX being originally defined as a weaker version that only drops nonzero-valued goods (Caragiannis et al. 2019). However, we adopt the stronger definition but label it as EFX as consistent with several recent works (see the survey of Amanatidis et al. (2023)). Notably, this also provides us with stronger results: in terms of existence, an EFX allocation exists if and only if an EFX_0 allocation exists (Chaudhury et al. 2021a).

We assume that the reader is familiar with basic notions of classic complexity theory (Papadimitriou 2007). All omitted proofs can be found in the full version of the paper.

We also make use of the concept of an *envy graph* (Lipton et al. 2004). The envy graph $\mathcal{G} = (N, E)$ for a (partial) allocation $\mathcal{A}$, is a directed graph where each agent is represented by a vertex. Moreover, for agents $i, j \in N$, $(i, j) \in E$ if agents agent i envies agent j. If there exists a cycle in the envy graph, we can perform a cyclic exchange of bundles, also known as *envy cycle elimination*. This process results in a new allocation where every agent swaps their bundle in exchange for an envied bundle, and consequently, no agent is worse off. Crucially, in our setting, if the initial allocation is EFX, after envy cycle elimination, the new allocation will remain EFX.

We first consider the standard EFX property.

3 EFX

Going back to our motivation on BoBW outcomes, we first look into the case with two additive agents. When $n = 2$, the minimum number of EFX allocations is 2 (Suksompong 2020). By randomising over these EFX allocations, an ex-ante EF and ex-post EFX outcome can always be computed, which we show with the following result.

Proposition 3.1. *When $n = 2$, an ex-ante* EF *and ex-post* EFX *outcome can always be computed.*

However, despite the usefulness of knowing the number of EFX allocations as motivated earlier, in general, counting the number of EFX allocations given an instance is $\#P$-complete. This might be the case even in instances where finding an EFX allocation is easy.

We first begin with a useful observation.

Lemma 3.2. *Let $a_0, \dots, a_n$ be a sequence of non-negative integers such that $\sum_{i=0}^{n} a_i \leq n!$. If we know the value $P = \sum_{i=0}^{n} a_i \cdot i^k$ for $k \geq \frac{n \ln(n)}{\ln(n/(n-1))}$, then we can deduce a_n.*

Proof. We will show that $n^k > (n-1)^k \cdot n!$ and thus $\left\lfloor \frac{P}{n^k} \right\rfloor = a_n$. Note that $k > \frac{n \ln(n)}{\ln(n/(n-1))} \iff k \ln \frac{n}{n-1} > n \ln(n) \implies k \ln \frac{n}{n-1} > \ln(n!) \iff k \ln n > k \ln(n-1) \ln(n!) \iff n^k > (n-1)^k \cdot n!$ ☐

Then our main result is as follows.

Theorem 3.3. *Given an instance, counting the number of* EFX *allocations is $\#P$-complete in general.*

Proof. We reduce from the $\#P$-complete problem of counting perfect matchings in a bipartite graph (Valiant 1979). The problem is as follows: given $G = (X \cup Y, E)$, with $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, count the number of perfect matchings in G.

First, if there exists some node with no neighbours, we can return 0 as there cannot be any perfect matching. Thus, we assume that all nodes in X contain at least 1 neighbour. Next, we construct an instance $\mathcal{I}$ with n agents and $n + k$ goods. Let $v_i(g_j) = 1$ if $j \leq n$ and $(x_i, y_j) \in E$, $v_i(g_j) = 0.5$ if $j \leq n$ and $(x_i, y_j) \notin E$, and $v_i(g_j) = 0$ otherwise. Then, our agents correspond to X, our first n goods correspond to Y and the remaining k goods are valued at 0.

Let $\mathcal{A}$ be the partial EFX allocation over the first n goods, and consider the envy graph induced by $\mathcal{A}$. Clearly, if any agent receives more than 1 good in $\mathcal{A}$, then $\mathcal{A}$ is not EFX and any complete allocation that extends $\mathcal{A}$ is also not EFX. If $\mathcal{A}$ corresponds to a perfect matching, then for all agents $i, j \in [n]$, we have that $v_i(A_i) = 1$ and $v_i(A_j) \leq 1$, and there is no envy in this case. If $\mathcal{A}$ does not correspond to a perfect matching, we will show that there must be an envied agent. Suppose $\mathcal{A}$ does not correspond to a perfect matching. Then there must exist an agent $i \in N$ such that $v_i(A_i) = \frac{1}{2}$. Furthermore, as each vertex in X has a neighbour, there is a $l \in [n]$ such that $(x_i, y_l) \in E$ and thus $v_i(y_l) = 1$. Suppose agent j receives good l under $\mathcal{A}$. Then $v_i(A_j) \geq 1$ and agent j is envied.

