

# Identifiability of Direct Effects from Summary Causal Graphs

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## Abstract

Dynamic structural causal models (SCMs) are a powerful framework for reasoning in dynamic systems about direct effects which measure how a change in one variable affects another variable while holding all other variables constant. The causal relations in a dynamic structural causal model can be qualitatively represented with an acyclic full-time causal graph. Assuming linearity and no hidden confounding and given the full-time causal graph, the direct causal effect is always identifiable. However, in many applications such a graph is not available for various reasons but nevertheless experts have access to the summary causal graph of the full-time causal graph which represents causal relations between time series while omitting temporal information and allowing cycles. This paper presents a complete identifiability result which characterizes all cases for which the direct effect is graphically identifiable from a summary causal graph and gives two sound finite adjustment sets that can be used to estimate the direct effect whenever it is identifiable.

## Introduction

Structural causal models (SCMs) are a powerful framework for representing and reasoning about causal relations between variables with a long history in many fields such as genetics (Wright 1920, 1921), econometrics (Haavelmo 1943), social sciences (Duncan 1975; Goldberger 1972), epidemiology (Hernan and Robins 2023), and artificial intelligence (Pearl 2000). In particular, SCMs are useful for reasoning about direct effects which measure how a change in one variable affects another variable while holding all other variables constant (Pearl 2012). The identification and estimation of direct effects are important in many applications, *e.g.*, epidemiologists are interested in measuring how smoking affects lung cancer risk without being mediated by genetic susceptibility (Zhou et al. 2021); ecologists are usually focus on understanding direct effects such as competition, herbivory, and predation (Connell 1961); IT experts can localize root causes of system failures by comparing the direct impact of different components on each other before and after the failure (Assaad, Ez-Zejjari, and Zan 2023).

In the framework of (non-dynamic) SCMs, assuming linearity and no hidden confounding and given a causal graph

which qualitatively represents causal relations between different variables, the direct effect between two variables is always identifiable and there exists a complete graphical tool, called the single-door criterion (Pearl 1998; Spirtes et al. 1998; Pearl 2000) that finds all possible adjustment sets that allow to estimate the direct effect from data. These results are directly applicable in dynamic SCMs (Rubenstein et al. 2018) given an acyclic full-time causal graph—which qualitatively represents all causal relations between different temporal instants of the dynamic SCM—and assuming consistency throughout time. However, in many dynamic systems, experts have difficulties in building a full-time causal graph (Aït-Bachir et al. 2023), while they can usually build a summary causal graph (Assaad, Devijver, and Gaussier 2022) which is an abstraction of the full-time causal graph where temporal information is omitted and cycles are allowed. So far, the problem of identifying direct effects has solely been tackled for summary causal graphs with no cycles exceeding a size of 2 (Assaad, Ez-Zejjari, and Zan 2023). Specifically, it has been shown that in the absence of cycles greater than 2, direct effects are always identifiable, and in these cases, an adjustment set has been proposed for estimating the direct effect from data. However, in many applications, there exist cycles with a size greater than 2 (Veilleux 1979; Staplin et al. 2016).

In this work, we focus on the identifiability of direct effects from summary causal graphs without restricting the size of cycles. Our main contribution is twofold. First, we give a complete identifiability result which characterizes all cases for which a direct effect is graphically identifiable from a summary causal graph. Then, we present two finite adjustment sets that can be used to estimate the direct effect from data whenever it is identifiable.

The remainder of the paper is organized as follows: in the next section, we give the definitions of direct effect and summary causal graph and recall all necessary graphical preliminaries. In the section that follows it, we present the complete identifiability result in addition to a weaker but interesting result that is implied by the complete identifiability result. Then, in another section, we provide two finite adjustment sets that can be used to estimate the direct effect from data whenever it is identifiable. Finally, in the last section, we conclude the paper while elaborating on promising directions for further research.

## Problem Setup

In this section, we first introduce some terminology, tools, and assumptions which are standard for the major part. Then, we formalize the problem we are going to solve. Without any opposite mention, in the following, a capital letter correspond to either to a variable or a vertex and a calligraphic letter correspond to a set with the exception of the set of natural numbers and the set of all integers which are respectively represented by  $\mathbb{N}$  and  $\mathbb{Z}$ .

In this work, we consider that a dynamic system can be represented by a linear dynamic SCM.

**Definition 1** (Linear Dynamic SCM). *Considering a finite set of observed times series  $\mathcal{V}$ , a linear dynamic SCM is a set of equations in which each instant  $t \in \mathbb{Z}$  of a time series (e.g.,  $Y_t$ ) is defined as a linear function of past instants of itself (e.g.,  $Y_{t-\gamma}$ ,  $\gamma > 0$ ), past or present instants of other times series (e.g.,  $X_{t-\gamma}$ ,  $X \neq Y$ ,  $\gamma \geq 0$ ) and of some unobserved noise (e.g.,  $\xi_{Y_t}$ ):*

$$Y_t := \sum_{\gamma > 0} \alpha_{Y_{t-\gamma}, Y_t} * Y_{t-\gamma} + \sum_{X \neq Y, \gamma \geq 0} \alpha_{X_{t-\gamma}, Y_t} * X_{t-\gamma} + \xi_{Y_t},$$

where any coefficient  $\alpha$  can be zero and for each  $Y \in \mathcal{V}$  the noises  $\{\xi_{Y_t}, t \in \mathbb{Z}\}$  are identically distributed.

Note that in nonlinear SCMs, the direct effect is not uniquely defined, potentially varying depending on the values taken by other variables (Pearl 2000). In contrast, linear SCMs do not encounter such issues. In the following, we give the definition of direct effects in the case of linear dynamic SCMs.

**Definition 2** (Direct Effect, (Pearl 2000)). *In a linear dynamic SCM, the direct effect of  $X_{t-\gamma_{xy}}$  on  $Y_t$  is fully specified by the structural coefficient<sup>1</sup>  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ .*

In the following, we explicitly state further assumptions we make in this paper.

