Balanced and Fair Partitioning of Friends

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Abstract

In the recently introduced model of *fair partitioning of friends*, there is a set of agents located on the vertices of an underlying graph that indicates the friendships between the agents. The task is to partition the graph into k balanced-sized groups, keeping in mind that the value of an agent for a group is equal to the number of edges they have in that group. The goal is to construct partitions that are "*fair*", i.e., no agent would like to replace an agent in a different group. We generalize the standard model by considering utilities for the agents that are beyond binary and additive. Having this as our foundation, our contribution is threefold: (a) we adapt several fairness notions that have been developed in the fair division literature to our setting; (b) we give several existence guarantees supported by polynomial-time algorithms; (c) we initiate the study of the computational (and parameterized) complexity of the model and provide an almost complete landscape of the (in)tractability frontier for our fairness concepts.

1 Introduction

You are the coordinator of your organization's annual banquet, and your task is to allocate seats on tables for the employees. As you aim for perfection, you want to ensure that every employee believes that they are part of one of the "best" tables of the banquet. In other words, you want each employee to value the company of their table "almost" as much as the company they would get if they replace an employee at a different table. However, you have the following constraints: (a) there are only k tables whose overall size exactly fits the participants; (b) friendships between employees vary; (c) you want to be "fair" to every employee. So, what seat allocation would bring all employees to the tables?

Scenarios like the one above appear in several other domains where the goal is to partition a group of people into k groups of almost the same size: students for team projects, employees for training groups, kids for camp houses, or clustering in general. Recently, Li et al. (2023) introduced an elegant framework in order to formally study those situations. In their model, there is a *friendship graph* where every node corresponds to an agent and every edge corresponds to a friendship between the agents. The task is to create k groups of almost equal size such that the resulting grouping is "fair". They have focused on two fairness notions: a version of the *core*, refined for this model, and *almost envy-free* partitions, denoted EFr, which is most relevant to this work. In an EFr partition, no agent could increase the number of friends by at least r by replacing an agent from another group. Li et al. showed that for general friendship graphs and $r = \mathcal{O}(\sqrt{\frac{n}{k}\log k})$, where n is the number of agents and k is the number of parts, EFr partitions always exist and can be computed in polynomial time. They also showed that EF1 partitions do not always exist.

Although the model of Li et al. is elegant and sets the foundation of the framework, it implicitly assumes that: (a) every agent values their friends equally, i.e., each friend counts the same, and friendships are symmetric; (b) the valuations are additive. Furthermore, the computational complexity of finding fair partitions remained unexplored. As someone can imagine, the aforementioned assumptions can be rather restrictive for a variety of real-life situations. Can we augment the model in order to handle a wider variety of situations? Moreover, the factor $r = \mathcal{O}(\sqrt{\frac{n}{k}\log k})$ for EFr in general graphs could be prohibitive to convince the agents that the partition is fair. Having said this, the space between general graphs and more restricted graph classes is vast. Can one exploit the structure of the friendship graph and provide stronger fairness guarantees? The goal of the paper is to shed some light on these questions.

Our Contribution

Our initial contribution is the generalization of the model of (Li et al. 2023), which goes beyond binary, symmetric, and additive utilities and thus, it can capture more real-life scenarios. Having this as our foundation, our contribution is threefold: (a) we adapt several fairness notions that have been developed in the fair division literature to our setting, thus, we can choose the most appropriate one depending on the scenario; (b) we give several existence guarantees supported by polynomial-time algorithms; (c) we initiate the study of the computational (and parameterized) complexity of the model and provide an almost complete landscape of the (in)tractability frontier for our fairness concepts.

More specifically, in Section 3, we define the fairness notions for our model, and we provide a complete taxonomy for their interconnections. Next, in Section 4, we provide a thorough study of the computational complexity of deciding the existence of fair allocations. Unfortunately, the main takeaway message is negative: for every fairness notion we study in this work, deciding whether a fair partition exists is computationally intractable, even in very simple cases, e.g., the friendship graph is a path (Theorem 2), or it is bipartite, with vertex cover of size 2, and $k = 2$ (Theorems 3 to 5). On the positive side, we show that the problem becomes *fixed-parameter tractable*, for *all* fairness notions if we parameterize by the vertex cover number (Theorem 6). Note that the vertex cover number is a very natural parameter for our problem. In many scenarios similar to our initial motivational story, there is a small set of super-stars/influencers/politicians, and many people want to be close to them, but they do not know each other.

The bulk of our positive results is given in Section 5, which focuses on tree-like friendship graphs. The main message is that the existence of a fair allocation and its tractability essentially depends on the fairness concept we adopt. In fact, we provide a strong dichotomy. For half of the fairness notions we consider, we prove that they *always* admit a solution, even if the utilities are monotone. Actually, we provide algorithms that compute such allocations that become polynomial-time if the utilities are additive (Theorem 9). For the remaining notions, though, the problem is intractable even on trees with binary utilities (Theorem 7). On the positive side, we complement our negative results with a "semi"-efficient algorithm, i.e., with a polynomial-time algorithm when k is constant, that works for all solution concepts with binary utilities on friendship graphs of constant treewidth (Theorem 8) and consequently on trees. The algorithm is based on dynamic programming and is the best possible given our previous hardness results.

Related Work

Our work, perhaps surprisingly, builds upon the foundations of three different subfields of AI research: *fair division*, *coalition formation*, and *clustering*.

In the *fair division of indivisible items* model (Brams and Taylor 1996; Bouveret, Chevaleyre, and Maudet 2016; Amanatidis et al. 2023), we are usually given a set of indivisible items and a set of agents together with their preferences over the items, and the goal is to find an allocation of items to the agents that is fair with respect to some well-defined notion of fairness. The crucial difference from our model is that, in our case, the set of agents and the set of items coincide. Additionally, we are given a friendship graph that further restricts the agents' preferences. Few papers have been published on fair division in the presence of an additional social network (Aziz et al. 2018; Beynier et al. 2019; Bredereck, Kaczmarczyk, and Niedermeier 2022; Eiben et al. 2023). However, in these works, the social network restricts the communication between agents and does not encode their preferences as in our case. A different line of work on fair division also considered the presence of graphs (Bouveret et al. 2017; Christodoulou et al. 2023; Zhou et al. 2024); however, here, the edges correspond to items and do not encode the preferences.