Thus, if $\mathcal{A}$ has no envied agents, then we can allocate the remaining k goods freely and still obtain an EFX allocation. This gives us n^k complete EFX allocations that has partial allocation $\mathcal{A}$ over the first n goods. Conversely, suppose agent i envies agent j under $\mathcal{A}$. Then, if we allocate one or more of the remaining goods to j, the resulting allocation cannot be EFX. After all, as all the remaining k goods are valued at 0, agent i still has a value of $\frac{1}{2}$ for their bundle and agent i has a value of 1 for agent j's bundle even after removing a good from it. Thus, with i envied agents, there are at most $(n-i)^k$ complete EFX allocations that has the partial allocation $\mathcal{A}$ over the first n goods.

Let a_i be the number of partial EFX allocations over the first n goods with i agents not envied in the envy graph. There are a total of $\sum_{i=0}^{n} a_i \cdot i^k$ complete EFX allocations. Since a_i is always non-negative and $\sum_{i=0}^{n} a_i < n!$, by our observation above, we can set k appropriately and deduce a_n. Moreover, as a_n is the number of partial EFX allocations over the first n goods such that all agents are unenvied, a_n is also the number of perfect matchings under G. Finally, when $y > 0$, we have that $\ln(1 + y) \geq y - 2y^2$, and $k \in \mathcal{O}(n^3)$. Our reduction can thus be done in polynomial time. □

Nevertheless, for the remainder of this section, we focus on restricted instances with few goods. We first note that the existence of EFX is only known for the case where $m \leq n + 3$. This was shown in a rather technical proof involving *champion graphs* and *lexicographic potential functions* (Mahara 2023). We posit that even when there are few goods, the existence of an EFX allocation is non-trivial; because if an EFX allocation always exists when $m = n + \lfloor f(n) \rfloor$ for any $f(n) \in \omega(1)$[3], then an EFX allocation always exists in general.

Theorem 3.4. *If there exists a function $f(n) \in \omega(1)$, such that there is always an* EFX *for any $m = n + \lfloor f(n) \rfloor$, then there is an* EFX *for every n, m.*

Next, we proceed to show our main results of this section, beginning with the simple cases when $m \leq n$ and when $m = n + 1$ where we show a tight bound. Note that the lower bounds apply for general monotone valuations, but the examples constructed for the upper bounds only assume additive valuations.

Proposition 3.5. *When $m \leq n$, every instance has at least $\frac{n!}{(n-m)!}$* EFX *allocations, and there exists an instance with at most $\frac{n!}{(n-m)!}$* EFX *allocations.*

Proof. For the lower bound, it is easy to see that any allocation where each agent receives at most one good is trivially EFX. There are $\binom{n}{m} \cdot m! = \frac{n!}{(n-m)!}$ such allocations where every agent receives at most one good.

For the upper bound, consider the case where every agent has a positive valuation for every good. In this case, it is necessary that all agents receive at most one good for the allocation to be EFX. □

Proposition 3.6. *When $m = n + 1$, every instance has at least n* EFX *allocations, and there exists an instance with at most n* EFX *allocations.*

Proof. We first prove the lower bound. Fix some arbitrary ordering of the agents, and label agents $1, \ldots, n$ in that order. Then, let each agent (in increasing order) pick their favourite good out of the remaining unallocated goods, and let last agent n get the remaining unallocated good. This yields an EFX allocation where every agent $i \in [n-1]$ receives one good and the agent n receives two goods. For all $i \in [n-1]$, letting agent i be the last agent in some ordering will yield an EFX allocation such that agent i receives two goods, and with any other agent $j \in [n] \setminus \{i\}$ receiving one good. Hence, there are at least n different EFX allocations.