**Assumption 1** (No hidden Confounding, (Spirtes, Glymour, and Scheines 2000; Pearl 2000)). *The noise variables in the linear dynamic SCM are assumed to be jointly independent (i.e.,  $\forall X, Y \in \mathcal{V} \forall t', t \in \mathbb{Z}, X_{t'} \neq Y_t \implies \xi_{X_{t'}} \perp \xi_{Y_t}$ ).*

**Assumption 2** (Stationarity). *The causal mechanisms of the system considered do not change over time and therefore,  $\forall X \neq Y \in \mathcal{V}, \forall t - \gamma \leq t \in \mathbb{Z}, \alpha_{X_{t-\gamma}, Y_t} = \alpha_{X_{t-\gamma+1}, Y_{t+1}}$ , and  $\forall t - \gamma < t \in \mathbb{Z}, \alpha_{Y_{t-\gamma}, Y_t} = \alpha_{Y_{t-\gamma+1}, Y_{t+1}}$ . There exists a maximum lag  $\gamma_{max}$  of a dynamic SCM defined as  $\gamma_{max} := \max\{\gamma \in \mathbb{N} | \exists X, Y \in \mathcal{V}, \alpha_{X_{t-\gamma}, Y_t} \neq 0\}$ .*

**Definition 3** (Full-Time Causal Graph). *Considering a finite set of times series  $\mathcal{V}$  and a dynamic SCM, one can define the full-time causal graph (FTCG)  $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$  associated to the dynamic SCM in the following way:*

$$\begin{aligned} \mathcal{V}_f &:= \{Y_t \mid \forall Y \in \mathcal{V}, \forall t \in \mathbb{Z}\}, \\ \mathcal{E}_f &:= \{X_{t-\gamma} \rightarrow Y_t \mid \forall X_{t-\gamma}, Y_t \in \mathcal{V}_f \\ &\quad \text{such that } \alpha_{X_{t-\gamma}, Y_t} \neq 0\}. \end{aligned}$$

In addition, for every vertex  $Y_t \in \mathcal{V}_f$  we note:

<sup>1</sup>In Sewall Wright's terminology the structural coefficient is called path coefficient (Wright 1920, 1921).

- $Par(Y_t, \mathcal{G}_f) = \{U_{t'} \in \mathcal{V}_f | U_{t'} \rightarrow Y_t \text{ in } \mathcal{E}_f\}$ ,
- $Anc(Y_t, \mathcal{G}_f) = \bigcup_{n \in \mathbb{N}} P_n$  where  $P_0 = \{Y_t\}$  and  $P_{k+1} = \bigcup_{U_{t'} \in P_k} Par(U_{t'}, \mathcal{G}_f)$ , and
- $Desc(Y_t, \mathcal{G}_f) = \bigcup_{n \in \mathbb{N}} C_n$  where  $C_0 = \{Y_t\}$  and  $C_{k+1} = \bigcup_{U_{t'} \in C_k} \{W_{t''} \in \mathcal{V}_f | U_{t'} \rightarrow W_{t''} \text{ in } \mathcal{E}_f\}$ .

**Assumption 3** (Acyclicity of the FTCTG). *Every FTCTG is acyclic.*

The FTCTG is the most natural way to represent a dynamic SCM but it is unpractical as it is infinite. Of course, given Assumption 2, it is possible to represent an FTCTG in a finite graph, but even in this case sometimes, it is difficult to construct this type of graphs from prior knowledge due to the uncertainty regarding temporal lags. Furthermore, causal discovery methods are not always efficient (Ait-Bachir et al. 2023) due to the strong assumptions they require that are not always satisfied in real applications, so constructing this type of graphs from data is not always a valid option. Therefore, it is much more reliable to construct an abstraction of this type of graphs which does not contain temporal information. This abstraction is usually referred to as a summary causal graph (Assaad, Devijver, and Gaussier 2022).

**Definition 4** (Summary Causal Graph). *Considering a finite set of times series  $\mathcal{V}$  and an FTCTG  $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ , one can define the summary causal graph (SCG)  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$  compatible with the FTCTG in the following way:*

$$\begin{aligned} \mathcal{V}_s &:= \mathcal{V}, \\ \mathcal{E}_s &:= \{X \rightarrow Y \mid \forall X, Y \in \mathcal{V}, \\ &\quad \exists t' \leq t \in \mathbb{Z} \text{ such that } X_{t'} \rightarrow Y_t \in \mathcal{E}_f\}. \end{aligned}$$

In addition, for every vertex  $Y \in \mathcal{V}_s$  we note:

- $Par(Y, \mathcal{G}_s) = \{U \in \mathcal{V}_s | U \rightarrow Y \text{ or } U \rightleftharpoons Y \text{ in } \mathcal{E}_s\}$ ,
- $Anc(Y, \mathcal{G}_s) = \bigcup_{n \in \mathbb{N}} P_n$  where  $P_0 = \{Y\}$  and  $P_{k+1} = \bigcup_{U \in P_k} Par(U, \mathcal{G}_s)$ , and
- $Desc(Y, \mathcal{G}_s) = \bigcup_{n \in \mathbb{N}} C_n$  where  $C_0 = \{Y\}$  and  $C_{k+1} = \bigcup_{U \in C_k} \{U \in \mathcal{V}_s | U \rightarrow W \text{ or } U \rightleftharpoons W \text{ in } \mathcal{E}_s\}$ .

Notice that an SCG may have cycles and in particular two arrows in opposite directions, i.e., if in the FTCTG we have  $X_{t'} \rightarrow Y_t$  and  $Y_{t''} \rightarrow X_t$  then in the SCG we have  $X \rightleftharpoons Y$ .

The abstraction of SCGs entails that, even though there is exactly one SCG compatible with a given FTCTG, there are in general several FTCTGs compatible with a given SCG. For example, the FTCTGs in Figure 1b and 1c correspond to the same SCG given in Figure 1a.

Now that we have defined direct effects and SCGs, we can formally define graphical identifiability.