Additively-Separable Hedonic Games (ASHGs) (Bogomolnaia and Jackson 2002) is an important special case of *hedonic games* (Drèze and Greenberg 1980; Aziz and Savani 2016), where we are given a set of agents, the utility each agent a receives from being in the same group as an agent b, and the task is to partition the agents into several groups, called coalitions, such that the outcome is stable. There are also several works that study fairness in the context of coalition formation (Aziz, Brandt, and Seedig 2013; Wright and Vorobeychik 2015; Peters 2016a,b; Ueda 2018; Barrot and Yokoo 2019; Kerkmann, Nguyen, and Rothe 2021), but none of these works requires coalitions of fixed size. Such settings, where either the number of coalitions or the coalitions' sizes are prescribed, were studied in recent works (Bilò, Monaco, and Moscardelli 2022; Sless et al. 2018; Cseh, Fleiner, and Harján 2019; Levinger, Azaria, and Hazon 2023). However, all of these papers suppose stability and not fairness as a solution concept. In this context, related are also *friend and enemies games* (Dimitrov et al. 2006; Ohta et al. 2017; Barrot et al. 2019; Flammini, Kodric, and Varricchio 2022; Skibski et al. 2022; Chen et al. 2023b), where the set of agents is given together with a friendship graph, where each edge represents that two agents are either friends or enemies. This is clearly different from our work, as we allow only friends, and, at the same time, the strength of friendships can be different for every pair of agents. Additionally, we require a fixed number of parts.

The third field where our results may find application is *clustering* (Jain 2010; Everitt et al. 2011) and specifically *data microaggregation* (Domingo-Ferrer 2009; Blažej et al. 2023). In related problems, the goal is usually to partition a set of data points into k *clusters* such that the clusters have some desirable properties that vary based on a particular application. The closest problem to our work is proportionally fair clustering (Chen et al. 2019; Micha and Shah 2020; Caragiannis, Micha, and Shah 2024). However, in contrast to our work, they do not suppose an underlying graph, and they require the solution to consist additionally of k representatives/centroids of every cluster.

Balanced partitioning of graphs is also a widely studied problem in graph theory. Arguably, the closest problem to what we study in this work is the so-called EQUITABLE CONNECTED PARTITION (ECP for short), where we are given a graph and a number of parts k, and the goal is to partition the graph into k connected subgraphs of almost equal sizes. This problem, heavily studied from the computational complexity perspective (Garey and Johnson 1979; Altman 1997; Enciso et al. 2009; Blažej et al. 2024), differs from ours in the sought solution concept; in ECP, all the parts need to be connected, while in our model, every solution partition needs to be fair, which does not imply connectedness. It should be pointed out that ECP has many applications, including redistricting theory (Williams Jr. 1995; Altman 1997; Landau, Reid, and Yershov 2009; Schutzman 2020; Ko et al. 2022; Chen et al. 2023a).

2 Preliminaries

By $\mathbb{N}$, we denote the set of positive integers $\{1, 2, 3, \ldots\}$. For every $i \in \mathbb{N}$, we let $[i] = \{1, \ldots, i\}$ and $[i]_0 = \{0\} \cup [i]$.

The input of our problem consists of a set $A = \{a_1, \dots, a_n\}$ of n *agents*. The relations between the agents are represented by an undirected graph $G = (A, E)$, called a *friendship graph*, where each edge $\{a, b\} \in E$ represents the friendship between agents a and b. Observe that the set of vertices and the set of agents coincide and that the friendship is always mutual. By $F(a) = \{b \mid \{a, b\} \in E\}$, we denote the set of *friends* of an agent $a \in A$. The number of friends, denoted by $\deg(a) = |F(a)|$, is called the *degree* of agent a. By Δ, we denote the maximum degree in our friendship graph, that is, $\Delta = \max_{a \in A} \deg(a)$. The *vertex cover number* of G, denoted by $\mathrm{vc}(G)$, is the size of the smallest vertex cover of G.

For every agent $a \in A$, we have a *utility function* $\mathrm{u}_a \colon A \to \mathbb{N} \cup \{0\}$ such that, for an agent $b \in A$, we have $\mathrm{u}_a(b) > 0$ if $b \in F(a)$ and 0 otherwise. If $\mathrm{u}_a(b) = \mathrm{u}_b(a)$ for every $a, b \in A$, we say that the utilities are *symmetric*. Additionally, if for every $b \in A$ it holds that $\mathrm{u}_a(b)$ is the same for every $a \in F(b)$, the utilities are called *objective*, and we write $\mathrm{u}_\forall(a) = x$ to denote that the utility of all $F(a)$ for a is the same $x > 0$. Finally, if $\mathrm{u}_a(b) \in \{0, 1\}$ for every $a, b \in A$, we say that the utilities are *binary*. Observe that binary utilities are both symmetric and objective.

Our goal is to find a *partition* of agents into $k \le n$ *groups* of almost the same sizes, which is fair with respect to some well-defined notion of fairness. A *k-partition* of A is a list $\pi = (\pi_1, \dots, \pi_k)$ such that $\bigcup_{i \in [k]} \pi_i = A$, $\pi_i \cap \pi_j = \emptyset$ for every pair of distinct $i, j \in [k]$, and no π_i is an empty set. Observe that a k-partition always exists. If k is clear from the context, we refer to a k-partition just as a partition. By $\pi(a)$, we denote the part in which agent $a \in A$ is placed in π. A k-partition π is called *balanced* if every pair of groups differs in their sizes by at most one. In other words, in a balanced partition we have $\lfloor \frac{n}{k} \rfloor \le \pi_i \le \lceil \frac{n}{k} \rceil$ for every $i \in [k]$. Otherwise, π is called *imbalanced*. Unless stated explicitly, we assume only balanced partitions. By Π_k, we denote the set of all (balanced) k-partitions.

To keep the notation clear, we extend utility functions from single agents to groups of agents. More specifically, let $S \subseteq A$ be a set of agents. Then, to denote the utility of an agent $a \in A$ in set S, we write $\mathrm{u}_a(S)$. A utility function is *additive*, if it holds that $\mathrm{u}_a(S) = \sum_{b \in S} \mathrm{u}_a(b)$. In some parts of the paper, we also study monotone utilities. A utility function is called *monotone*, if for every pair of sets S, T it holds that $S \subset T \implies \mathrm{u}(S) \le \mathrm{u}(T)$. Unless stated otherwise, we assume that the utilities under consideration are additive. Similarly, we set the utility of an agent $a \in A$ in a partition π to be $\mathrm{u}_a(\pi) = \mathrm{u}_a(\pi(a))$.

Parameterized Complexity. Parameterized complexity studies the complexity of a problem with respect to its input size, n, and the size of a parameter k. A problem is called *fixed-parameter tractable* (FPT) with respect to a parameter k if it can be solved in time $f(k) \cdot \mathrm{poly}(n)$, where f is a computable function. A less favorable, but still positive, outcome is a so-called XP *algorithm*, which has running-time $n^{\mathcal{O}(f(k))}$. Showing that a problem is W[1]-hard rules out the existence of a fixed-parameter algorithm under the well-established assumption that W[1] $\neq$ FPT. For a comprehensive introduction to parameterized complexity, we refer the interested reader to (Cygan et al. 2015).

