Linear Equations with Min and Max Operators: Computational Complexity

Krishnendu Chatterjee[1], **Ruichen Luo**[1], **Raimundo Saona**[1], **Jakub Svoboda**[1]

[1]Institute of Science and Technology Austria
Am Campus 1
Klosterneuburg, 3400 Austria
krishnendu.chatterjee@ist.ac.at, rluo@ist.ac.at, raimundo.saona@gmail.com, jakub.svoboda@ist.ac.at

Abstract

We consider a class of optimization problems defined by a system of linear equations with min and max operators. This class of optimization problems has been studied under restrictive conditions, such as, (C1) the halting or stability condition; (C2) the non-negative coefficients condition; (C3) the sum up to 1 condition; and (C4) the only min or only max operator condition. Several seminal results in the literature focus on special cases. For example, turn-based stochastic games correspond to conditions C2 and C3; and Markov decision process to conditions C2, C3, and C4. However, the systematic computational complexity study of all the cases has not been explored, which we address in this work. Some highlights of our results are: with conditions C2 and C4, and with conditions C3 and C4, the problem is NP-complete, whereas with condition C1 only, the problem is in UP intersects coUP. Finally, we establish the computational complexity of the decision problem of checking the respective conditions.

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1 Introduction

Optimization Problems. Optimization problems are in the core of most applications of artificial intelligence. Prominent examples include: learning algorithms, where stochastic gradient descent approaches are fundamental (Bottou 2010); game theory, where the connection between matrix games and linear programming was established in the seminal work of von Neumann and Morgenstern; sequential decision making, where the solution is obtained via Markov decision processes which can be solved via linear programming (Puterman 2014). Hence, understanding the computational complexity of different classes of optimization problems is a fundamental question in artificial intelligence and learning and an active research area, for example, linear programming (Dantzig 2002), convex optimization (Nesterov 2013), semi-definite programming (Gärtner and Matousek 2012), to name a few.

Linear Equations with Min and Max Operators. In this work, we consider the class of optimization problems defined by a system of linear equations along with min and max operators. Since linear equalities are very basic operations, and min and max operators are the most fundamental optimization operators, this corresponds to a very natural class of optimization problems, for example, this is a natural generalization of linear equations with Boolean variables.

Restrictive Conditions. Given the generality of the above class, the associated decision problems are computationally hard. Hence, several natural sub-classes of the above optimization problem have been studied in the literature and we first recall the restrictive conditions: (C1) the halting or the stability condition; (C2) the non-negative coefficients condition; (C3) the sum up to 1 condition; and (C4) the min-only or max-only operator condition. Condition C1 has a natural interpretation in Markov chains, Markov Decision Processes, and Stochastic Games as a geometric convergence to termination irrespective of the action of the players. See Section 3 for concrete examples illustrating all these conditions. For example, in the example "co-evolution in ecosystem", condition C1 is equivalent to the condition that all species will go extinct when there is no external intervention.

Classical Complexity Results from the Literature. Several seminal results consider subsets of the above conditions. First, conditions C2 and C3 together represent stochasticity or probability distributions. With these two conditions and both max and min operators, we obtain the very well-studied class of simple stochastic games (SSGs). SSGs have been studied in the seminal work of Condon (1992) as an important sub-class of Shapley games (Shapley 1953). The computational problem of SSGs is known to be in NP $\cap$ coNP (Condon 1992), as well as UP $\cap$ coUP[1] (Chatterjee and Fijalkow 2011). The existence of polynomial-time algorithms for SSGs is a major and long-standing open problem in game theory. We mention a problem is SSG-hard if there is a polynomial-time reduction from the SSGs; hence for SSG-hard problems polynomial-time algorithms require a major breakthrough. If we have conditions C2, C3, and C4, then this corresponds to the class of Markov decision processes (MDPs), where there is a single player, as opposed to SSGs where there are two adversarial players. Hence, un-

[1]A language is in UP (or coUP) if a given Yes (or No) answer can be verified in polynomial time, and the verifier machine only accepts at most one answer for each problem instance (Hemaspaandra and Rothe 1997).

der conditions C2, C3, and C4 the problems are polynomial-time solvable via linear programming.

Classical Results from the Literature Related to Various Connections. It was shown in (Condon 1992) that, if we have conditions C2 and C3, then these two conditions together imply condition C1 as well. Moreover, recent results show that robust versions of MDP problems can be reduced to SSGs (Chatterjee et al. 2024). Thus, robust versions of optimization problems with only max or only min operators naturally give rise to both operators.

Open Problems. While there are several studies that consider special sub-classes of the above problem, a systematic study of the computational complexities for all cases has been missing in the literature and is the focus of this work. For example, condition C1 is a natural stability condition, and the complexity of this problem without the restriction of C2 and/or C3 has not been addressed in the literature.

Our Contributions. In this work, we present the computational complexity landscape for the class of optimization problem of linear equations with min and max operators under all subsets of restrictive conditions. In particular, some highlights of our results are as follows: (i) with conditions C2 and C4, and with conditions C3 and C4, the problem is NP-complete; (ii) with conditions C1 only, the problem is in UP $\cap$ coUP; (iii) even with conditions C1, C3, and C4, the problem is SSGs-hard and hence proving the existence of a polynomial-time algorithm requires a major breakthrough. These complexity results are summarized in Table 1. Finally, we consider the complexity of the decision problem of checking the conditions. We show that checking condition C1 is coNP-hard, but checking conditions C1 and C2 can be achieved in polynomial time. These complexity results are summarized in Table 2.

Technical Contributions. Our main technical contributions are as follows. First, we show that if we have only condition C1, but not condition C2 and C3, then several classical properties of SSGs, e.g., monotonicity of solutions and the $\min\max = \max\min$ property do not hold. We provide illuminating examples (see Example 1–Example 3) to illustrate this aspect. Second, even though these fundamental properties break, we still show the existence of a unique solution under condition C1 (see Lemma 6), which allows us to establish the UP $\cap$ coUP result. Details and proofs omitted due to lack of space are presented in the Appendix of the arXiv version.