**Definition 5** (Graphical Identifiability of a Direct Effect from an SCG). *The direct effect of a time instant  $X_{t-\gamma_{xy}}$  on another time instant  $Y_t$ , i.e.,  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ , in a linear dynamic SCM is said to be identifiable from an SCG if the quantity  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  can be computed uniquely from the observed distribution without any further assumption on the distribution and without knowing the FTCTG.*

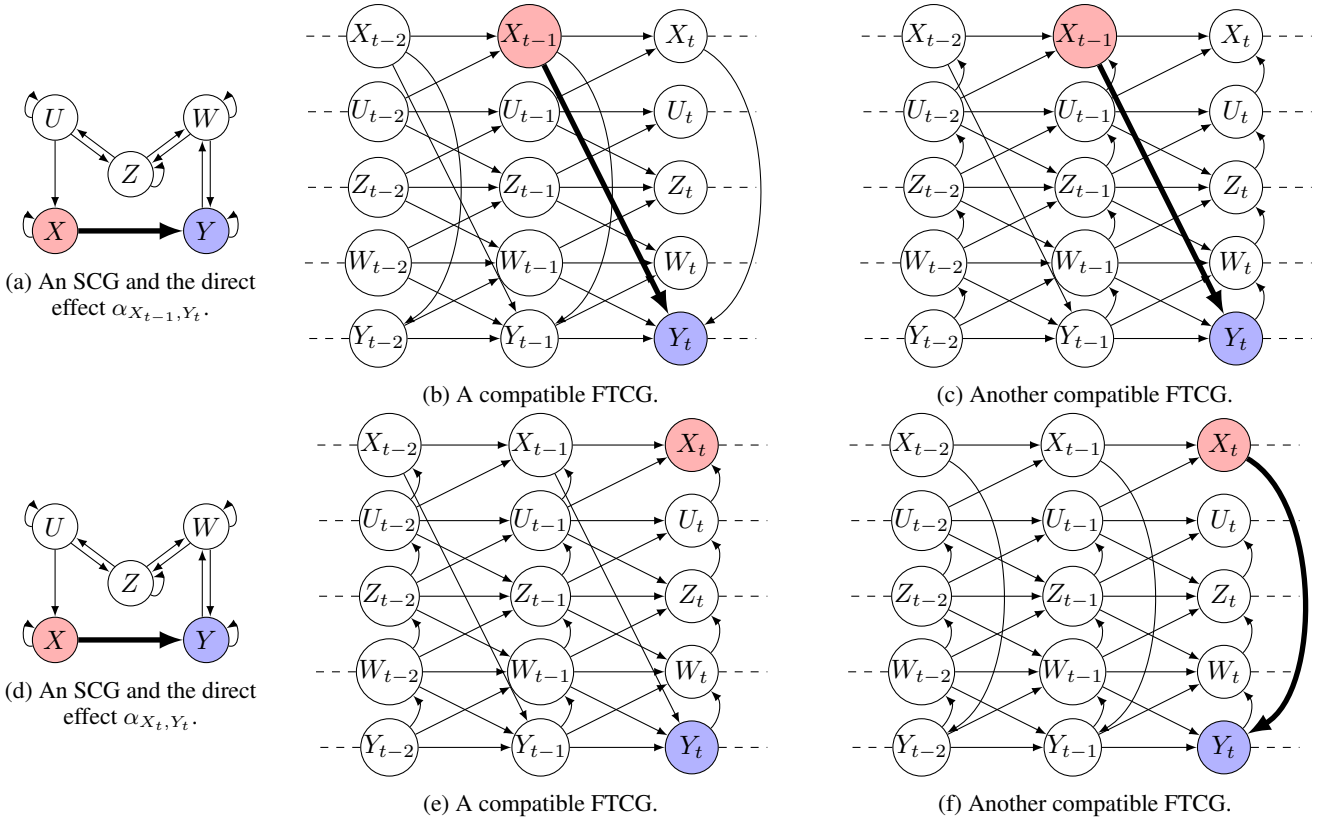


Figure 1: An example of an SCG in (a and b) with four of its compatible FTCCGs in (b), (c), (e) and (f). The pair of red and blue vertices in the SCG and in the FTCCGs represents the direct effect of interest, *i.e.*,  $\alpha_{X_t, Y_t}$  and  $\alpha_{X_{t-1}, Y_t}$ . In this example,  $\alpha_{X_{t-1}, Y_t}$  is non-identifiable given the SCG because for this direct effect, in the FTCCG in (b),  $X_t$  should be in every valid adjustment set but  $X_t$  should not be in any valid adjustment set in the FTCCG in (c). Similarly,  $\alpha_{X_t, Y_t}$  is non-identifiable given the SCG because for this direct effect (equal to zero), in the FTCCG in (e), at least one vertex in  $\{U_t, Z_t, W_t\}$  should be in every valid adjustment set but none of the vertices in  $\{U_t, Z_t, W_t\}$  should be in any valid adjustment set in the FTCCG in (f).

Note that given the true FTCCG, computing the direct effect uniquely from the observed distribution without any further assumption on the distribution usually consists of removing all confounding bias and non-direct effects by adjusting on a set that do not create any selection bias. Removing confounding bias usually necessitates adjusting on some suitable ancestors of the cause or the effect, removing non-direct effects usually consists of adjusting on some suitable ancestors of the effect that are not ancestors of the cause, and creating selection bias consists of adjusting on descendants of the effect. In the linear setting, given such an adjustment set  $Z_f$ , the direct effect  $\alpha_{X_{t-\gamma}, Y_t}$  can be estimated using the partial linear regression coefficient  $r_{X_{t-\gamma}, Y_t, Z_f}$  where  $r_{X_{t-\gamma}, Y_t, Z_f}$  represents the correlation between  $X_{t-\gamma}$  and  $Y_t$  after  $Z_f$  is "partialled out" (Pearl 2000). The difficulty of this paper lies in finding such an adjustment set  $Z_f$  using the SCG and without knowing the true FTCCG. This consists in finding an adjustment set that is valid for every FTCCG compatible with the SCG.

In the following, we recall several preliminaries related to FTCCGs and SCGs.

**Definition 6** (Paths in FTCCGs). A path between two vertices

$X_{t'}$  to  $Y_t$  is an ordered sequence of vertices denoted as  $\pi_f = \langle V_{t_1}^1, \dots, V_{t_n}^n \rangle$  such that  $V_{t_1}^1 = X_{t'}$ ,  $V_{t_n}^n = Y_t$  and  $\forall 1 \leq i < n$ ,  $V_{t_i}^i$  and  $V_{t_{i+1}}^{i+1}$  are adjacent (*i.e.*,  $V_{t_i}^i \rightarrow V_{t_{i+1}}^{i+1}$  or  $V_{t_i}^i \leftarrow V_{t_{i+1}}^{i+1}$ ) and  $\forall 1 \leq i < j \leq n$ ,  $V_{t_i}^i \neq V_{t_j}^j$ . In cases where orientations are crucial in the context under discussion, in the path  $\pi_f$ , we will replace commas with orientations. In this paper, a path  $\pi_f$  from  $X_{t'}$  to  $Y_t$  is said to be non-direct if  $\pi_f \neq \langle X_{t'} \rightarrow Y_t \rangle$ .