3 Fairness Concepts

In this section, we formally define the studied notions of fairness and settle their basic properties and relationships. We mirror the respective definitions for the standard fair division of indivisible items (Brams and Taylor 1996; Bouveret, Chevaleyre, and Maudet 2016; Amanatidis et al. 2023).

The first notion of fairness is based on envy between pairs of agents and was already studied by Li et al. (2023).

Definition 1. *A k-partition π is called* envy-free (EF) *if, for every pair of agents $a, b \in A$,* $\mathrm{u}_a(\pi(a)) \ge \mathrm{u}_a(\pi(b) \setminus \{b\})$.

It is easy to see that envy-free partitions are not guaranteed to exist: assume an instance where the friendship graph is a path with three agents, binary utilities, and $k = 2$. If the smaller part consists of a degree-two agent, then this agent has envy towards any agent in the larger part. Similarly, if the smaller part consists of a degree-one agent, this agent is envious towards the other degree-one agent.

As illustrated by the previous example, envy-freeness is generally not guaranteed to exist, even in very simple instances. Therefore, we introduce several relaxations of EF, which are very popular and heavily studied in the standard fair division literature.

The first relaxation is inspired by the work of Plaut and Roughgarden (2020) and imposes that any envy between two agents can be eliminated by the removal of an arbitrary agent from the target part.

Definition 2. *A k-partition π is called* envy-free up to any agent (EFX_0) *if, for every pair of agents $a, b \in A$ and every agent $c \in \pi(b) \setminus \{b\}$ we have* $\mathrm{u}_a(\pi(a)) \ge \mathrm{u}_a(\pi(b) \setminus \{b, c\})$.

It is easy to observe that every EF partition is also EFX_0. To see that this is not the case in the opposite direction, recall the counterexample for the existence guarantee of EF; if we are allowed to remove any agent in the target part, we obtain that any 2-partition for this instance is EFX_0.

By the definition of EFX_0, the envy of a towards b should be eliminated by the removal of any *agent* c, even if this agent is not a friend of agent c. That is, $\mathrm{u}_a(c) = 0$. This may be counter-intuitive in some scenarios, and therefore, we also introduce a more restricted variant, which requires that the envy is eliminated by the removal of arbitrary *friend*. This notion of fairness is in line with the envy-freeness up to any good, which is due to Caragiannis et al. (2019).

Definition 3. *A k-partition π is called* envy-free up to any friend (EFX) *if, for every pair of agents $a, b \in A$ and every agent $c \in (\pi(b) \setminus \{b\}) \cap F(a)$ it holds that* $\mathrm{u}_a(\pi(a)) \ge \mathrm{u}_a(\pi(b) \setminus \{b, c\})$.

Both above-defined relaxations can also be understood such that the envy between two agents can be eliminated by removing a least-valued friend (or agent in the case of EFX_0) from the target part. We can give an even more relaxed envy-based notion that allows the removal of the *most valued* friend. This is an adaptation of EF1 from the standard fair division literature (Lipton et al. 2004; Budish 2011).

Definition 4. *A k-partition π is called* envy-free up to one agent (EF1) *if, for every pair of agents $a, b \in A$, there exists an agent $c \in \pi(b) \setminus \{b\}$ such that $\mathrm{u}_a(\pi(a)) \geq \mathrm{u}_a(\pi(b) \setminus \{b, c\})$.*

It was already proved by (Li et al. 2023) that, even if the utilities are binary, there are instances with no EF1 partition for any $k \geq 2$. Consequently, the same non-existence result also holds for EFX and EFX_0.

The fairness notions introduced so far were based on comparing the agent's current part with all other parts they can move to. The following set of fairness notions deviates from comparison with other agents and is based solely on certain threshold values that should be guaranteed for every agent in each fair outcome.

Definition 5. *For every agent $a \in A$, we set $\text{PROP-share}(a) = \mathrm{u}_a(F(a))/k$. A k-partition π is called* proportional (PROP) *if for every agent $a \in A$ it holds that $\mathrm{u}_a(\pi(a)) \geq \text{PROP-share}(a)$.*

It is again easy to see that proportional partitions are not guaranteed to exist: assume again the instance where the friendship graph is a path on three agents, utilities are binary, and $k = 2$. The PROP-share of the degree-two agent is exactly one, and therefore, this agent is necessarily in part with one of its friends. Consequently, the other friend is alone in the second part; the utility of this agent is clearly zero, but its PROP-share is $1/2$. Hence, no partition is proportional for this instance.

A more relaxed fairness notion compared to PROP is the maxi-min share, denoted MMS.

Definition 6. *For every agent $a \in A$, we set $\text{MMS-share}(a) = \max_{\pi' \in \Pi_k} \min_{i \in [k]} \mathrm{u}_a(\pi'_i)$. A k-partition π is called* maxi-min share (MMS) *if for every agent $a \in A$ it holds that $\mathrm{u}_a(\pi(a)) \geq \text{MMS-share(a)}$.*

In our first result, we show that MMS partitions are not guaranteed to exist for any number of parts $k \geq 2$.

Proposition 1. *For every $k \geq 2$, an MMS partition is not guaranteed to exist, even if the utilities are binary.*

On a positive note, there is a simple structural condition that guarantees the existence of MMS partitions. Informally, if the agents do not have too many friends compared to the number of parts, an MMS partition always exists and can be found efficiently.

Observation 1. *If $k > \Delta$, an MMS partition is guaranteed to exist and can be found in linear time.*

Verifying Fairness

Now, we discuss the complexity of verifying whether a given partition is fair with respect to each notion of interest. It is easy to see that, given a partition, it can be verified in polynomial time whether this partition is EF, EFX_0, EFX, or EF1. We just enumerate all pairs of agents and check whether the corresponding fairness criterion is violated or not. Similarly, it is easy to determine the PROP-share for each agent and check whether their utility in the given partition exceeds this value. For MMS, though, the situation is not as positive as for the other fairness notions we study.

Theorem 1. *It is* NP*-complete to decide whether the MMS-share of an agent $a \in A$ is at least a given $\sigma \in \mathbb{N}$, even if $k = 2$ and the utilities are symmetric.*

On the other hand, if we restrict ourselves to binary utilities, MMS-shares can be computed efficiently.