2 Definitions

In this section, we present the basic definitions. We define the linear equations with min and max operators (LEMM), the associated decision problem, and finally the conditions. We use the following notation: for an integer k, we denote $[1, k] \cap \mathbb{N}$ by $[k]$.

Definition 1 (Linear Equations with Min and Max Operators (LEMM)). *Consider $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b})$ such that the following conditions hold: (a) $n_1, n_2, n \in \mathbb{N}$, (b) $n \geq n_1 + n_2$, (c) $\emptyset \subsetneq \mathcal{N}(i) \subseteq [n]$ for $i \in [n_1 + n_2]$, (d) $\mathbf{q}_k \in \mathbb{R}^n$ for $n_1 + n_2 < k \leq n$, and (e) $\mathbf{b} = [0, \ldots, 0, b_{n_1+n_2+1}, \ldots, b_n]^\top \in \mathbb{R}^n$. A vector $\mathbf{x} = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ that satisfies the following system of linear equations with min and max operators (LEMM) is called feasible:*

$$\begin{cases} x_i = \min\limits_{l \in \mathcal{N}(i)} x_l, \quad 1 \leq i \leq n_1, & \text{(1a)} \\ x_j = \max\limits_{l \in \mathcal{N}(j)} x_l, \quad n_1 < j \leq n_1 + n_2, & \text{(1b)} \\ x_k = \mathbf{q}_k^\top \mathbf{x} + b_k, \quad n_1 + n_2 < k \leq n. & \text{(1c)} \end{cases}$$

In Equation (1), $x_1, \ldots, x_{n_1}$ in (1a) are the min variables, $x_{n_1+1}, \ldots, x_{n_1+n_2}$ in (1b) are the max variables, and $x_{n_1+n_2+1}, \ldots, x_n$ in (1c) are the affine variables.

Definition 2 (LEMM Decision Problem). *The LEMM decision problem is: given an LEMM with $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b})$, an index $i \in [n]$, and a threshold $\beta \in \mathbb{R}$, determine whether there exists a feasible solution $\mathbf{x}$ to the LEMM, s.t. $x_i < \beta$.*

Remark 1 (Generalities). *We discuss the generality of the LEMMs.*

- *Any finite nested min, max, and linear operators can be separated by substitution. For instance, consider the following equation*
$$z_i = \min_{j \in \mathcal{N}} f_{i,j}(z),$$
where $f_{i,j}$ contains finite (nested) min, max, and linear operators. It can be rewritten as
$$z_i = \min_{j \in \mathcal{N}} x_{i,j}, \qquad x_{i,j} = f_{i,j}(z).$$
- *Boolean variables can be encoded using the min and/or max, and linear operators. Hence, LEMMs generalize linear equations with Boolean variables.*

While the LEMM decision problem is quite general, which implies computational hardness, several restrictive sub-classes have been considered in the literature. We first introduce some notations and then describe the conditions.

Notations. We introduce the following notations. Let the indicator vector

$$\mathbf{e}_i \coloneqq [\delta_{i,1}, \ldots, \delta_{i,n}]^\top, \quad i \in [n],$$

where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise. Denote the set system of linear equations $\mathbf{x} = \mathbf{Q}\mathbf{x} + \mathbf{b}$ induced by forcing each min and max variable x_i to be equal to $x_{\ell(i)}$, i.e., fixing one choice for each min and max variable, by

$$\begin{aligned} \mathcal{Q} &\coloneqq \{\mathbf{Q}_i\}_{i \in \mathcal{I}} \\ &\coloneqq \Big\{ [\mathbf{e}_{\ell(1)}, \ldots, \mathbf{e}_{\ell(n_1+n_2)}, \mathbf{q}_{n_1+n_2+1}, \ldots, \mathbf{q}_n]^\top \\ &\qquad \Big| \ \ell(j) \in \mathcal{N}(j) \text{ for all } j \in [n_1 + n_2] \Big\}. \end{aligned}$$

The convex hull of $\mathcal{Q}$ is denoted as $\mathbf{conv}(\mathcal{Q})$:

$$\left\{ \sum_{i \in \mathcal{I}} \alpha_i \mathbf{Q}_i \ \middle| \ \sum_{i \in \mathcal{I}} \alpha_i = 1 \text{ and } \forall\, i \in \mathcal{I},\ \mathbf{Q}_i \in \mathcal{Q} \wedge \alpha_i \geq 0 \right\}.$$

Condition C1 (Halting or Stability). *For all $\mathbf{Q} \in \mathbf{conv}(\mathcal{Q})$,*

$$\lim_{m \to \infty} \mathbf{Q}^m = \mathbf{0}_{n \times n}.$$

{C2,C4}, **{C3,C4}**, {C2}, {C3}, {C4}, $\emptyset$	NP-COMPLETE	Theorem 1
{C1,C3,C4}, **{C1,C3}**, **{C1,C4}**, **{C1}**	UP $\cap$ COUP (SSG-hard)	Corollary 3 & 6
{C1,C2}	UP $\cap$ COUP (SSG-hard)	Corollary 3 & 6
{C1,C2,C3}, {C2,C3}	UP $\cap$ COUP (SSG-hard)	Proposition 1
{C1,C2,C3,C4}, {C2,C3,C4}, **{C1,C2,C4}**	PTIME	Corollary 5

Table 1: The complexity of the LEMM decision problems under all subsets of conditions. If $X \subseteq Y$ represents two subsets of conditions, then the LEMM decision problem under X that has fewer conditions is more general. The table describes all the results, with our main results in bold. Moreover, for the problems in a row there is a polynomial-time equivalence.