**Definition 7** (Paths in SCGs). A path between two vertices  $X$  to  $Y$  is an ordered sequence of vertices denoted as  $\pi_s = \langle V^1, \dots, V^n \rangle$  such that  $V^1 = X$ ,  $V^n = Y$  and  $\forall 1 \leq i < n$ ,  $V^i$  and  $V^{i+1}$  are adjacent (*i.e.*,  $V^i \rightarrow V^{i+1}$  or  $V^i \leftarrow V^{i+1}$  or  $V^i \rightleftharpoons V^{i+1}$ ) and  $\forall 1 \leq i < j \leq n$ ,  $V^i \neq V^j$ . In cases where orientations are crucial in the context under discussion, in the path  $\pi_s$ , we will replace commas with orientations. In this paper, a path  $\pi_s$  from  $X$  to  $Y$  is said to be non-direct if  $\pi_s \neq \langle X \rightarrow Y \rangle$ .

A path is said to be directed if it only contains  $\rightarrow$  and  $\rightleftharpoons$ .

**Definition 8** (Cycles in SCGs). In an SCG, a cycle is an ordered sequence of vertices  $\pi_s = \langle V^1, \dots, V^n \rangle$  such that:

- $V^1 = V^n$ ,

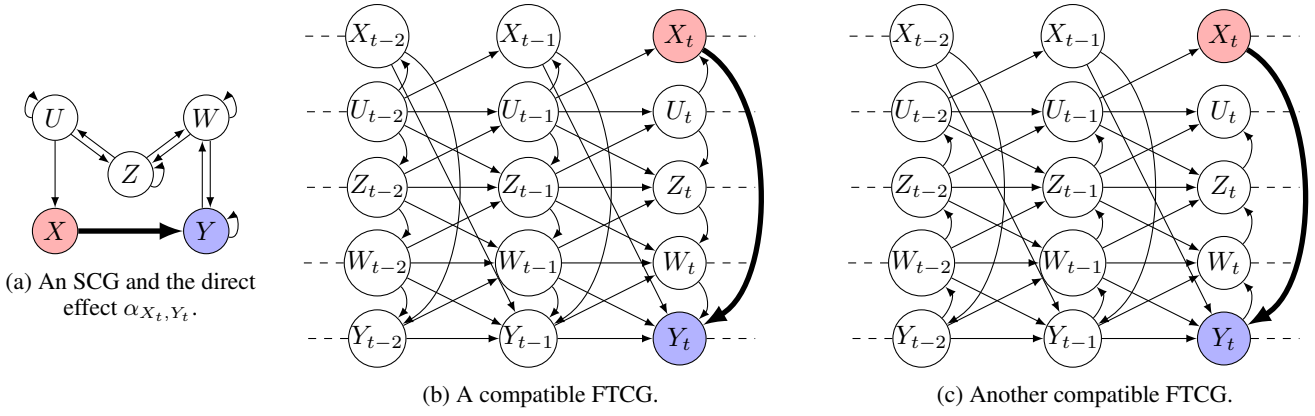


Figure 2: An example of an SCG in (a) with two of its compatible FTCCGs in (b) and (c). The pair of red and blue vertices in the SCG and in the FTCCGs represents the direct effect of interest, *i.e.*,  $\alpha_{X_t, Y_t}$ . In this example,  $\alpha_{X_t, Y_t}$  is non-identifiable given the SCG because for this direct effect at least one vertex in  $\{U_t, Z_t, W_t\}$  should be in every valid adjustment set for the first FTCCG but none of the vertices in  $\{U_t, Z_t, W_t\}$  should be in any valid adjustment set for the second FTCCG.

- $\forall 1 \leq i < n$ ,  $V^i \rightarrow V^{i+1}$  or  $V^i \rightleftharpoons V^{i+1}$ , and
- $\forall 1 \leq i < j \leq n$ ,  $V^i = V^j \implies i = 1$  and  $j = n$ .

The set of cycles with endpoints  $Y \in \mathcal{V}_s$  in an SCG  $\mathcal{G}_s$  is written  $Cycles(Y, \mathcal{G}_s)$ .

An important graphical notion used in causal reasoning is the notion of blocked path (Pearl 1998) for which the classical definition was introduced for directed acyclic graphs and thus can be directly used for FTCCGs under Assumption 3. Note that an adjustment set that removes all confounding bias and non-direct effects and that do not create any selection bias between  $X_{t-\gamma_{xy}}$  and  $Y_t$  consists of non descendants of  $Y_t$  different from  $X_{t-\gamma_{xy}}$  that block all non-direct paths between  $X_{t-\gamma_{xy}}$  and  $Y_t$ . In the following, we firstly give the definition of blocked path in FTCCGs and then we give a similar notion for SCGs.

**Definition 9** (Blocked Path in FTCCGs). *In an FTCCG  $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ , a path  $\pi_f = \langle V_{t_1}^1, \dots, V_{t_n}^n \rangle$  is said to be blocked by a set of vertices  $\mathcal{Z}_f \subseteq \mathcal{V}_f$  if:*

1.  $V_{t_1}^1 \in \mathcal{Z}_f$  or  $V_{t_n}^n \in \mathcal{Z}_f$ , or
2.  $\exists 1 < i < n$  such that  $V_{t_{i-1}}^{i-1} \leftarrow V_{t_i}^i$  or  $V_{t_i}^i \rightarrow V_{t_{i+1}}^{i+1}$  and  $V_{t_i}^i \in \mathcal{Z}_f$ , or
3.  $\exists 1 < i < n$  such that  $\rightarrow V_{t_i}^i \leftarrow$  and  $Desc(V_{t_i}^i, \mathcal{G}_f) \cap \mathcal{Z}_f = \emptyset$ .

A path which is not blocked is said to be active. When the set  $\mathcal{Z}_f$  is not specified, it is implicit that we consider  $\mathcal{Z}_f = \emptyset$ .