Observation 2. *If the utilities are binary, the MMS-share of any agent $a \in A$ can be computed in linear time.*

Relationships Between Studied Fairness Notions

The definitions of envy-based fairness concepts clearly indicate the relationships between them: EF $\Rightarrow$ EFX_0, EFX_0 $\Rightarrow$ EFX, and EFX $\Rightarrow$ EF1, and that none of the implications is actually an equivalence. On the other hand, though, the relationship between our two share-based fairness concepts and their connections with envy-based notions is not straightforward. The rest of the section is devoted to clarifying these relationships, which are concisely depicted in Figure 1.

First, we show that the existence of an EF partition does not imply the existence of a PROP partition.

Proposition 2. *For every $k \geq 2$, there is an instance that admits an EF k-partition and no PROP k-partition.*

Next, we show that, while for $k = 2$ every PROP partition (if it exists) is also an EF, this no longer holds when $k \geq 3$.

Proposition 3. *If $k = 2$, every PROP partition is also EF. For every $k \geq 3$, there is a PROP partition that is not EF, even if the utilities are binary.*

Then, we study the connection between PROP and MMS.

Proposition 4. *For every $k \geq 2$, every PROP k-partition is MMS. Also, for every $k \geq 2$, there are k-partitions that are MMS and not PROP, even if the utilities are binary.*

Next, we show the analog of Proposition 3, this time though for MMS and EF1.

Proposition 5. *If $k = 2$, every MMS partition is also EF1. For every $k \geq 3$, there is an MMS k-partition which is not EF1, even if the utilities are binary.*

Our last observation concludes the section and states that EF1 does not imply MMS.

Observation 3. *For every $k \geq 2$, there exists an instance that admits an EF1 partition and no MMS partition, even if the utilities are binary.*

4 Algorithms and Complexity

In this section, we provide the complexity landscape for the fair partitioning of friends with respect to all of the above-defined notions of fairness. A basic overview of our results is provided in Figure 1.

We start with the case of the most general fairness notions, which are envy-freeness and proportionality. Specifically, we show that in this setting, even if the friendship graph is as simple as a path, the problem is intractable.

Theorem 2. *It is* NP*-complete and* W[1]*-hard when parameterized by the number of parts k to decide whether an EF, EFX_0, or PROP allocation exists, even if G is a path and the utilities are objective. For EF and EFX_0, the hardness holds even if the utilities are encoded in unary.*

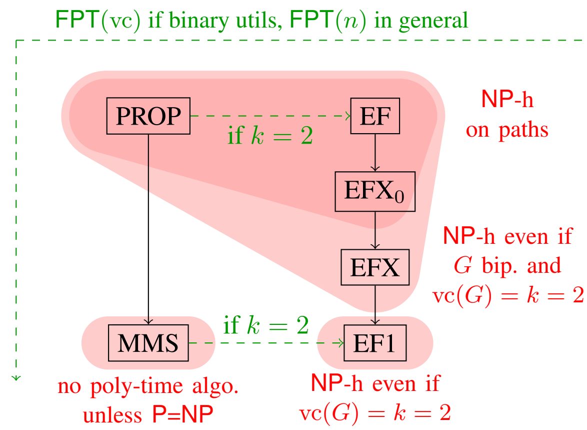

Figure 1: A basic overview of our algorithmic and complexity results for unrestricted friendship graphs. An arrow from notion A to B denotes that if a partition is fair with respect to A, then it is also fair with respect to B. Here, k is the number of parts, n is the number of agents, and vc denotes the vertex cover number of the friendship graph. By $\mathsf{FPT}(x)$ we mean that deciding whether fair partition exists is in FPT with respect to the parameter x.

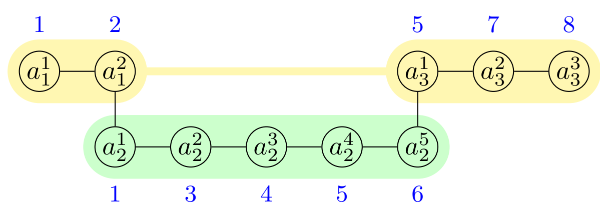

Figure 2: An illustration of the construction used to prove Theorem 2. In this example, we assume an instance of UNARY BIN PACKING with $S = (2, 5, 3)$, $B = 2$, and $c = 5$. Next to each agent a_i^j, we depict (in blue) the value of $\text{u}_\forall(a_i^j)$. With differently colored backgrounds, we highlight one possible balanced 2-partition for this instance.

Proof Sketch. We prove the theorem by a reduction from the UNARY BIN PACKING problem. In this problem, we are given a multi-set $S = (s_1, \dots, s_N)$ of positive integers, a number of bins B, and a capacity c of every bin. Without loss of generality, we can assume that $\sum_{i \in [N]} s_i = c \cdot B$. The goal is to find an allocation $\alpha\colon S \to [B]$ such that for every $i \in [B]$ it holds that $\sum_{j\colon \alpha(s_j)=i} s_j = c$. The UNARY BIN PACKING problem is known to be NP-complete and W[1]-hard when parameterized by the number of bins B, even if all the integers are encoded in unary, $c \geq 4$, $B \geq 3$, and every $s_i \in S$ is at least two (Jansen et al. 2013).

We describe our reduction for EF. Given an instance $\mathcal{I} = (S, B, c)$, we create an equivalent instance $\mathcal{J}$ of our problem as follows. For every element $s_i \in S$, we introduce a set A_i of s_i agents $a_i^1, \dots, a_i^{s_i}$ and connect each two consecutive agents; that is, we add an edge $\{a_i^j, a_i^{j+1}\}$ for every $j \in [s_i - 1]$. It is easy to see that these agents induce a path. To finalize the construction of G, we add an edge $\{a_i^{s_i}, a_{i+1}^1\}$ for every $i \in [N-1]$. Observe that G is a single path with exactly $c \cdot B$ agents. Next, we define the utilities. First, we set $\text{u}_\forall(a_1^i) = i$ for every $i \in [s_1]$. For every $i \in [2, N]$, we define the utilities recursively as follows

$$\text{u}_\forall(a_i^j) = \begin{cases} \text{u}_\forall(a_{i-1}^{s_{i-1}-1}) & \text{if } j = 1, \\ \text{u}_\forall(a_i^{j-1}) + 2 & \text{if } j = 2, \text{ and} \\ \text{u}_\forall(a_i^{j-1}) + 1 & \text{otherwise.} \end{cases}$$

Figure 2 illustrates our construction. The basic idea behind the construction is that once an agent $a_i^{s_i}$, $i \in [N]$, is assigned to some part π_j in a partition π, all a_i^j, $j \in [s_i - 1]$ are necessarily assigned to the same part, as otherwise, π is not EF. To complete the construction, we set $k = B$. □

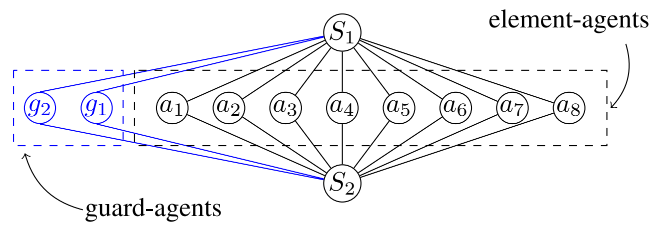

Figure 3: An illustration of the construction used in Theorem 3. For guard-agents, which are depicted in blue, we have $\text{u}_\forall(g_1) = \text{u}_\forall(g_2) = 1$, and $\text{u}_\forall(S_1) = \text{u}_\forall(S_2) = B$, where B is half of the sum of all integers given in an original instance of the EQUITABLE PARTITION problem. The element-agents are in one-to-one correspondence with integers of the EQUITABLE PARTITION problem.