For the upper bound, consider an instance where agents have valuations as follows: for each $i \in [n]$ and $j \in [n+1]$, $v_i(g_j) = 2$ if $i = j$, $v_i(g_j) = 1$ if $j = n + 1$, and $v_i(g_j) = 0$ otherwise. In all EFX allocations, for all agents $i \in [n]$, agent i will receive g_i. There are exactly n EFX allocations, differing only by which agent receives g_{n+1}. □

We now proceed to show an upper bound of n^2 and a lower bound of n for the case of $m = n+2$, with the following two theorems—the first showing the upper bound, and the next, which is more involved, showing the lower bound.

Theorem 3.7. *When $m = n+2$, there exist an instance with at most n^2* EFX *allocations.*

Proof. Consider an instance where agents have valuations over goods defined as follows: for each $i \in [n]$ and $j \in [n+2]$, $v_i(g_j) = 3$ if $i = j$, $v_i(g_j) = 1$ if $j \in \{n+1, n+2\}$, and $v_i(g_j) = 0$ otherwise. Suppose some agent $i \in [n]$ receives at least two goods (including good g_j, for some $j \neq i$). If $j \leq n$, then agent j will envy agent i even after removing the least-valued good (in j's view) from agent i's bundle (which will not be g_j), and hence the allocation is not EFX. Thus, without loss of generality, suppose $j = n + 1$, i.e., agent i receives g_{n+1}. We show that in every EFX allocation, for all agents $x \in [n]$, agent x must receive g_x. Suppose for a contradiction there exists an EFX allocation $\mathcal{A}$ and an agent $x \in [n]$ such that $g_x \notin A_x$. Let $g_x \in A_y$ for some other agent $y \in [n] \setminus \{x\}$. From our previous observation, $A_y = \{g_x\}$. This means $v_y(A_y) = 0$. However, we know that $g_{n+1} \in A_i$, and hence $v_y(A_i \setminus \{g\}) \geq 1$, contradicting the fact that $\mathcal{A}$ is an EFX allocation.

Thus, there are exactly n^2 EFX allocations differing only by which agent receives the goods g_{n+1} and g_{n+2}.

□

Theorem 3.8. *When $m = n+2$, every instance has at least n* EFX *allocations.*

We draw inspiration from the algorithms of Amanatidis, Markakis, and Ntokos (2020) that together imply the existence of EFX allocations when $m = n + 2$, with the following algorithm (Algorithm 1).

We then show that the above algorithm returns an EFX allocation. It is clear that the partial allocation at State 1 is EFX as agents 1 to $n-1$ are allocated only one good each, and they weakly prefer their allocated good to any good in P. Hence, after envy cycle elimination, the partial allocation at State 2 is also EFX, and there exists at least one non-envied agent (let it be agent i). If agent i has two goods (and

[3] A function $f(n)$ is in $\omega(1)$ if and only if $\lim_{n\to\infty} \frac{f(n)}{1} = \infty$.

Algorithm 1: Returns an EFX allocation when $m = n + 2$.

Input: Set of agents N, set of goods G, valuation profile $\mathbf{v} = (v_1, \dots, v_n)$.
Order the agents from 1 to n.
Initialize the partial allocation, $A \leftarrow (\emptyset \dots \emptyset)$
Initialize a set of unallocated goods, $P \leftarrow G$
for $i = 1, \dots, n-1$ **do**
  Let $g \in \arg\max_{g' \in P} v_i(g')$ be a good in P that agent i values the most
  Allocate the good g to agent i, $A_i \leftarrow \{g\}$
  Remove g from the set of unallocated goods, $P \leftarrow P \setminus \{g\}$
end for
Create *virtual goods* s, l such that $\forall i \in N$, $v_i(s) = \min_{g \in P} v_i(g)$ and $v_i(l) = v_i(P) - v_i(s)$
Allocate good l to agent n, $A_n \leftarrow \{l\}$ ▷ *State 1*
Perform envy cycle elimination and allocate s to the non-envied agent ▷ *State 2*
Allocate the final owner of l her two most valued goods in P, and the final owner of s the remaining good in P.

hence l), all agents will weakly prefer the good they are allocated over any two goods in P. Hence, no agent will envy agent i (after dropping their least-valued good from agent i's bundle) if they were to receive all goods in P. If i has one good g, all agents will weakly prefer their bundle over either g or s. Hence, no agent will envy agent i—after dropping their least-valued good from i's bundle—if she were to receive both g and s, and the allocation is EFX.

Next, we show—with the following lemma—an important structure of some EFX allocation.