{C1}, **{C1,C3}**, **{C1,C4}**, **{C1,C3,C4}**	CONP-hard	Corollary 10
{C1,C2}, **{C1,C2,C4}**, {C1,C2,C3}, {C1,C2,C3,C4}	PTIME	Corollary 8

Table 2: The complexity of the condition decision problems for all subsets of conditions in the presence of condition C1. In the absence of condition C1, all other conditions can be checked in linear time. Our results are in bold.

Condition C2 (Non-Negative Coefficients)**.** *For all $n_1 + n_2 < k \leq n$, we have that $\mathbf{q}_k \geq 0$ and $b_k \geq 0$.*

Condition C3 (Sum up to 1)**.** *For all $n_1 + n_2 < k \leq n$, we have that $\mathbf{q}_k^\top \mathbf{1} + b_k \leq 1$.*

Condition C4 (Max-Only or Min-Only)**.** *Either $n_1 = 0$ or $n_2 = 0$.*

We will use the following subset notation for the LEMM decision problem under various subsets of the conditions, *e.g.*, "the LEMM decision problem under {C1,C2}" means "the LEMM decision problem under conditions C1 and C2".

Remark 2 (Relevance of the Conditions in Previous Studies)**.** *We clarify the relevance of the conditions.*

1. *The classical turn-based version of stochastic games played by two adversarial players with reachability objectives (which is referred to as simple stochastic games or SSGs) has stochastic or probabilistic transitions (Condon 1990). The stochastic transitions lead to conditions C2 and C3, and stochastic games require both min and max operators for the two players. Hence, SSGs naturally correspond to LEMM with conditions C2 and C3.*
2. *In the context of SSGs (*i.e.*, if conditions C2 and C3 hold), then we also obtain condition C1 without loss of generality (Condon 1992).*
3. *For SSGs, the existence of a polynomial-time algorithm is a major open problem, and we say a problem is SSG-hard if there is a polynomial-time reduction from SSGs. For SSG-hard problems, polynomial-time algorithms require a major breakthrough.*
4. *The intuitive description of condition C1 is as follows: in the absence of min and max operators, it is similar to Markov chains, and the condition implies that eventually recurrent states are reached almost-surely. This represents reaching the stable distribution almost-surely. In the presence of mix and max choices, this represents that irrespective of the choices, almost-surely stability is achieved. This is also referred to as the halting or stopping condition in the literature.*

We consider the problem of checking the conditions.

Definition 3 (Condition Decision Problem)**.** *The condition decision problem is: given an LEMM and a subset of conditions, determine whether all the conditions are satisfied.*

Remark 3. *For the condition decision problem, the conditions* C2, C3, and C4 *are easy to check. The main condition decision problem is in the presence of condition* C1.

3 Motivating Examples

In this section we present motivating examples that can be modeled in the LEMM framework.

Neural Network Verification. Consider the following *neural network verification* problem. Multilayer perceptrons (MLP) receive input and forward it by multiple layers of fully connected neurons with activation functions (e.g., ReLU or Maxout) (Goodfellow et al. 2013). The last layer is interpreted as a decision of the MLP. For a bounded region of input, we want to decide whether the decision of the MLP is in a bounded region.

Formally, for input $\mathbf{x}^{\text{in}} \in [0,1]^d$, we are to decide whether the output satisfies $x^{\text{out}} < \beta$. The above problem can be posed as the following LEMM:

$$\begin{cases} \mathbf{x}^{\text{in}} = \max\{0, \min\{\mathbf{x}^{\text{in}}, 1\}\}\,, \\ \mathbf{x}^{\text{Layer 1}} = f(\mathbf{Q}^{\text{Layer 1}}\mathbf{x}^{\text{in}} + \mathbf{b}^{\text{Layer 1}})\,, \\ \vdots \\ \mathbf{x}^{\text{Layer } n} = f(\mathbf{Q}^{\text{Layer } n}\mathbf{x}^{\text{Layer } n-1} + \mathbf{b}^{\text{Layer } n})\,, \\ x^{\text{out}} = \mathbf{q}^{\text{out}\top}\,\mathbf{x}^{\text{Layer } n} + b^{\text{out}}\,, \end{cases}$$

where f can be any piecewise linear activation function (*e.g.*, ReLU or Maxout). This problem is known to be NP-complete (Katz et al. 2017), and corresponds to a LEMM decision problem with no restriction. □

Capital Preservation. Motivated by the classical portfolio management problem (Markowitz 1952; Puterman 2014), the capital preservation problem models the periodic management of assets under uncertainty. Consider a cyclic evolution with T periods, each period corresponds to an opportunity for the controller to affect the evolution of the assets, and for the market to affect the assets. In each period, these effects come in turn, so at period $t \in [T]$ the controller can choose between different transformations that convert the assets from the previous stage in expectation in a linear fashion; i.e., having x assets, we can get y assets given by

$$y \leftarrow \max\{q_{1,i}\, x + b_{1,i} \mid i \in [m]\} .$$

Then, the uncertainty of the market is modeled by one of many possible linear transformations; i.e., the controller obtains at the end of period t an amount z given by

$$z \leftarrow \min\{q_{2,j}\, y + b_{2,j} \mid j \in [\ell]\} .$$

Note that the effect of inflation on the value of the assets can be incorporated in, for example, $q_{2,j}$. Following this dynamic, the controller chooses how to evolve the amount of assets at each period, and the cycle restarts.

Formally, this problem corresponds to the LEMM

$$\begin{cases} x_{1,1} = \max\{q^1_{1,i}\, x_{T,2} + b^1_{1,i} \mid i \in [m]\} , \\ x_{1,2} = \min\{q^1_{2,j}\, x_{1,1} + b^1_{2,j} \mid j \in [\ell]\} , \\ \quad \vdots \\ x_{T,1} = \max\{q^T_{1,i}\, x_{T-1,2} + b^T_{1,i} \mid i \in [m]\} , \\ x_{T,2} = \min\{q^T_{2,j}\, x_{T,1} + b^T_{2,j} \mid j \in [\ell]\} . \end{cases}$$

We argue that we have conditions C1 and C2 in the following sense: condition C1 ensures that the controller does not get infinite returns in expectation; and condition C2 implies that the assets are non-negative. □

Co-Evolution in Ecosystem. Branching processes (Athreya and Ney 2004) describe the evolution of population where the next state depends only on the previous state. We consider the following *co-evolution* extension of the problem. There are k different species in an ecosystem. Evolution is the combined consequence of internal interactions (*e.g.*, reproduction or predation) and external interventions (*e.g.*, immigration or fertility control). The population of the next generation is modeled as a piece-wise linear function of the current population. We want to find the stable distribution of the ecosystem.