If we move away from paths, then our problem becomes intractable even for $k = 2$, and the friendship graph is bipartite with vertex cover number 2. Figure 3 depicts the graph we create; we reduce from EQUITABLE PARTITION, which can be thought as a BIN PACKING problem with two bins.

Theorem 3. *It is NP-complete to decide whether a PROP, EF, EFX_0, or EFX partition exists, even if $k = 2$, G is a bipartite graph, $\text{vc}(G) = 2$, and the utilities are objective.*

While Theorem 1 shows that we cannot compute in polynomial time the MMS-share of an agent, it does not exclude the existence of polynomial-time algorithms for finding an MMS allocation. In fact, for the instance constructed in Theorem 1, where the friendship graph was a star, we could efficiently compute a solution; see Proposition 6 for formal argument. However, a polynomial-time algorithm is unlikely to exist in general.

Theorem 4. *Unless P = NP, there is no polynomial-time algorithm that computes an MMS allocation, if it exists, even if G is a bipartite graph and $\text{vc}(G) = 2$.*

Next, we establish hardness for EF1 as well by slightly modifying the construction from Figure 3.

Theorem 5. *It is NP-complete to decide whether an EF1 partition exists, even if $\text{vc}(G) = 2$ and $k = 2$.*

Maybe surprisingly, our above hardness results are tight. Specifically, if we restrict the friendship graph even more, we show that for some of the studied fairness notions, a fair partition is guaranteed to exist. For those where such a guarantee is not possible, we show a simple condition that allows us to decide such instances in linear time. The result also contrasts Theorem 1 in the sense that under this restriction, verifying whether a given partition is MMS is computationally hard, but, at the same time, it is easy to find one.

Proposition 6. *If the friendship graph is of vertex cover number* 1*, a* ϕ *partition is guaranteed to exist and can be found in linear time for every* $\phi \in \{EFX, EF1, MMS\}$*. Under the same restriction, the existence of EF, EFX*$_0$*, or PROP partition can be decided in linear time.*

As our previous results clearly indicate, if we allow for arbitrary utilities, then, under the standard theoretical assumptions, there is no hope for tractable algorithms, even for graphs of constant vertex cover number. In the following, we show that if we additionally restrict the utilities, the situation is much more positive.

The proof of Theorem 6 is rather technical and involves two levels of guessing combined with an ILP formulation. First, it guesses what the solution looks like on the vertices of a vertex cover. Then, it guesses the neighborhood structure, and finally, it tries to verify the fairness for the independent vertices.

Theorem 6. *If the utilities are binary, then for every notion of fairness* $\phi \in \{EF, EFX_0, EFX, EF1, PROP, MMS\}$*, there is an* FPT *algorithm parameterized by the vertex cover number* vc *that decides whether a partition which is fair with respect to* ϕ *exists.*

The last result of this section is an algorithm FPT for parameterization by the number of agents. Although the algorithm is not complicated, it highlights the difference between our model and the setting of standard fair division of indivisible items, where the existence of fair outcomes may be algorithmically challenging already for a constant number of agents (Berger et al. 2022; Ghosal et al. 2023; Chaudhury, Garg, and Mehlhorn 2024; Deligkas et al. 2024).

Observation 4. *For every fairness notion* $\phi \in \{EF, EFX_0, EFX, EF1, PROP, MMS\}$*, the problem of deciding whether a* ϕ*-fair partition exists is in* FPT *when parameterized by the number of agents* n*.*

5 Tree-Like Friendship Graph

As is clear from the results of the previous section, without an additional restriction of the input instances, we are not able to guarantee any existential result or positive algorithm. Therefore, in this section, we focus on the natural restriction of the underlying friendship graph; specifically, we focus on situations where the social friendship graph is similar to a tree. It should be noted that such restriction is heavily studied in the related literature; see, e.g., (Bouveret et al. 2017; Farhadi et al. 2019; Hanaka and Lampis 2022; Xiao, Qiu, and Huang 2023; Hosseini, Narang, and Was 2024).

In Theorem 2, we showed that the situation is, from the computational complexity perspective, hopeless if we require EF, EFX$_0$, or PROP partition and the utilities can be arbitrary, even if the friendship graph is a path. On the other hand, if the utilities are binary, we can decide the existence of fair partitions in polynomial time.

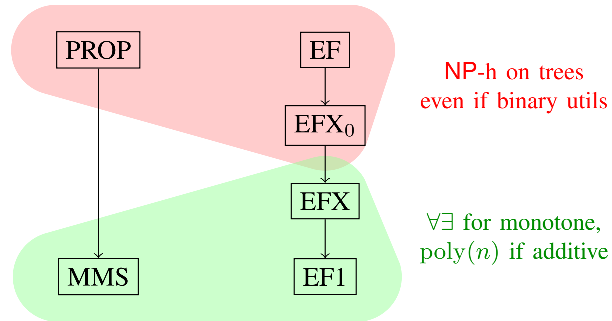

Figure 4: A basic overview of our complexity results for trees.

Proposition 7. *If the utilities are binary and* G *is a path, there is a linear-time algorithm that decides whether an EF or PROP partition exists. In the same setting, EFX*$_0$ *partitions are guaranteed to exist and can be found in linear time.*

Then, a natural question arises. Can we generalize the result from Proposition 7 to a more general class of graphs? In the following result, we show that this is not the case. Specifically, we show that it is computationally hard to decide whether an EF or PROP partition exists for trees.