Lemma 3.9. *When $m = n + 2$, for each agent $i \in N$, there exists an* EFX *allocation $\mathcal{A}$ whereby each agent receives at least one good, and either*

1. *i receives three goods; or*
2. *i and some agent $j \neq i$ each receives two goods, and one of $\{i, j\}$ receives their most valued good in $A_i \cup A_j$.*

We then introduce the concept of a *join-graph*, a directed graph where the vertices are the set of agents and the edges represents different EFX allocations.

Definition 3.10 (Join-graph)**.** A *join-graph* $G = (N, E)$ is a directed graph where an edge $(i, j) \in E$ if one of the following two criteria is satisfied.

- Criterion 1: $i \neq j$ and there is an EFX allocation $\mathcal{A}$ where
  - each agent receives at least one good;
  - agent i and j each receives two goods; and
  - agent j receives her favourite good in $A_i \cup A_j$.
- Criterion 2: $i = j$ and there is an EFX allocation $\mathcal{A}$ where
  - each agent receive at least one good; and
  - agent i receives three goods.

Then, we prove another lemma.

Lemma 3.11. *For the join-graph $G = (N, E)$ and any instance with $n + 2$ goods, there exists at least $|E|$* EFX *allocations.*

Then, our final lemma below proves our result.

Lemma 3.12. *For each connected component $G_x = (N_x, E_x)$ in the join-graph $G = (V, E)$, $|E_i| \geq |N_i|$. Hence, $|E| \geq |N|$ and there are at least n* EFX *allocations.*

Proof. Suppose there is a connected component $G_x = (N_x, E_x)$ such that $|E_x| < |N_x|$. First observe that in a such connected component G_x, if $|E_x| < |N_x|$ then there are no in cycles in G_x. This means that there are no self-loops, and thus no EFX allocation in which any agent in N_x receives three goods. There also must be a source agent i such that for all agents j, $(j, i) \notin E$. From Lemma 3.9, in an instance with $m = n + 2$, there is at least one incoming or outgoing edge for every agent in the join-graph. As i has no incoming edge in the join-graph, then there must be an agent j such that $(i, j) \in E$.

Let $\mathcal{A}$ be the EFX allocation that corresponds to (i, j) and let $A_i = \{x_1, x_2\}$ and $A_j = \{y_1, y_2\}$. Without loss of generality, let $v_j(y_1) \geq v_j(y_2)$. Also let $P = \{x_1, x_2, y_2\}$ and create virtual goods s, l such that $v_i(s) = \min_{g \in P} v_i(g)$ and $v_i(l) = v_i(P) - v_i(s)$. Consider the partial allocation $\mathcal{A}'$ where j receives only y_1, i receives l, and all other agents receives the same good as they did in $\mathcal{A}$. Note that $\mathcal{A}'$ is EFX. As $(i, i) \notin E$, there is an agent k that envies agent i under the partial allocation $\mathcal{A}'$; otherwise, we can allocate P to i and still get an EFX allocation.

Now, there exists a path in the envy graph for $\mathcal{A}'$ from k to a non-envied agent. Let this path be $y_0, \dots y_\alpha$ where $y_0 = k$, y_α is a non-envied agent and y_i envies y_{i-1} for all $i \in [\alpha]$. Note that if k is non-envied, $y_0 = y_\alpha = k$.

If there is a path from k to a non-envied agent other than i, let $A'_{y_\alpha} = \{z\}$. As y_α is non-envied, we can create an EFX allocation by allocating s to y_α. Then, since $(y_\alpha, i) \notin E$, i values z the most in $P \cup \{z\}$. Construct an allocation $\mathcal{B}$ by setting $B_{y_i} = A'_{y_{i-1}}$ for $i \in [\alpha]$, $B_i = \{s, z\}$, $B_k = l$ and allocating all other agents their good in A'. Observe that $\mathcal{B}$ is EFX and i receives their favourite good in $P \cup \{z\}$, $(k, i) \in E$, giving us a contradiction.