Formally, let $\mathbf{x} \in \mathbb{R}^k_{\geq 0}$ be the current population, and the population of the next generation is given by

$$\mathbf{x}' = \max\{\mathbf{Q}\mathbf{x} + \mathbf{b}, 0\} ,$$

in which $\mathbf{Q} \in \mathbb{R}^{k \times k}$ models the internal interactions and $\mathbf{b} \in \mathbb{R}^k$ models the external interventions. Hence the stable distribution of the ecosystem can be posed as the LEMM

$$\mathbf{x} = \max\{\mathbf{Q}\mathbf{x} + \mathbf{b}, 0\} .$$

We argue that we naturally have conditions C1 and C4: condition C1 implies that all species will go extinct without external intervention; and condition C4 holds because the LEMM is max-only. □

The above examples show that LEMMs with restrictions can model classical optimization problems from the literature and motivate the computational complexity landscapes of various subsets of conditions.

4 Complexity of LEMM Decision Problems

We start with basic results and previous results from the literature, then we present our complexity bounds, and finally the equivalence between some problems. In what follows, we consider that inputs of the problems are given as rational numbers as usual.

4.1 Basic and Previous Results

Basic Fact 1. *If $X \subseteq Y$ represents two subsets of conditions, then the LEMM decision problem under X is no easier than the LEMM decision problem under Y. Hence any complexity lower bound for Y also holds for X, and any complexity upper bound for X also holds for Y. See Figure 1 in the arXiv version for the lattice structure of the LEMM decision problem under different conditions.*

Lemma 1. *The LEMM decision problem without any restriction is in* NP.

Proof. For all LEMM, index $i \in [n]$, and threshold $\beta \in \mathbb{R}$, if there is a feasible solution $\mathbf{x}$ *s.t.* $x_m < \beta$, then $\mathbf{x}$ is a certificate that can be verified in polynomial time. □

Lemma 2. *Let $\mathbf{Q} \in \mathbb{R}^{n\times n}$. If $\lim_{m\to\infty} \mathbf{Q}^m = \mathbf{0}_{n\times n}$, then $(\mathbf{I}-\mathbf{Q})$ is invertible. Further, if $\mathbf{Q} \geq 0$, then $(\mathbf{I}-\mathbf{Q})^{-1} \geq 0$.*

Remark 4. *Condon (1992) proves a version of this lemma under conditions C1, C2 and C3, and we observe that the lemma generalizes without any restrictions.*

Proposition 1 (Complexity Results from the Literature)**.** *The following assertions hold:*

1. *The LEMM decision problem under* {C2,C3} *is in* UP ∩ coUP.
2. *The LEMM decision problems under* {C1,C2,C3} *and under* {C2,C3} *are polynomially equivalent.*
3. *The LEMM decision problem under* {C2,C3,C4} *is in* PTIME.

Explanation. Items 1 and 2 follow mainly from (Condon 1992) and (Chatterjee and Fijalkow 2011). Item 3 follows from single-player stochastic games being MDPs, which can be solved via linear programming.

4.2 Complexity Lower Bounds

In this section, we present the complexity lower bounds. First, we recall the classic NP-complete partition problem.

Lemma 3 (Garey and Johnson (1979))**.** *The partition problem is: given a set of positive integers $\{a_1, \ldots, a_k\} \in \mathbb{Z}^k_{\geq 1}$, determine whether the set can be partitioned into two subsets with equal sums. This problem is* NP*-complete.*

Lemma 4. *The LEMM decision problem under* {C2,C4} *is* NP*-hard.*

Main Proof Idea. We show a reduction from the partition problem of $\{a_1, \ldots, a_m\}$ to an LEMM as follows. First, we construct a min-only LEMM with non-negative coefficients, in which the min variables can only take value ± 1. Then, we associate each integer a_i with a min variable x_i. When x_i takes value $+1$, we put a_i into the first subset; and when it takes value -1, we put a_i into the second subset. Finally, we use the last affine variable x_{2n_1+2} to encode the constraint that the sum of the two subsets is equal. $\square$

Lemma 5. *The LEMM decision problem under* {C3,C4} *is* NP-*hard.*

Main Proof Idea. From Lemma 4 and Basic Fact 1, we know that the LEMM decision problem under {C4} is NP-hard. Then, we show that from the LEMM under {C4}, we can construct another equivalent LEMM that satisfies conditions C3 and C4. $\square$

We conclude this section with a theorem.

Theorem 1. *The LEMM decision problems under conditions* {C2,C4}*,* {C3,C4}*,* {C2}*,* {C3}*,* {C4}*, or* $\emptyset$ *are all* NP-*complete. Hence, we establish the first row of Table 1.*

Proof. The NP-upper bound follows from Lemma 1 and Basic Fact 1. The NP-hardness follows from Lemma 4, Lemma 5, and Basic Fact 1. $\square$

4.3 Complexity Upper Bounds

In this section, we will establish the following main results.

Theorem 2. *The LEMM decision problem under* {C1} *is in* UP ∩ coUP.

Corollary 3. *The* UP ∩ coUP *complexity results in the second, third, and fourth row of Table 1 are established.*

Explanation. Proposition 1 (item 1 and item 2) establishes the results of the fourth row of Table 1. Theorem 2 and Basic Fact 1 establish the results for the second and third row.

Theorem 4. *The LEMM decision problem under* {C1,C2,C4} *is in* PTIME.