Theorem 7. *It is* NP*-complete and* W[1]*-hard when parameterized by the number of parts* k *to decide whether PROP, EF, or EFX*$_0$ *partition exists, even if the utilities are binary and* G *is a tree.*

Proof Sketch. Again, we reduce from UNARY BIN PACKING, and we provide a parameterized reduction. The idea behind the construction is to create a gadget for each item $s_i \in S$ such that the whole gadget is part of the same part in every solution. For connectivity reasons, we add one connecting gadget that connects all these gadgets and, in every solution, forms one extra part. Formally, given an instance $\mathcal{I} = (S, B, c)$ of the UNARY BIN PACKING problem, we create an equivalent instance of our problem as follows. For every item $s_i \in S$, we create an *item-gadget* X_i, which is a star with center c_i and $s_i - 1$ leaves $v_1^i, \ldots, v_{s_i-1}^i$. Additionally, we add one *connecting-gadget* X_g, which is again a star with center c^* and $c - 1$ leaves $v_1^*, \ldots, v_{c-1}^*$. Moreover, there is an edge $\{c_i, c^*\}$ for every $i \in [N]$. To finalize the construction, we set $k = B + 1$, and we recall that the utilities are binary. Figure 5 depicts the construction. ☐

Observe that the tree constructed in the proof of Theorem 7 is very shallow. In particular, it can be shown that these trees have constant treedepth (3, to be precise), which shows a strong intractability result for this particular structural restriction and, therefore, also for more general parameters such as the celebrated tree-width of the friendship graph.

Our following result complements the lower-bound given in Theorem 7. Specifically, we show that if G is a tree, then

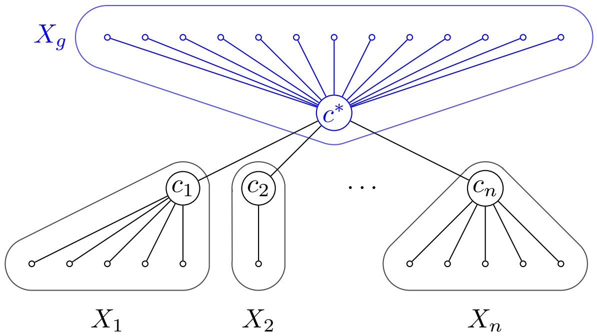

Figure 5: An illustration of the construction from the proof of Theorem 7. In blue, we highlight the connecting gadget X_g with $c-1$ leaves. Item-gadgets X_i, $i \in [N]$, are in one-to-one correspondence with the elements of S.

for every constant k, it can be decided whether a fair partition exists in polynomial time. In fact, the algorithm is even more general and shows XP-tractability when parameterized by the treewidth of the friendship graph and the number of parts k, combined. Informally, the tree-width specifies how similar a graph is to a tree; see (Cygan et al. 2015, Chapter 7) for a formal introduction to the topic.

Theorem 8. *If the utilities are binary and G is a graph of tree-width* tw*, then for every fairness notion $\phi \in \{EF, EFX_0, EFX, EF1, PROP, MMS\}$, there is an XP algorithm parameterized by k and* tw *that decides whether a partition π, which is fair with respect to ϕ, exists.*

It is not hard to see that our above algorithms almost directly work for any symmetric and objective utilities – not necessarily binary. In fact, with a little bit more of effort, the algorithm can be generalized to settings where the co-domain of every utility function is of constant size.

None of the previous results rule out better algorithms for weaker fairness notions such as EFX, EF1, or MMS. The next set of results shows strong existence guarantees and tractable algorithms that can find a fair outcome for these fairness notions, even if we assume the most general types of valuations; namely, we show that fair outcomes are guaranteed to exist for arbitrary *monotone* utilities.

We give an algorithm that outputs an MMS and EFX (end hence EF1) k-partition for any monotone utility function. In addition, if the utilities are additive, this algorithm can be implemented in linear time.

Theorem 9. *If G is a forest and $k \in \mathbb{N}$, there always exists a k-partition that is MMS and EFX, even if the utility functions are monotone. Moreover, if the utilities are additive, we can find such a partition in linear time.*

Proof Sketch. The main idea of the algorithm is to root each tree in the forest in some arbitrary vertex and then process the agents in a BFS order. When processing an agent a, we are computing a "preliminary" assignment of all the children of the agent and finalizing the assignment of a to the parts of π. Loosely speaking, we first determine the list $\vec{\ell} = (\ell_1, \ldots, \ell_k)$, where ℓ_i, $i \in [k]$, determines how many of $|\text{children}(a)|$ many newly assigned agents should be assigned to π_i in order to keep the parts always balanced. Note that, since we are keeping "being balanced" as an invariant, then $|\ell_i - \ell_j| \leq 1$ for all $i, j \in [k]$. Afterward, a decides on (1) whether to stay in the same part of π or change to $\pi(p_a)$, where p_a is the parent of a (if this change is allowed by $\vec{\ell}$), and (2) how to distribute the children among the parts according to $\vec{\ell}$. If a was originally in π_j and chooses to change to $\pi_i = \pi(p_a)$, then to preserve balancedness, we have to add $\ell_i - 1$ children of a to π_i and $\ell_j + 1$ children of a to π_j. Since (a) the number of neighbors of a in each part is balanced, (b) agent a is choosing which children are in the same part with it, and (c) agent a is allowed to change to the part of its parent, it is not too difficult to show that this partial assignment is MMS and EFX for agent a. From this point onward, the fairness for the agent a is guaranteed: any change between parts for the children of a can only increase the utility of a. Hence, MMS and EFX for a are preserved. ☐

6 Discussion

Our results indicate that the problem of fair partitioning of friends heavily depends on the friendship graph and the fairness concept we adopt. Although we have provided a rather complete picture of the complexity of the problem, we believe that there are several directions for future work. Firstly, are there any other graph classes that allow the problem to be tractable? Our preliminary results indicate that grid-graphs are such a class. What if friendship graphs are directed? Some of our results apply here, but there exist unsettled questions that deserve to be studied. Does the presence of enemies, i.e., agents get negative utilities, drastically change the landscape of the problem?

Of course, someone can wonder if the balance constraint is required to obtain our results. As we prove, this is not always the case: our algorithm from Theorem 9 can be extended for any fixed list of group sizes. We note that such a model was studied by Bilò, Monaco, and Moscardelli (2022); however, their solution concept was swap-stability.

Theorem 10. *If G is a forest and we are given the desired sizes for each part, an MMS and EFX partition is guaranteed to exist, even for monotone utilities. If the utilities are additive, such a partition can be found in polynomial time.*

Finally, we should highlight that other fairness notions exist, even more relaxed than ours, whose existence of fair partition remains unknown: PROP1 and EFr, for $r \geq 2$, or approximations of MMS are among the best candidates.

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References

Altman, M. 1997. The Computational Complexity of Automated Redistricting: Is Automation the Answer? *Rutgers Computer and Law Technology Journal*, 23(1): 81–142.

Amanatidis, G.; Aziz, H.; Birmpas, G.; Filos-Ratsikas, A.; Li, B.; Moulin, H.; Voudouris, A. A.; and Wu, X. 2023. Fair division of indivisible goods: Recent progress and open questions. *Artificial Intelligence*, 322: 103965.