In the other case, the only path from k to a non-envied agent is a path to i. This is only possible if $y_{\alpha-1}$ is only envied by agent i. Hence, we can construct an allocation $\mathcal{B}$ by setting $B_{y_i} = A'_{y_{i-1}}$ for $i \in [\alpha - 1]$, $B_i = A'_{y_{\alpha-1}} \cup \{s\}$, $B_k = l$ and allocating all other agents their good in $\mathcal{A}'$. Observe that $\mathcal{B}$ is EFX and i receives their favourite good in $B_i \cup B_k$, $(k, i) \in E$, giving us a contradiction. ☐

To end this section, we show a tight bound for the special case of identical valuations.

Theorem 3.13. *When $m \geq n$, under identical valuations, every instance has at least $n!$* EFX *allocations, and there exists an instance with at most $n!$* EFX *allocations.*

Proof. For the lower bound, we know that there always exists an EFX allocation under identical valuations (Plaut and Roughgarden 2020). When $m \geq n$, there is always some EFX allocation $\mathcal{A}$ such that each agent receives at least one good. All allocations that are a permutation of $\mathcal{A}$ (i.e., bundles remains the same, but may be allocated to a different agent) also satisfies EFX, giving us at least $n!$ EFX allocations.

For the upper bound, consider the instance where $v(g_j) = 1$ if $j \leq n-1$ and $v(g_j) = 0$, otherwise. Then, there are only $n!$ EFX allocations. The proof of correctness for this upper bound is deferred to the full version of this paper. □

Importantly, we obtain the following corollary, contrasting the number of EFX allocations with the minimum number of EF1 allocations which is known to be exponential in m even for a fixed n (Suksompong 2020).

Corollary 3.14. *For any m, the number of* EFX *allocations can be as few as $n!$.*

Next, we consider two variants of EFX, and study similar questions. The first variant we will consider is an "up to any good" relaxation of *weighted envy-freeness (WEF)*.

4 Weighted EFX

In recent years, there has been a growing number of works in weighted fair division, an extension of the standard fair division model to one where agents have differing entitlements (Aziz, Moulin, and Sandomirskiy 2020; Babaioff, Ezra, and Feige 2021b; Babaioff, Nisan, and Talgam-Cohen 2021; Chakraborty et al. 2021; Chakraborty, Segal-Halevi, and Suksompong 2022; Farhadi et al. 2019; Hoefer, Schmalhofer, and Varricchio 2023; Scarlett, Teh, and Zick 2023). In this model, each agent $i \in N$ has an additional *weight* parameter $w_i > 0$ representing her entitlement.

This model would allow us to better capture settings such as inheritance division or divorce settlements where entitlements are typically unequal. Moreover, weighted fair division also extends the well-studied setting of *apportionment*, which is used in political systems to allocate parliamentary seats (Balinski and Young 2001; Pukelsheim 2014). Chakraborty et al. (2021) primarily studied the generalization of EF1 to *weighted EF1 (WEF1)*. Weighted EFX (WEFX), a natural generalization of EFX, has also been looked into, but there has been strong negative results.

In this section, we first show the existence of a WEFX and *Pareto-optimal (PO)* allocation under binary additive valuation for any number of agents, resolving an open question in the area. We also show that even in this restricted instance where WEFX allocations exists, there can be instances with only one WEFX allocation. In addition, we provide stronger negative results for the non-existence of WEFX, by further showing that a weaker version of WEFX (i.e., weak WEFX) also does not exist even in very restricted settings, thereby considerably tightening the existential gap known in the literature. We first state the definition of WEFX.[4]

Definition 4.1 (WEFX). An allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is *weighted envy-free up to any good (WEFX)* if for all agents $i, j \in N$ and all $g \in A_j$, $\frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j \setminus \{g\})}{w_j}$.

Hajiaghayi, Springer, and Yami (2023) showed an impossibility result—that an WEFX allocation may not exist even for $n = 2$ under additive valuations. An immediate question is then: are there any (more restricted) settings where a WEFX allocation exists? And if so, how many are there?

Indeed, we first show that in the restricted setting of binary additive valuations—which has also been well studied in the fair division literature (Aleksandrov et al. 2015; Amanatidis et al. 2021; Bouveret and Lemaître 2016; Freeman et al. 2019; Halpern et al. 2020; Suksompong and Teh 2022)—with any number of agents, not only does an WEFX allocation exist, but we can find an allocation satisfying both WEFX and PO. This combination is surprisingly elusive—a weighted generalization of *maximum Nash welfare* has been shown to not always satisfy WEF1 and PO under binary additive valuations.

We first state the definition of PO.