Corollary 5. *The* PTIME *complexity results in the fifth row of Table 1 are established.*

Explanation. Proposition 1 (item 3) and Basic Fact 1 establishes the results for the LEMM decision problems under {C1,C2,C3,C4} and {C2,C3,C4}, and Theorem 4 establishes the result for LEMMs under {C1,C2,C4}.

Classic Properties. Before presenting our proofs, we explain the key difficulties in establishing the results as compared to previous analysis in the literature (*e.g.*, Condon (1992); Ludwig (1995); Chatterjee et al. (2023)). For halting stochastic games (*i.e.*, the LEMM under {C1,C2,C3}), the "monotonicity", "minimax equality", and "solution from subsolution" properties defined below hold and were used in previous analysis. We show that these key properties do not hold under condition C1 only. First, we define these properties.

- **Monotonicity.** For LEMMs $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \cdot)$ with a unique solution, we say that the LEMMs are monotonic if, for all $\mathbf{b}_1 \geq \mathbf{b}_2$, if $\mathbf{x}^1$ is the solution of the LEMM with $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b}_1)$, and $\mathbf{x}^2$ is the solution of the LEMM with $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b}_2)$, then $\mathbf{x}^1 \geq \mathbf{x}^2$.
- **Minimax Equality.** Consider LEMMs $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b})$ such that, for all assignments $\ell\colon [n_1+n_2] \mapsto [n]$ where
$$\ell(k) \in \mathcal{N}(k),\ \forall k \in [n_1+n_2]\,,$$
defining
$$\mathbf{Q}^{(\ell)} = \left[\mathbf{e}_{\ell(1)}, \ldots, \mathbf{e}_{\ell(n_1+n_2)}, \mathbf{q}_{n_1+n_2+1}, \ldots, \mathbf{q}_n\right]^\top,$$
we have that $(\mathbf{I} - \mathbf{Q}^{(\ell)})$ is invertible. Then, taking $\mathbf{x}^{(\ell)} = (\mathbf{I} - \mathbf{Q}^{(\ell)})^{-1}\mathbf{b}$, the LEMM satisfies the minimax equality if, for all $k \in [n]$,
$$\min_{\substack{\ell(i)\in\mathcal{N}(i),\\ 1\leq i\leq n_1}} \max_{\substack{\ell(j)\in\mathcal{N}(j),\\ n_1<j\leq n_1+n_2}} x_k^{(\ell)} = \max_{\substack{\ell(j)\in\mathcal{N}(j),\\ n_1<j\leq n_1+n_2}} \min_{\substack{\ell(i)\in\mathcal{N}(i),\\ 1\leq i\leq n_1}} x_k^{(\ell)}.$$
- **Solution from Subsolution.** Consider LEMMs $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b})$ and $1 \leq i \leq n_1$ (resp. $n_1 < j \leq n_1 + n_2$), where, for all $l \in \mathcal{N}(i)$ (resp. $l \in \mathcal{N}(j)$), modifying the i-th equation in the LEMM by
$$x_i = x_l \text{ (or } x_j = x_l)$$
we obtain a new LEMM with a unique solution $\mathbf{x}^l$ and let
$$l^* = \arg\min_{l\in\mathcal{N}(i)} x_i^l \text{ (or } l^* = \arg\max_{l\in\mathcal{N}(j)} x_j^l).$$
Then, we say the original LEMM has a solution from its subsolution if, for all $l \in \mathcal{N}(i)$ (resp. $l \in \mathcal{N}(j)$), we have that $\mathbf{x}^{l^*}$ is the solution of the original LEMM.

We present illuminating counterexamples showing that the above classic properties break when condition C1 holds, and conditions C2 and C3 do not hold.

Example 1 (Non-Monotonicity)**.** *Consider the LEMM*
$$\begin{cases} x_1 = \max\{x_2,\ x_3\}, \\ x_2 = -x_3 + 1, \\ x_3 = 0.5x_3 + 0.2, \end{cases}$$
which satisfies condition C1 and has a unique solution $\mathbf{x} = [0.6, 0.6, 0.4]^\top$*. Increasing the value of* b_3*, we have the following LEMM*
$$\begin{cases} x_1' = \max\{x_2',\ x_3'\}\,, \\ x_2' = -x_3' + 1\,, \\ x_3' = 0.5x_3' + 0.25\,, \end{cases}$$
with a unique solution $\mathbf{x}' = [0.5, 0.5, 0.5]^\top$*, so these LEMMs do not satisfy the monotonicity property.* $\square$

Example 2 ($\min\max \neq \max\min$)**.** *Consider the LEMM*
$$\begin{cases} x_1 = \min\{x_3, x_4\}, \\ x_2 = \max\{x_5, x_6\}, \\ x_3 = -0.18x_1 + 0.72x_2, \\ x_4 = 0.36x_1 - 0.18x_2, \\ x_5 = 0.36x_1 - 0.54x_2 + 2, \\ x_6 = -0.18x_1 - 0.36x_2 + 2, \end{cases}$$

which satisfies condition C1*, while we have*

$$\min_{l(1)\in\{3,4\}} \max_{l(2)\in\{5,6\}} x_2^{(l)} = x_2^{(l(1)=3,\ l(2)=5)} > 1.5\,,$$

and

$$\max_{l(2)\in\{5,6\}} \min_{l(1)\in\{3,4\}} x_2^{(l)} = x_2^{(l(1)=3,\ l(2)=6)} < 1.4\,.$$