The classical definition of blocked path is usually used in directed acyclic graph and since the SCG compatible with an FTCCG can be cyclic, one needs to adapt it. (Spirtes 1993) explains that under the linearity assumption the notion of blocked path is readily extended. Moreover, (Forré and Mooij 2017) introduced a more recent and general (non-parametric and allow for hidden confounding) adaptation called  $\sigma$ -blocked path. Here we will focus on the definition used in (Spirtes 1993) since we assume linearity but we adapt it such that in SCGs extremity vertices of a path do not necessarily block the path.

**Definition 10** (Blocked Path in SCGs). *In an SCG  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ , a path  $\pi_s = \langle V^1, \dots, V^n \rangle$  is said to be blocked by a set of vertices  $\mathcal{Z}_s \subseteq \mathcal{V}_s$  if:*

1.  $\exists 1 < i < n$  such that  $V^{i-1} \leftarrow V^i$  or  $V^i \rightarrow V^{i+1}$  and  $V^i \in \mathcal{Z}_s$ , or
2.  $\exists 1 < i \leq j < n$  such that  $\langle V^{i-1} \rightarrow V^i \rightleftharpoons \dots \rightleftharpoons V^j \leftarrow V^{j+1} \rangle$  and  $Desc(V^i, \mathcal{G}_s) \cap \mathcal{Z}_s = \emptyset$ .

A path which is not blocked is said to be active. When the set  $\mathcal{Z}_s$  is not specified, it is implicit that we consider  $\mathcal{Z}_s = \emptyset$ .

Condition 1 in Definition 10 is a direct adaptation of condition 2 in Definition 9. Condition 2 of Definition 10 is explained by the fact that for a path  $\pi_s = \langle V^1, \dots, V^n \rangle$  in an SCG  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$  and a set of vertices  $\mathcal{Z}_s \subseteq \mathcal{V}_s$ , if  $\exists 1 < i \leq j < n$  such that

$$\langle V^{i-1} \rightarrow V^i \rightleftharpoons \dots \rightleftharpoons V^j \leftarrow V^{j+1} \rangle \text{ and } Desc(V^i, \mathcal{G}_s) \cap \mathcal{Z}_s = \emptyset$$

then  $\forall \pi_f = \langle V_{t_1}^1, \dots, V_{t_n}^n \rangle$ ,  $\exists 1 < i \leq k \leq j < n$  such that  $\rightarrow V_{t_k}^k \leftarrow$  and  $Desc(V_{t_k}^k, \mathcal{G}_s) \cap \mathcal{Z}_s = \emptyset$

therefore  $Desc(V_{t_k}^k, \mathcal{G}_f) \cap \mathcal{Z}_f = \emptyset$  where  $\mathcal{Z}_f \subseteq \{V_{t'} | V \in \mathcal{Z}_s, t' \in \mathbb{Z}\}$ . Notice that there is no adaptation of condition 1 as having  $V^1 \in \mathcal{Z}_s$  or  $V^n \in \mathcal{Z}_s$  does not mean that instants of interests  $V_{t_1}^1$  or  $V_{t_n}^n$  which are endpoints of compatible paths of interests are in  $\mathcal{Z}_f \subseteq \{V_{t'} | V \in \mathcal{Z}_s, t' \in \mathbb{Z}\}$ . Moreover, in this paper we are interested in the direct effect of  $V_{t_1}^1 = X_{t-\gamma_{xy}}$  on  $V_{t_n}^n = Y_t$  and therefore we cannot adjust on them and thus we will always have  $V_{t_1}^1, V_{t_n}^n \notin \mathcal{Z}_f$ .

Note that a set  $\mathcal{Z}_s$  that blocks all paths between two vertices  $X$  and  $Y$  in an SCG does not necessarily have a compatible finite set  $\mathcal{Z}_f \subseteq \{V_{t'} | V \in \mathcal{Z}_s, t' \in \mathbb{Z}\}$  that block all paths between two vertices  $X_{t-\gamma_{xy}}$  and  $Y_t$  in every FTCCG compatible with the given SCG. For example, in Figure 1d,  $U$  blocks all paths between  $X$  and  $Z$  but in Figure 1e,  $\exists i \in \mathbb{N}$  such that  $\{U_{t-i}, \dots, U_t\}$  blocks all paths between  $X_t$  and  $Z_t$  because there will always be an active path between them passing by  $U_{t-i-1}$ .

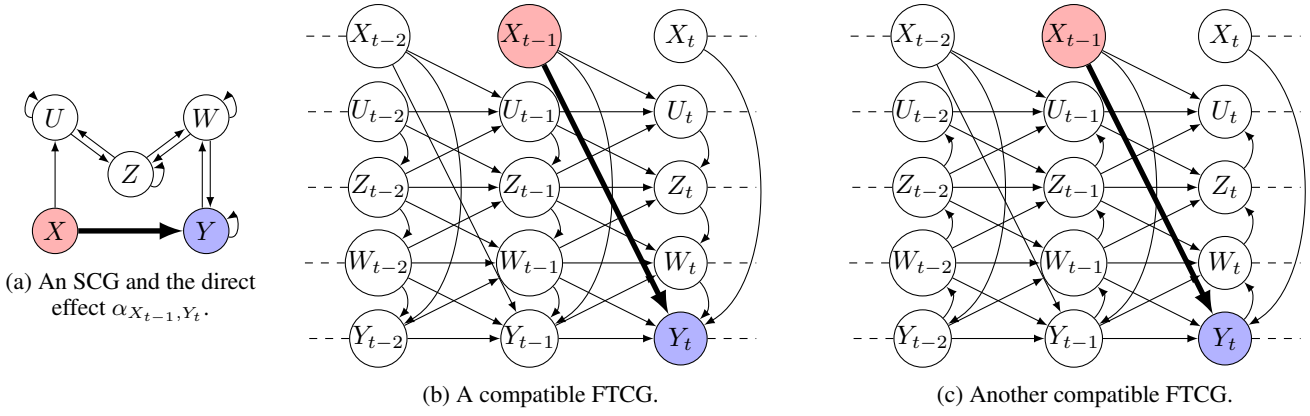


Figure 3: An example of an SCG in (a) with two of its compatible FTCCGs in (b) and (c). The pair of red and blue vertices in the SCG and in the FTCCGs represents the direct effect of interest, i.e.,  $\alpha_{X_{t-1}, Y_t}$ . In this example,  $\alpha_{X_{t-1}, Y_t}$  is non-identifiable given the SCG because for this direct effect at least one vertex in  $\{U_t, Z_t, W_t\}$  should be in every valid adjustment set for the first FTCCG but none of the vertices in  $\{U_t, Z_t, W_t\}$  should be in any valid adjustment set for the second FTCCG.