Definition 4.2 (PO). An allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is *Pareto-optimal (PO)* if there does not exist another allocation $\mathcal{A}'$ such that $v_i(A'_i) \geq v_i(A_i)$ for all $i \in N$, and $v_i(A'_i) > v_i(A_i)$ for some $i \in N$.

We show the existence of WEFX and PO allocations through two methods. Firstly, we show that by appropriately adapting the definiton of *leximin* to account for the weights, a weighted leximin allocation (that maximises a version of weighted egalitarian welfare function) is WEFX and PO. The benefit of this is that it is a *welfare function*, which is widely studied in the literature, and may potentially offer additional fairness or efficiency guarantees. We then provide a matching-based algorithm that can compute a WEFX and PO allocation in polynomial time.

Theorem 4.3. *Under binary additive valuations, for any number of agents, a* WEFX *and* PO *allocation exists and can be computed in polynomial-time.*

However, while an allocation satisfying WEFX exists in this restricted setting, there are not many of them, as illustrated with the following result.

Theorem 4.4. *Under binary additive valuations with $n = 2$,*

- *when $m \leq 2$, every instance has at least* 2 WEFX *allocations, and there exists an instance with at most* 2 WEFX *allocations;*
- *when $m \geq 3$, every instance has at least* 1 WEFX *allocations, and there exists an instance with at most* 1 WEFX *allocation.*

A follow-up question is: does the existence of WEFX allocation hold for valuation functions that are more general than binary additive functions? To address this, we consider two more general valuation classes commonly studied in the literature—namely *binary submodular* and *restricted additive* valuations.

A valuation profile $(v_1, \ldots, v_n)$ is said to be *binary submodular* (or *matroid-rank*) if for all $i \in N, g \in G$ and $S, T \subseteq G$, $v_i(g|S) = 0$ or $v_i(g|S) = 1$ and $v_i(S) + v_i(T) \geq v_i(S \cup T) + v_i(S \cap T)$ (Babaioff, Ezra, and Feige 2021a; Barman and Verma 2021; Benabbou et al. 2021; Goko et al. 2022; Montanari et al. 2024; Suksompong and Teh 2023); and *restricted additive* (or *generalized binary*) if $v_1, \ldots, v_n$ are additive and there exists a function $h : G \to \mathbb{R}_{\geq 0}$ such that $v_i(g) \in \{0, h(g)\}$ for all $i \in N$ and $g \in G$ (Akrami,

[4]Note that we are actually considering WEFX_0, a stronger variant of WEFX that can even drop zero-valued goods.

Rezvan, and Seddighin 2022; Camacho et al. 2023). We show that considering the above-defined generalizations of valuation functions beyond binary additive functions would yield negative results when $n = 2$, and not only for WEFX, but for a weaker version of WEFX, called *weak WEFX (WWEFX)*. This weaker relaxation has also been studied for WEF1 (Chakraborty et al. 2021; Chakraborty, Segal-Halevi, and Suksompong 2022).

Our results essentially close the (WEFX) existential gap of existing works that show a WEFX allocation may not exist for two agents with general, additive valuations (Hajiaghayi, Springer, and Yami 2023), and highlights that binary additive valuations may possibly be the best that one can hope for in terms of achieving WEFX (or even for WWEFX). We first formally define WWEFX.

Definition 4.5 (WWEFX)**.** An allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is *weak weighted envy-free up to any good (WWEFX)* if for all agents $i, j \in N$ and all $g \in A_j$, it holds that $\frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j \setminus \{g\})}{w_j}$ or $\frac{v_i(A_i \cup \{g\})}{w_i} \geq \frac{v_i(A_j)}{w_j}$.

Then, our (negative) results are as follows.

Proposition 4.6. *Under binary submodular valuations, a* WWEFX *allocation may not exist, even when $n = 2$.*

Proposition 4.7. *Under restricted additive valuations, a* WWEFX *allocation may not exist, even when $n = 2$.*

To end off this section, we show that a constant-factor approximation to WEFX when $n = 2$ exists. Formally, for $\alpha \in [0, 1]$, we say that an allocation $\mathcal{A}$ is α-WEFX if for all agents $i, j \in N$ and all $g \in A_j$, $\frac{v_i(A_i)}{w_i} \geq \alpha \cdot \frac{v_i(A_j \setminus \{g\})}{w_j}$.