This violates the minimax equality. □

Example 3 (Misleading Solution from Subsolution)**.** *Consider the LEMM*

$$\begin{cases} x_1 = \min\{x_2,\ x_3\}\,, \\ x_2 = -0.1x_1 + 0.8x_4 + 2.2\,, \\ x_3 = 0.1x_1 + 0.5x_4 + 2.2\,, \\ x_4 = -0.5x_1 + 1.2x_4 - 1.4\,, \end{cases}$$

which satisfies condition C1. Choosing $x_1 = x_2$*, the subsolution is* $\mathbf{x}^2 = [-26/3, -26/3, -6, -44/3]^\top$*, while choosing* $x_1 = x_3$*, the subsolution is* $\mathbf{x}^3 = [-114/7, -162/7, -114/7, -236/7]^\top$*. Even when* $x_1^2 > x_1^3$*, the* larger *subsolution* $\mathbf{x}^2$ *is the solution of the original LEMM. This violates the solution from subsolution property, which states that* x^3 *should be the solution to the original LEMM.* □

Novelty. We present our key technical lemma, which shows that condition C1 already ensures the existence of a unique solution. In the context of halting SSGs (*i.e.*, {C1,C2,C3}), the proof of NP ∩ coNP is based on the minimax equality since, given the choices of the min (resp. max) variables as the certificate of the YES (resp. NO) answer, it can be verified by solving the remaining max-only (resp. min-only) subproblem via linear programming (Condon 1992; Ludwig 1995; Chatterjee et al. 2023). Our main novelty is to present a new analysis technique that establishes the desired complexity results even though the classic properties that were crucial in the previous analysis do not hold.

Lemma 6 (Key Lemma)**.** *Under condition* C1*, there exists a unique solution to the LEMM.*

Main Proof Idea. We prove by induction on $n_1 + n_2$.

(i) *Base Case.* The base case $n_1 + n_2 = 0$ follows from Lemma 2.

(ii) *Inductive Case.* Assume by induction that the existence and uniqueness hold when $n_1 + n_2 = m$, and we consider the case $n_1 + n_2 = m + 1$. Without loss of generality, let $x_{m+1} = \max\{x_i, x_j\}$. By contradiction, assume that the LEMM either has no solution or has 2 different solutions. For the reduced LEMM obtained by fixing x_{m+1} as x_j (resp. x_i), we get its unique solution $\mathbf{x}^{(0)}$ (resp. $\mathbf{x}^{(1)}$) by the induction hypothesis. Then, for any $\alpha \in (0,1)$, fixing x_{m+1} as $\alpha x_i + (1-\alpha)x_j$, we get a unique solution $\mathbf{x}^{(\alpha)}$ for the reduced LEMM by induction hypothesis. Moreover, by continuity, there is α^* such that $\mathbf{x}_i^{(\alpha^*)} = \mathbf{x}_j^{(\alpha^*)}$. This is a contradiction because $\mathbf{x}^{(\alpha^*)}$ becomes the unique solution for the original LEMM. □

Proof. Without loss of generality, assume that $|\mathcal{N}(i)| = 2$, for all $i \in [n_1 + n_2]$. We proceed by induction on $(n_1 + n_2)$.

(i) *Base Case.* For $n_1 + n_2 = 0$, by Lemma 2, there exists a unique solution $\mathbf{x} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{b}$, where $\mathbf{Q}$ is the only element in $\mathcal{Q}$.

(ii) *Inductive Case.* By induction, all LEMMs with $n_1 + n_2 = m$ have a unique solution. We prove it is the case for all LEMMs with $n_1 + n_2 = m + 1$. By contradiction, consider an LEMM $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b})$ with $n_1 + n_2 = m + 1$ that has no solution or has 2 different solutions. Without loss of generality $\mathcal{N}(m+1) = \{i, j\}$, and $x_{m+1} = \max\{x_i, x_j\}$ (if x_{m+1} is a min variable, we only switch some inequalities). For all $\alpha \in [0,1]$, consider replacing this equation by $x_{m+1} = \alpha\, x_i + (1-\alpha)\, x_j$. Then, by the induction hypothesis, the modified LEMM has a unique solution denoted $\mathbf{x}^{(\alpha)}$.

Note that $\mathbf{x}^{(0)}$ is a solution to the original system with $(m+1)$ min and max variables if and only if ${x^{(0)}}_i \geq {x^{(0)}}_j$. Similarly, $\mathbf{x}^{(1)}$ is a solution to the original system if and only if ${x^{(1)}}_i \leq {x^{(1)}}_j$. Since the LEMM has no solution or has 2 different solutions, the following holds:

$$\left({x^{(0)}}_i - {x^{(0)}}_j\right) \cdot \left({x^{(1)}}_i - {x^{(1)}}_j\right) < 0\,. \qquad (2)$$

We introduce the following notations for $\alpha \in [0,1]$.

$$\mathbf{E} := [\mathbf{e}_1, \cdots, \mathbf{e}_m, \mathbf{0}, \mathbf{e}_{m+2}, \cdots, \mathbf{e}_n]^\top\,,$$

$$\mathcal{Q}^{(\alpha)} := \left\{\mathbf{E}\mathbf{Q} + \mathbf{e}_{m+1}\,(\alpha\mathbf{e}_i + (1-\alpha)\mathbf{e}_j)^\top \;\middle|\; \mathbf{Q} \in \mathcal{Q}\right\}\,,$$

$$\mathcal{M}^{(\alpha)} := \left\{\mathbf{M} \in \mathcal{Q}^{(\alpha)} \;\middle|\; \mathbf{x}^{(\alpha)} = \mathbf{M}\mathbf{x}^{(\alpha)} + \mathbf{b}\right\}\,.$$

Intuitively, $\mathbf{E}\mathbf{Q} + \mathbf{e}_{m+1}\,(\alpha\mathbf{e}_i + (1-\alpha)\mathbf{e}_j)^\top$ corresponds to replacing the $(m+1)$-th row of $\mathbf{Q}$ by $(\alpha\mathbf{e}_i + (1-\alpha)\mathbf{e}_j)^\top$.