Aziz, H.; Bouveret, S.; Caragiannis, I.; Giagkousi, I.; and Lang, J. 2018. Knowledge, Fairness, and Social Constraints. In *Proc. of the 32nd AAAI Conference on Artificial Intelligence, AAAI '18*, 4638–4645.

Aziz, H.; Brandt, F.; and Seedig, H. G. 2013. Computing desirable partitions in additively separable hedonic games. *Artificial Intelligence*, 195: 316–334.

Aziz, H.; and Savani, R. 2016. Hedonic Games. In Brandt, F.; Conitzer, V.; Endriss, U.; Lang, J.; and Procaccia, A. D., eds., *Handbook of Computational Social Choice*, 356–376. Cambridge University Press.

Barrot, N.; Ota, K.; Sakurai, Y.; and Yokoo, M. 2019. Unknown Agents in Friends Oriented Hedonic Games: Stability and Complexity. In *Proc. of the 33rd AAAI Conference on Artificial Intelligence, AAAI '19*, 1756–1763.

Barrot, N.; and Yokoo, M. 2019. Stable and Envy-free Partitions in Hedonic Games. In *Proc. of the 28th International Joint Conference on Artificial Intelligence, IJCAI '19*, 67–73.

Berger, B.; Cohen, A.; Feldman, M.; and Fiat, A. 2022. Almost Full EFX Exists for Four Agents. In *Proc. of the 36th AAAI Conference on Artificial Intelligence, AAAI '22*, 4826–4833.

Beynier, A.; Chevaleyre, Y.; Gourvès, L.; Harutyunyan, A.; Lesca, J.; Maudet, N.; and Wilczynski, A. 2019. Local envy-freeness in house allocation problems. *Autonomous Agents and Multi-Agent Systems*, 33(5): 591–627.

Bilò, V.; Monaco, G.; and Moscardelli, L. 2022. Hedonic Games with Fixed-Size Coalitions. In *Proc. of the 36th AAAI Conference on Artificial Intelligence, AAAI '22*, 9287–9295.

Blažej, V.; Ganian, R.; Knop, D.; Pokorný, J.; Schierreich, Š.; and Simonov, K. 2023. The Parameterized Complexity of Network Microaggregation. In *Proc. of the 37th AAAI Conference on Artificial Intelligence, AAAI '23*, 6262–6270.

Blažej, V.; Knop, D.; Pokorný, J.; and Schierreich, Š. 2024. Equitable Connected Partition and Structural Parameters Revisited: N-Fold Beats Lenstra. In *Proc. of the 49th International Symposium on Mathematical Foundations of Computer Science, MFCS '24*, volume 306 of *LIPIcs*, 29:1–29:16.

Bogomolnaia, A.; and Jackson, M. O. 2002. The Stability of Hedonic Coalition Structures. *Games and Economic Behavior*, 38(2): 201–230.

Bouveret, S.; Cechlárová, K.; Elkind, E.; Igarashi, A.; and Peters, D. 2017. Fair Division of a Graph. In *Proc. of the 26th International Joint Conference on Artificial Intelligence, IJCAI '17*, 135–141.

Bouveret, S.; Chevaleyre, Y.; and Maudet, N. 2016. Fair Allocation of Indivisible Goods. In Brandt, F.; Conitzer, V.; Endriss, U.; Lang, J.; and Procaccia, A. D., eds., *Handbook of Computational Social Choice*, 284–310. Cambridge University Press.

Brams, S. J.; and Taylor, A. D. 1996. *Fair division - from cake-cutting to dispute resolution*. Cambridge University Press. ISBN 978-0-521-55644-6.

Bredereck, R.; Kaczmarczyk, A.; and Niedermeier, R. 2022. Envy-free allocations respecting social networks. *Artificial Intelligence*, 305: 103664.

Budish, E. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6): 1061–1103.

Caragiannis, I.; Kurokawa, D.; Moulin, H.; Procaccia, A. D.; Shah, N.; and Wang, J. 2019. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation*, 7(3): 1–32.

Caragiannis, I.; Micha, E.; and Shah, N. 2024. Proportional Fairness in Non-Centroid Clustering. In *Proc. of the 38th Annual Conference on Neural Information Processing Systems, NeurIPS '24*.

Chaudhury, B. R.; Garg, J.; and Mehlhorn, K. 2024. EFX Exists for Three Agents. *Journal of the ACM*, 71(1): 4:1–4:27.

Chen, F.; Biswas, S.; Chen, Z.; Lei, S.; Ramakrishnan, N.; and Lu, C. 2023a. Exploring Tradeoffs in Automated School Redistricting: Computational and Ethical Perspectives. In *Proc. of the 37th AAAI Conference on Artificial Intelligence, AAAI '23*, 15912–15920.

Chen, J.; Csáji, G.; Roy, S.; and Simola, S. 2023b. Hedonic Games With Friends, Enemies, and Neutrals: Resolving Open Questions and Fine-Grained Complexity. In *Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems, AAMAS '23*, 251–259.

Chen, X.; Fain, B.; Lyu, L.; and Munagala, K. 2019. Proportionally Fair Clustering. In *Proc. of the 36th International Conference on Machine Learning, ICML '19*, volume 97 of *Proceedings of Machine Learning Research*, 1032–1041.

Christodoulou, G.; Fiat, A.; Koutsoupias, E.; and Sgouritsa, A. 2023. Fair allocation in graphs. In *Proc. of the 24th ACM Conference on Economics and Computation, EC '23*, 473–488.

Cseh, Á.; Fleiner, T.; and Harján, P. 2019. Pareto Optimal Coalitions of Fixed Size. *Journal of Mechanism and Institution Design*, 4(1): 87–108.

Cygan, M.; Fomin, F. V.; Kowalik, L.; Lokshtanov, D.; Marx, D.; Pilipczuk, M.; Pilipczuk, M.; and Saurabh, S. 2015. *Parameterized Algorithms*. Springer. ISBN 978-3-319-21274-6.

Deligkas, A.; Eiben, E.; Korchemna, V.; and Schierreich, Š. 2024. The Complexity of Fair Division of Indivisible Items with Externalities. In *Proc. of the 38th AAAI Conference on Artificial Intelligence, AAAI '24*, 9653–9661.

Dimitrov, D.; Borm, P.; Hendrickx, R.; and Sung, S. C. 2006. Simple Priorities and Core Stability in Hedonic Games. *Social Choice and Welfare*, 26(2): 421–433.

Domingo-Ferrer, J. 2009. Microaggregation. In Liu, L.; and Özsu, M. T., eds., *Encyclopedia of Database Systems*, 1736–1737. Boston, MA: Springer. ISBN 978-0-387-39940-9.