### Complete Graphical Identifiability from SCGs

In this section, we start by presenting the complete identifiability result followed by its proof as well as several toy examples to demonstrate it. Then we give a weaker but interesting result that is implied by the identifiability result.

**Theorem 1.** Let  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$  be an SCG that represents a linear dynamic SCM verifying Assumptions 1,2,3,  $\gamma_{max} \geq 0$  a maximum lag and  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  the direct effect of  $X_{t-\gamma_{xy}}$  on  $Y_t$  such that  $X, Y \in \mathcal{V}_s$  and  $X \neq Y$ . The direct effect  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  is non-identifiable from  $\mathcal{G}_s$  if and only if  $X \in Par(Y, \mathcal{G}_s)$  and one of the following conditions holds:

1.  $X \in Desc(Y, \mathcal{G}_s)$  and  $\exists C \in Cycles(X, \mathcal{G}_s)$  with  $Y \notin C$ , or
2. There exists an active non-direct path  $\pi_s = \langle V^1, \dots, V^n \rangle$  from  $X$  to  $Y$  in  $\mathcal{G}_s$  such that  $\langle V^2, \dots, V^{n-1} \rangle \subseteq Desc(Y, \mathcal{G}_s)$ , and one of the following conditions holds:
  - (a)  $\gamma_{xy} = 0$ , or
  - (b)  $\gamma_{xy} > 0$ ,  $n \geq 3$  and  $\nexists 1 \leq i < n, V^i \leftarrow V^{i+1}$  (i.e.,  $\forall i, V^i \rightarrow V^{i+1}$  or  $V^i \rightleftarrows V^{i+1}$ ).

In the following, we provide an additional definition that will be utilized in proving Theorem 1, followed by the proof itself. A more detailed proof of Theorem 1 is also available in the supplementary materials.

**Definition 11** (A First Finite Adjustment Set). Consider an SCG  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ , a maximal lag  $\gamma_{max}$ , two vertices  $X$  and  $Y$  with  $X \in Par(Y, \mathcal{G}_s)$  and a lag  $\gamma_{xy}$ .  $\mathcal{Z}_f^{Def11} = \mathcal{A}_{\leq t} \cup \mathcal{D}_{< t}$  is an adjustment set relative to  $(X_{t-\gamma_{xy}}, Y_t)$  such that:

$$\begin{aligned} \mathcal{D}_{< t} &= \{V_{t'} \mid V \in Desc(Y, \mathcal{G}_s), \\ &\quad t - \gamma_{max} \leq t' < t\} \setminus \{X_{t-\gamma_{xy}}\} \quad \text{and} \\ \mathcal{A}_{\leq t} &= \{V_{t'} \mid V \in \mathcal{V}_s \setminus Desc(Y, \mathcal{G}_s), \\ &\quad t - \gamma_{max} \leq t' \leq t\} \setminus \{X_{t-\gamma_{xy}}\}. \end{aligned}$$

*Proof.* Firstly, let us prove the backward implication.

Let  $\gamma_{xy} \geq 0$  and  $X \in Par(Y, \mathcal{G}_s)$ .

- Suppose Condition 1 holds. Let  $C = \langle V^1, \dots, V^n \rangle$  be a cycle on  $X$  with  $Y \notin C$ . If  $\gamma_{xy} > 0$  and since  $X \in Par(Y, \mathcal{G}_s)$ , there exists a compatible FTCCG  $\mathcal{G}_f^1$  where there exists a directed path  $\pi_f^1 = \langle V_{t-\gamma_{xy}}^1 = X_{t-\gamma_{xy}}, V_t^2, \dots, V_t^n = X_t, Y_t \rangle$ . Obviously, each set of vertices  $\mathcal{Z}_f^1 \subset \mathcal{V}_f^1$  that blocks all non-direct paths between  $X_{t-\gamma_{xy}}$  and  $Y_t$  have to contain at least one vertex from  $\langle V_t^2, \dots, V_t^{n-1} \rangle$ . Similarly, since  $X \in Desc(Y, \mathcal{G}_s)$ , there exists a compatible FTCCG  $\mathcal{G}_f^2$  where  $\{V_t^2, \dots, V_t^n = X_t\} \subseteq Desc(Y_t, \mathcal{G}_f^2)$ . Thus each set of vertices  $\mathcal{Z}_f^2 \subset \mathcal{V}_f^2$  that blocks all non-direct paths between  $X_{t-\gamma_{xy}}$  and  $Y_t$  should not contain any vertex from  $\langle V_t^2, \dots, V_t^n \rangle$  in order to be a valid adjustment set. If  $\gamma_{xy} = 0$ , then since  $X \in Desc(Y, \mathcal{G}_s)$  there exists  $\langle V^n, \dots, V^1 \rangle$  a directed path from  $Y$  to  $X$  and there exists  $\pi_f = \langle V_t^1, \dots, V_t^n \rangle$  in a compatible FTCCG. If  $n = 2$ , then  $\pi_f = \langle V_t^1 \leftarrow V_t^2 \rangle$  cannot be blocked and if  $n \geq 3$ , then every set  $\mathcal{Z}_f$  that blocks this path contains a descendant of  $Y_t$  in another compatible FTCCG.
- Suppose Condition 2 holds. Let  $\pi_s = \langle V^1, \dots, V^n \rangle$  as described. If  $\pi_s = \langle X \rightleftarrows Y \rangle$  and  $\gamma_{xy} = 0$ , then there exists a compatible FTCCG  $\mathcal{G}_f$  in which the path  $\pi_f = \langle X_t \leftarrow Y_t \rangle$  exists and cannot be blocked. Else,  $n \geq 3$  and  $\pi_s$  is active so there exists a compatible FTCCG  $\mathcal{G}_f^1$  in which the path  $\pi_f = \langle V_{t-\gamma_{xy}}^1, V_t^2, \dots, V_t^{n-1}, V_t^n \rangle$  exists and is active. Notice that there exists another compatible FTCCG  $\mathcal{G}_f^2$  in which  $\{V_t^2, \dots, V_t^{n-1}\} \subseteq Desc(Y_t, \mathcal{G}_f^2)$  and every set  $\mathcal{Z}_f$  that blocks  $\pi_f$  in  $\mathcal{G}_f^1$  contains a vertex in  $Desc(Y_t, \mathcal{G}_f^2)$ .