We show that $\alpha \in [0,1] \mapsto \mathbf{x}^{(\alpha)}$ is continuous. By the continuity of linear, min, and max operators, there exists $\epsilon > 0$, such that, for all $\alpha' \in \mathcal{B}(\alpha;\epsilon) := (\alpha-\epsilon, \alpha+\epsilon) \cap [0,1]$, defining $\mathcal{M}^{(\alpha;\alpha')}$ by

$$\left\{\mathbf{E}\mathbf{M} + \mathbf{e}_{m+1}\,(\alpha'\mathbf{e}_i + (1-\alpha')\mathbf{e}_j)^\top \;\middle|\; \mathbf{M} \in \mathcal{M}^{(\alpha)}\right\}\,,$$

we have $\mathcal{M}^{(\alpha')} \subseteq \mathcal{M}^{(\alpha;\alpha')}$. Therefore, for all sequences $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\alpha;\epsilon)$ that converges to α, we have that

$$\begin{aligned} |\mathbf{x}^{(\alpha_n)} - \mathbf{x}^{(\alpha)}| &= \sup_{\mathbf{M}\in\mathcal{M}^{(\alpha_n)}} |(\mathbf{I}-\mathbf{M})^{-1}\mathbf{b} - \mathbf{x}^{(\alpha)}| \\ &\leq \sup_{\mathbf{M}\in\mathcal{M}^{(\alpha;\alpha_n)}} |(\mathbf{I}-\mathbf{M})^{-1}\mathbf{b} - \mathbf{x}^{(\alpha)}|\,. \end{aligned}$$

This bound is equal to the supremum over $\mathbf{M} \in \mathcal{M}^{(\alpha)}$ of

$$\left|\begin{array}{l} \left(\mathbf{I} - \mathbf{E}\mathbf{M} - \mathbf{e}_{m+1}\,(\alpha_n\mathbf{e}_i + (1-\alpha_n)\mathbf{e}_j)^\top\right)^{-1}\mathbf{b} \\ \quad - \left(\mathbf{I} - \mathbf{E}\mathbf{M} - \mathbf{e}_{m+1}\,(\alpha\mathbf{e}_i + (1-\alpha)\mathbf{e}_j)^\top\right)^{-1}\mathbf{b} \end{array}\right|,$$

which converges to 0 as n grows. Hence, $\alpha \in [0,1] \mapsto \mathbf{x}^{(\alpha)}$ is continuous.

Going back to Equation (2), by continuity, there exists $\alpha^\star \in [0,1]$ such that ${x^{(\alpha^\star)}}_i - {x^{(\alpha^\star)}}_j = 0$. Therefore, the two LEMMs after the modification $x_{m+1} = x_i$ and $x_{m+1} = x_j$

have the same solution $\mathbf{x}^{(\alpha^*)}$, *i.e.*, $\mathbf{x}^{(0)} = \mathbf{x}^{(1)} = \mathbf{x}^{(\alpha^*)}$, so the original LEMM with $(m+1)$ min and max variables has a unique solution $\mathbf{x}^{(\alpha^*)}$. This yields the desired contradiction, which concludes the proof by induction. □

Finally, we prove Theorem 2 and Theorem 4.

Proof of Theorem 2. By Lemma 6, we use the unique solution of the LEMM as a (unique) certificate of the answer YES or NO, which is verifiable in polynomial time. □

Main Proof Idea of Theorem 4. Consider an LEMM $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b})$ satisfying conditions C1, C2 and C4. By Lemma 6, there exists a unique solution. Without loss of generality, assume it only has max-variables. Then, the solution is given by solving a linear program. □

4.4 Equivalence Between Sub-Classes of Problems

We establish the equivalence between the LEMM decision problems in the second row of Table 1.

Lemma 7. *There is a polynomial time reduction from LEMMs under* $\{$C1$\}$ *to LEMMs under* $\{$C1,C3,C4$\}$.

Main Proof Idea. From LEMMs under condition C1, we construct a new, equivalent, min-only LEMM (that satisfies conditions C1 and C4). To do so, we introduce negative copies of the variables, so the $\max$ equations can be rewritten as $\min$ equations over the negative copies. We note that extra positive copies are required to keep the periodicity and thus the halting condition. Finally, we introduce a dummy variable to ensure condition C3. □

Corollary 6. *The LEMM decision problems under* $\{$C1,C3,C4$\}$, $\{$C1,C3$\}$, $\{$C1,C4$\}$, *or* $\{$C1$\}$ *are polynomially equivalent and SSG-hard. Hence, we establish the equivalence of the problems in the second row of Table 1 and their SSG-hardness.*

Proof. The equivalence follows from Lemma 7. The SSG-hardness of $\{$C1$\}$ follows from Proposition 1 and Basic Fact 1. □

5 Complexity of Condition Decision Problems

In this section, we establish complexity results for the condition decision problem. In what follows, we consider that inputs of the problems are given as rational numbers as usual. We start with a remark on the basic and previous complexity results and a basic fact.

Lemma 8 (Basic and Previous Complexity Results)**.** *The condition decision problems for* $\{$C2$\}$, $\{$C3$\}$, *or* $\{$C4$\}$ *are all easily solvable in linear time. Moreover, the condition decision problem for* $\{$C1,C2,C3$\}$ *is in* PTIME *(Baier and Katoen 2008).*

Basic Fact 2. *If* $\{$C1$\} \subseteq X \subseteq Y$ *represents two subsets of conditions, then the condition decision problem for* X *is no easier than the condition decision problem for* Y. *Hence any complexity lower bound for* Y *also holds for* X, *and any complexity upper bound for* X *also holds for* Y.

We present the following characterization, which implies that checking conditions C1 and C2 can be done in polynomial time via linear programming.

Lemma 9 (Technical Lemma)**.** *Consider the LEMM with* $(n_1, n_2, n, \mathcal{N}, \mathbf{q}, \mathbf{b})$ *such that* $\mathbf{q}_k \geq 0$, *for all* $k \in [n_1 + n_2 + 1, n]$. *Then, condition C1 holds if and only if*

$$\exists \mathbf{x} \in \mathbb{R}^n_{\geq 0}\ \forall \mathbf{Q} \in \mathcal{Q} \quad \mathbf{x} \geq \mathbf{Q}\mathbf{x} + \mathbf{1}\,.$$

Theorem 7. *The condition decision problem for* $\{$C1,C2$\}$ *is in* PTIME.