Suksompong (2025) raised as an open question the existence of such a constant α, even for the case of $n = 2$. Here, we show the existence of a $\frac{1}{4}$-WEFX allocation, thereby improving upon the $\frac{w}{2\sqrt[3]{m}}$-WEFX result by Hajiaghayi, Springer, and Yami (2023), where w is the larger weight of the two agents. Our result is as follows.

Theorem 4.8. *When $n = 2$, there always exists a* $\frac{1}{4}$-WEFX *allocation.*

5 EFX+

In the standard definition of EFX (see Definition 2.2), a good is dropped from the right-hand side (i.e., the 'other agent's' bundle). One could also consider a variant whereby the good is added on the left-hand side instead (i.e., the 'evaluating agent's' bundle)—we call this *EFX+*. Under additive valuations, EFX+ is equivalent to EFX. However, under general monotone valuations, EFX+ and EFX are incomparable (i.e., one does not necessarily imply the other). Conceptual variants of this nature can be also found for envy-freeness properties studied in weighted fair division (Chakraborty et al. 2021; Chakraborty, Segal-Halevi, and Suksompong 2022; Montanari et al. 2024). It would thus be interesting to consider EFX+, which is defined as follows.

Definition 5.1 (EFX+)**.** An allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is *EFX+* if for all agents $i, j \in N$ and all $g \in A_j$, $v_i(A_i \cup \{g\}) \geq v_i(A_j)$.

We first give an example to illustrate the non-equivalence (and incomparability) of EFX and EFX+ under general monotone valuations. Consider an instance with $n = 2$ agents and $m = 3$ goods such that for both agents, $v(g_1) < v(g_2) < v(g_3) < v(g(\{g_2, g_3\}) < v(g(\{g_1, g_2\}) < v(g(\{g_1, g_3\})$. Then, the allocation that gives g_1 to an agent and g_2, g_3 to the other would satisfy EFX+ but not EFX; whereas the allocation that gives g_3 to an agent and g_1, g_2 to the other would satisfy EFX but not EFX+.

Next, we show that for two agents, EFX+ allocations exist, and that there is always a minimum of at least two EFX+ allocations. To prove this, we employ the 'cut-and-choose' protocol, where one agent partitions the set of goods into two 'almost equal' bundles, and the other agent chooses their favourite bundle. The non-trivial part of this protocol involves showing that the first agent always has a valid cut. In a slight variation of *leximin++* partial order (Plaut and Roughgarden 2020), we introduce the notion of *leximax*. Our result is as follows.

Theorem 5.2. *When $n = 2$, there are at least two* EFX+ *allocations.*

We note that the use of *leximax* can be used to show that EFX+ allocations also exists for any number of agents with identical valuations. While an EFX+ allocation always exists with two agents (or trivially when $m \leq n$), perhaps surprisingly (and in contrast to EFX), an EFX+ allocation may fail to exist with just three agents and four goods, as illustrated in the following result.

Proposition 5.3. *When $n = 3$ and $m = 4$, an* EFX+ *allocation may not exist.*

6 Conclusion

In this work, we investigate the minimum number of EFX allocations in various restricted instances, and with respect to WEFX and EFX+. For EFX, we obtain tight bounds in the easy cases when $m \leq n + 1$, and a lower bound of n and upper bound of n^2 allocations when $m = n + 2$. For the setting with $m = n+2$, we found instances where there are just n allocations when $n \leq 4$ (see Remark 1 in the full version of this paper). However, when $n = 5$, even after extensively generating random additive instances, our (empirical) upper bound on the number of EFX allocations is 14. Thus, we present the following conjecture.

Conjecture 6.1. *With n agents, there are at least n EFX allocations.*

We also showed the existence and polynomial time computability of WEFX and PO allocations for any number of agents under binary additive valuations, and a $\frac{1}{4}$-approximation to WEFX for the case of $n = 2$ agents, thereby resolving open problems in the area.

We also note that the negative results in the two variants considered highlights that EFX is surprisingly brittle. For instance, in Proposition 4.7, we observe that even in the case of $n = 2$ agents with slightly differing entitlements, and under restricted additive valuations, a WWEFX allocation may fail to exist. In Proposition 5.3, by slightly tweaking the definition of EFX to EFX+, an EFX+ allocation may not exist with just $n = 3$ agents and $m = 4$ goods.

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