Drèze, J. H.; and Greenberg, J. 1980. Hedonic coalitions: Optimality and stability. *Econometrica*, 48(4): 987–1003.

Eiben, E.; Ganian, R.; Hamm, T.; and Ordyniak, S. 2023. Parameterized complexity of envy-free resource allocation in social networks. *Artificial Intelligence*, 315: 103826.

Enciso, R.; Fellows, M. R.; Guo, J.; Kanj, I. A.; Rosamond, F. A.; and Suchý, O. 2009. What Makes Equitable Connected Partition Easy. In *Proc. of the 4th International Workshop on Parameterized and Exact Computation, IWPEC '09*, volume 5917 of *LNCS*, 122–133.

Everitt, B. S.; Landau, S.; Leese, M.; and Stahl, D. 2011. *Cluster Analysis*. Wiley, 5th edition. ISBN 978-0-470-74991-3.

Farhadi, A.; Ghodsi, M.; Hajiaghayi, M. T.; Lahaie, S.; Pennock, D. M.; Seddighin, M.; Seddighin, S.; and Yami, H. 2019. Fair Allocation of Indivisible Goods to Asymmetric Agents. *Journal of Artificial Intelligence Research*, 64: 1–20.

Flammini, M.; Kodric, B.; and Varricchio, G. 2022. Strategyproof mechanisms for Friends and Enemies Games. *Artificial Intelligence*, 302: 103610.

Garey, M. R.; and Johnson, D. S. 1979. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman. ISBN 0-7167-1044-7.

Ghosal, P.; HV, V. P.; Nimbhorkar, P.; and Varma, N. 2023. EFX Exists for Four Agents with Three Types of Valuations. *CoRR*, abs/2301.10632.

Hanaka, T.; and Lampis, M. 2022. Hedonic Games and Treewidth Revisited. In *Proc. of the 30th Annual European Symposium on Algorithms, ESA '22*, volume 244 of *LIPIcs*, 64:1–64:16.

Hosseini, H.; Narang, S.; and Was, T. 2024. Fair Distribution of Delivery Orders. In *Proc. of the 33rd International Joint Conference on Artificial Intelligence, IJCAI '24*, 2825–2833.

Jain, A. K. 2010. Data clustering: 50 years beyond K-means. *Pattern Recognition Letters*, 31(8): 651–666.

Jansen, K.; Kratsch, S.; Marx, D.; and Schlotter, I. 2013. Bin packing with fixed number of bins revisited. *Journal of Computer and System Sciences*, 79(1): 39–49.

Kerkmann, A. M.; Nguyen, N.; and Rothe, J. 2021. Local fairness in hedonic games via individual threshold coalitions. *Theoretical Computer Science*, 877: 1–17.

Ko, S.; Taylor, E.; Agarwal, P. K.; and Munagala, K. 2022. All Politics is Local: Redistricting via Local Fairness. In *Proc. of the 35th Annual Conference on Neural Information Processing Systems, NeurIPS '22*.

Landau, Z.; Reid, O.; and Yershov, I. 2009. A fair division solution to the problem of redistricting. *Social Choice and Welfare*, 32(3): 479–492.

Levinger, C.; Azaria, A.; and Hazon, N. 2023. Social Aware Coalition Formation with Bounded Coalition Size. In *Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems, AAMAS '23*, 2667–2669.

Li, L.; Micha, E.; Nikolov, A.; and Shah, N. 2023. Partitioning Friends Fairly. In *Proc. of the 37th AAAI Conference on Artificial Intelligence, AAAI '23*, 5747–5754.

Lipton, R. J.; Markakis, E.; Mossel, E.; and Saberi, A. 2004. On approximately fair allocations of indivisible goods. In *Proc. of the 5th ACM Conference on Electronic Commerce, EC '04*, 125–131.

Micha, E.; and Shah, N. 2020. Proportionally Fair Clustering Revisited. In *Proc. of the 47th International Colloquium on Automata, Languages, and Programming, ICALP '20*, volume 168 of *LIPIcs*, 85:1–85:16.

Ohta, K.; Barrot, N.; Ismaili, A.; Sakurai, Y.; and Yokoo, M. 2017. Core Stability in Hedonic Games among Friends and Enemies: Impact of Neutrals. In *Proc. of the 26th International Joint Conference on Artificial Intelligence, IJCAI '17*, 359–365.

Peters, D. 2016a. Complexity of Hedonic Games with Dichotomous Preferences. In *Proc. of the 30th AAAI Conference on Artificial Intelligence, AAAI '16*, 579–585.

Peters, D. 2016b. Graphical Hedonic Games of Bounded Treewidth. In *Proc. of the 30th AAAI Conference on Artificial Intelligence, AAAI '16*, 586–593.

Plaut, B.; and Roughgarden, T. 2020. Almost Envy-Freeness with General Valuations. *SIAM Journal on Discrete Mathematics*, 34(2): 1039–1068.

Schutzman, Z. 2020. Trade-offs in Fair Redistricting. In *Proc. of the 3rd AAAI/ACM Conference on AI, Ethics, and Society, AIES '20*, 159–165.

Skibski, O.; Suzuki, T.; Grabowski, T.; Sakurai, Y.; Michalak, T. P.; and Yokoo, M. 2022. Measuring power in coalitional games with friends, enemies and allies. *Artificial Intelligence*, 313: 103792.

Sless, L.; Hazon, N.; Kraus, S.; and Wooldridge, M. J. 2018. Forming k coalitions and facilitating relationships in social networks. *Artificial Intelligence*, 259: 217–245.

Ueda, S. 2018. Envy based fairness in hedonic games. In *Proc. of the 2018 Genetic and Evolutionary Computation Conference Companion, GECCO '18*, 1852–1853.

Williams Jr., J. C. 1995. Political redistricting: A review. *Papers in Regional Science*, 74(1).

Wright, M.; and Vorobeychik, Y. 2015. Mechanism Design for Team Formation. In *Proc. of the 29th AAAI Conference on Artificial Intelligence, AAAI '15*, 1050–1056.

Xiao, M.; Qiu, G.; and Huang, S. 2023. MMS Allocations of Chores with Connectivity Constraints: New Methods and New Results. In *Proc. of the 2023 International Conference on Autonomous Agents and Multiagent Systems, AAMAS '23*, 2886–2888.

Zhou, Y.; Wei, T.; Li, M.; and Li, B. 2024. A Complete Landscape of EFX Allocations on Graphs: Goods, Chores and Mixed Manna. In *Proc. of the 33rd International Joint Conference on Artificial Intelligence, IJCAI '24*, 3049–3056.