Secondly, let us prove the forward implication. Let  $\mathcal{G}_f$  be an FTCCG,  $\gamma_{xy} \geq 0$  a lag,  $X_{t-\gamma_{xy}}$  and  $Y_t$  two vertices and  $\mathcal{G}_s$  the SCG compatible with  $\mathcal{G}_f$ . If  $X \notin Par(Y, \mathcal{G}_s)$  then there is no direct effect from  $X_{t-\gamma_{xy}}$  to  $Y_t$ , i.e.,  $\alpha_{X_{t-\gamma_{xy}}, Y_t} = 0$ . Suppose that  $X \in Par(Y, \mathcal{G}_s)$  and that the conditions of Theorem 1 are not verified and let us show that  $\mathcal{Z}_f^{Def11}$  is a

valid adjustment set to estimate the direct effect  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  by showing that  $\mathcal{Z}_f^{\text{Def 11}}$  verifies:

- $\mathcal{Z}_f^{\text{Def 11}} \cap (\text{Desc}(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$ , and
- $\mathcal{Z}_f^{\text{Def 11}}$  blocks every non-direct path from  $X_{t-\gamma_{xy}}$  to  $Y_t$  in every compatible FTCC  $\mathcal{G}_f$  of maximal lag at most  $\gamma_{max}$ .

The first point,  $\mathcal{Z}_f^{\text{Def 11}} \cap (\text{Desc}(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$  is immediate. In order to prove the second point, let  $\pi_f = \langle V_{t^1}^1, \dots, V_{t^n}^n \rangle$  be a non-direct path from  $X_{t-\gamma_{xy}}$  to  $Y_t$ . Notice the following properties:

- If  $\exists i_{max} = \max\{1 < i < n \mid t^i < t\}$  then  $t - \gamma_{xy} \leq t^{i_{max}} < t$  and  $V_{t^{i_{max}}}^{i_{max}} \rightarrow V_{t^{i_{max}+1}}^{i_{max}+1}$  thus  $\pi_f$  is blocked by  $\mathcal{Z}_f^{\text{Def 11}}$  as  $V_{t^{i_{max}}}^{i_{max}} \in \mathcal{Z}_f^{\text{Def 11}}$ .
- Else, if  $I = \{1 < i < n \mid t^i > t\} \neq \emptyset$  then  $\exists i \in I$  such that  $\rightarrow V_{t^i}^i \leftarrow$  in  $\pi_f$  and  $\pi_f$  is  $\mathcal{Z}_f^{\text{Def 11}}$ -blocked as  $\text{Desc}(V_{t^i}^i, \mathcal{G}_f) \cap \mathcal{Z}_f^{\text{Def 11}} = \emptyset$ .
- Else, if  $\exists i_{max} = \max\{1 < i < n \mid V^i \notin \text{Desc}(Y, \mathcal{G}_s)\}$  then  $t^{i_{max}} = t$  and  $V_{t^{i_{max}}}^{i_{max}} \rightarrow V_{t^{i_{max}+1}}^{i_{max}+1}$  thus  $\pi_f$  is blocked by  $\mathcal{Z}_f^{\text{Def 11}}$  as  $V_{t^{i_{max}}}^{i_{max}} \in \mathcal{Z}_f^{\text{Def 11}}$ .

Therefore, if  $\pi_f$  is  $\mathcal{Z}_f^{\text{Def 11}}$ -active then  $\langle V^2, \dots, V^n \rangle \subseteq \text{Desc}(Y, \mathcal{G}_s)$  and either  $\pi_f = \langle X_{t-\gamma_{xy}} \leftarrow Y_t \rangle$  which forces  $\gamma_{xy} = 0$  or  $n \geq 3$  and  $t^2 = \dots = t^{n-1} = t$ . Notice that  $\pi_f$  and  $\pi_s = \langle V^1, \dots, V^n \rangle$  are active and non-direct.

- Suppose  $\pi_f = \langle X_t \leftarrow Y_t \rangle$  ( $\gamma_{xy} = 0$ ). Given that  $X \in \text{Par}(Y, \mathcal{G}_s)$ , this means that  $\langle X \rightleftharpoons Y \rangle$  is a path in  $\mathcal{G}_s$  and thus Condition 2a is verified.
- Suppose  $n \geq 3$  and  $t^2 = \dots = t^{n-1} = t$ . We make the following *observation*: if  $\gamma_{xy} > 0$ , then  $t - \gamma_{xy} = t^1 < t^2 = t$  so  $V_{t^1}^1 \rightarrow V_{t^2}^2$  and given that  $\pi_f$  is active,  $\forall 1 \leq i < n$ ,  $V_{t^i}^i \rightarrow V_{t^{i+1}}^{i+1}$  and  $\nexists 1 \leq i < n$ ,  $V^i \leftarrow V^{i+1}$ . In addition, since  $\pi_f$  is a path and  $t^2 = \dots = t^{n-1} = t$ , either  $\pi_s$  is a path or  $\gamma_{xy} > 0$  and  $\exists 1 < i < n$  such that  $V^i = X$  and  $\forall 1 \leq j_1 < j_2 \leq n$ ,  $V^{j_1} = V^{j_2} \implies j_1 = 1$  and  $j_2 = i$ .
  - Suppose  $\pi_s$  is a path. In the case  $\gamma_{xy} = 0$ , Condition 2a would be verified. In the case  $\gamma_{xy} > 0$ , by the previous *observation*, Condition 2b would be verified.
  - Suppose  $\gamma_{xy} > 0$  and  $\exists 1 < i < n$  such that  $V^i = X$  and  $\forall 1 \leq j_1 < j_2 \leq n$ ,  $V^{j_1} = V^{j_2} \implies j_1 = 1$  and  $j_2 = i$ . Notice that  $X = V^i \in \text{Desc}(Y, \mathcal{G}_s)$  and that, using the previous *observation*,  $Y \notin \langle V^1, \dots, V^i \rangle \in \text{Cycles}(X, \mathcal{G}_s)$ . Thus Condition 1 is verified.