Proof. Condition C2 can be easily checked. By Lemma 9, under condition C2, we can check condition C1 via the feasibility of a system of linear inequalities. □

Corollary 8. *The condition decision problems for* $\{$C1,C2$\}$, $\{$C1,C2,C4$\}$, $\{$C1,C2,C3$\}$, *or* $\{$C1,C2,C3,C4$\}$ *are in* PTIME. *Hence, we establish the second row of Table 2.*

Proof. It follows from Theorem 7 and Basic Fact 2. □

Finally, we show that the condition decision problem for condition C1 is coNP-hard, even under conditions C3 and C4.

Theorem 9. *The condition decision problem for* $\{$C1,C3,C4$\}$ *is* coNP*-hard.*

Main Proof Idea. We show the reduction from the SAT problem in conjunctive normal form with variables $v_1, \ldots, v_r$ and clauses $c_1, \ldots, c_m$ to LEMM as follows.

First, we construct the variables: (i) the max variables $x_1, \ldots, x_m$ associated with $c_1, \ldots, c_m$, (ii) the max variables $x_{m+1}, \ldots, x_{m+r}$ associated with $v_1, \ldots, v_r$, and their negative copies $x_{m+r+1}, \ldots, x_{m+2r}$, as well as (iii) an (affine) variable x_{m+2r+1} and its negative copy x_{m+2r+2}.

Then, we construct the transitions: (i) From x_i ($1 \leq i \leq m$), we can go to x_{m+l} if v_l is a literal in c_i, or to the negative copy of x_{m+l} if $\bar{v}_l$ is literal in c_i. (ii) From x_j ($m \leq j \leq m + r$), we can go to x_{m+2r+1} or its negative copy.

Finally, we show that "condition C1 does not hold" if and only if "from each x_i ($i \in [m]$), we can reach x_{m+2r+1} (or its negative copy) via some x_{m+l} (or its negative copy)". This corresponds to "for each c_i ($i \in [m]$), some assignment of v_l makes c_i true", *i.e.*, the original SAT instance is satisfiable. □

Corollary 10. *The condition decision problems for* $\{$C1$\}$, $\{$C1,C3$\}$, $\{$C1,C4$\}$, *or* $\{$C1,C3,C4$\}$ *are* coNP*-hard. Hence, we establish the first row of Table 2.*

Proof. It follows from Theorem 9 and Basic Fact 2. □

6 Conclusions

In this work, we consider LEMMs with various conditions. While previous results from the literature consider specific cases, our results establish the computational complexity landscape for all possible subsets of conditions and the complexity of checking the conditions. Exploring our approach (*e.g.*, with condition C1 only, or with conditions C1 and C2) in practical applications is an interesting direction for future work.

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References

Athreya, K. B.; and Ney, P. E. 2004. *Branching processes*. Courier Corporation.

Baier, C.; and Katoen, J.-P. 2008. *Principles of model checking*. MIT press.

Bottou, L. 2010. Large-scale machine learning with stochastic gradient descent. In *Proceedings of the 19th International Conference on Computational Statistics (COMPSTAT)*, 177–186.

Chatterjee, K.; and Fijalkow, N. 2011. A reduction from parity games to simple stochastic games. In *Proceedings of the Second International Symposium on Games, Automata, Logics and Formal Verification (GandALF)*, volume 54 of *EPTCS*, 74–86.

Chatterjee, K.; Kafshdar Goharshady, E.; Karrabi, M.; Novotny, P.; and Žikelić, D. 2024. Solving Long-run Average Reward Robust MDPs via Stochastic Games. In *Proceedings of the Thirty-Third International Joint Conference on Artificial Intelligence (IJCAI-24)*, 6707–6715.

Chatterjee, K.; Meggendorfer, T.; Saona, R.; and Svoboda, J. 2023. Faster algorithm for turn-based stochastic games with bounded treewidth. In *Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 4590–4605.

Condon, A. 1990. On Algorithms for Simple Stochastic Games. *Advances in computational complexity theory*, 13: 51–72.

Condon, A. 1992. The Complexity of Stochastic Games. *Information and Computation*, 96(2): 203–224.

Dantzig, G. B. 2002. Linear programming. *Operations research*, 50(1): 42–47.

Garey, M. R.; and Johnson, D. S. 1979. *Computers and intractability*, volume 174. freeman San Francisco.

Gärtner, B.; and Matousek, J. 2012. *Approximation algorithms and semidefinite programming*. Springer Science & Business Media.

Goodfellow, I.; Warde-Farley, D.; Mirza, M.; Courville, A.; and Bengio, Y. 2013. Maxout networks. In *International conference on machine learning*, 1319–1327.

Hemaspaandra, L. A.; and Rothe, J. 1997. Unambiguous computation: Boolean hierarchies and sparse Turing-complete sets. *SIAM Journal on Computing*, 26(3): 634–653.

Katz, G.; Barrett, C.; Dill, D. L.; Julian, K.; and Kochenderfer, M. J. 2017. Reluplex: An efficient SMT solver for verifying deep neural networks. In *Proceedings of the 29th International Conference on Computer Aided Verification (CAV)*, 97–117.

Ludwig, W. 1995. A subexponential randomized algorithm for the simple stochastic game problem. *Information and computation*, 117(1): 151–155.

Markowitz, H. 1952. Portfolio Selection. *The Journal of Finance*, 7(1): 77.

Nesterov, Y. 2013. *Introductory lectures on convex optimization: A basic course*, volume 87. Springer Science & Business Media.

Puterman, M. L. 2014. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. Wiley.

Shapley, L. S. 1953. Stochastic Games. *Proceedings of the National Academy of Sciences*, 39(10): 1095–1100.

von Neumann, J.; and Morgenstern, O. 1953. *Theory of games and economic behavior*. Princeton university press.