In conclusion, when the conditions of Theorem 1 are not verified, there is no non-direct  $\mathcal{Z}_f^{\text{Def 11}}$ -active path  $\pi_f$  between  $X_{t-\gamma_{xy}}$  and  $Y_t$ .  $\square$

To graphically illustrate the backward implication of Theorem 1, we consider three different examples which respectively correspond to Figures 1, 2, and 3 and to Conditions 1, 2a, and 2b of the theorem.

**Example 1.** Given the SCG in Figure 1a,  $\alpha_{X_{t-1}, Y_t}$  is non-identifiable since Condition 1 of Theorem 1 is satisfied. This can be illustrated by looking at the two FTCCs in Figures 1b and 1c which are compatible with the given SCG. In the first FTCC, it is obvious that it is important to adjust on  $X_t$  in order to identify  $\alpha_{X_{t-1}, Y_t}$  (to block the path  $\langle X_{t-1}, X_t, Y_t \rangle$ ). However, in the other FTCC,  $X_t$  is a descendant of  $Y_t$  therefore it is important not to adjust on it. Which means that without knowing which is the true FTCC and given only the SCG,  $\alpha_{X_{t-1}, Y_t}$  is non-identifiable. Notice that  $\alpha_{X_{t-1}, Y_t}$  remains non-identifiable if we remove the cycle of size 2 on  $X$  and we replace  $X \leftarrow U$  by  $X \rightleftharpoons U$  since  $X \rightleftharpoons U$  induces a cycle of size 3. This cycle means that there might exist an active path  $\langle X_{t-1} \rightarrow U_t \rightarrow X_t \rightarrow Y_t \rangle$  in one FTCC and that  $\{U_t, X_t\}$  can be descendants of  $Y_t$  in another FTCC.

Similarly, given the SCG in Figure 1d,  $\alpha_{X_t, Y_t}$  is non-identifiable because in the first FTCC, at least one vertex in  $\{U_t, Z_t, W_t\}$  should be in every valid adjustment set but in the second FTCC, these vertices are descendants of  $Y_t$ .

**Example 2.** Given the SCG in Figure 2a (which is similar to the SCGs considered in Example 1 but where the cycle of size 2 on  $X$  is removed so Condition 1 of Theorem 1 is no longer satisfied),  $\alpha_{X_t, Y_t}$  is non-identifiable since Condition 2a of Theorem 1 is satisfied. This can be illustrated as in Example 1. Notice that in this case  $\alpha_{X_{t-1}, Y_t}$  becomes identifiable as the path  $\langle X_{t-1}, X_t, Y_t \rangle$  can no longer exist.

**Example 3.** Given the SCG in Figure 3a (which is similar to the SCGs considered in Example 2 but where the orientation between  $X$  and  $U$  is reversed),  $\alpha_{X_{t-1}, Y_t}$  is non-identifiable since Condition 2b of Theorem 1 is satisfied. This can be illustrated by looking at the two FTCCs in Figures 3b and 3c. In the first FTCC, at least one vertex in  $\{U_t, Z_t, W_t\}$  should be in every valid adjustment set but in the second FTCC, these vertices are descendants of  $Y_t$ . Notice that we have the same result for  $\alpha_{X_t, Y_t}$ .

We give additional examples of SCGs in Figure 4 where the direct effect  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  is identifiable for all  $\gamma_{xy}$ . Note that  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  remains identifiable in Figures 4a, 4b, 4c, 4d even if we replace  $X \rightarrow U$  or  $X \leftarrow U$  by  $X \rightleftharpoons U$ .

It is important to highlight that Theorem 1 encompasses the identifiability result presented in (Assaad, Ez-Zejjari, and Zan 2023). Specifically, when there are no cycles of size greater than 2 in the SCG, conditions 1 and 2 of the theorem are not satisfied, indicating the identifiability of the direct effect. Interestingly, Theorem 1 also shows that under some conditions we can directly know that there exist at least one non-identifiable instantaneous direct effect. This is given by the following Corollary.

**Corollary 1.** Let  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$  be an SCG. If there exists a cycle in  $\mathcal{G}_s$  of length strictly greater than 2, then  $\exists \alpha_{X_t, Y_t}$  which is non-identifiable.

*Proof.* Let  $\langle V^1, \dots, V^n \rangle$  be a cycle in  $\mathcal{G}_s$  with  $n \geq 3$ .  $\alpha_{V_t^1, V_t^2}$  from  $V_t^1$  to  $V_t^2$  is non-identifiable because the path  $\langle V^n, V^{n-1}, \dots, V^2 \rangle$  from  $V^1 = V^n$  to  $V^2$  in the SCG, verifies Condition 2a of Theorem 1.  $\square$



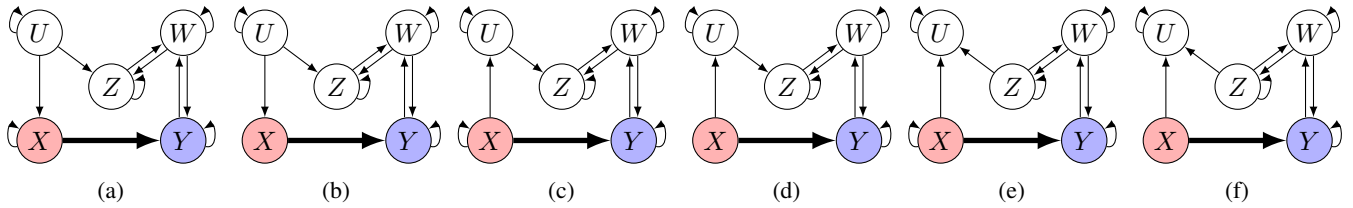


Figure 4: Examples of SCGs with 5 vertices where the direct effect  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  is identifiable for all  $\gamma_{xy}$ . Red and blue vertices respectively represent the cause and the effect we are interested in and the thick edge corresponds to the edge between them. All SCGs share the same skeleton, the edges  $X \rightarrow Y$ ,  $Y \leftrightarrow W$ , and  $Z \leftrightarrow W$  and the cycles of size 2 on  $Y, W, Z$  and  $U$ .

## Two Sound Adjustment Sets

In this section, we provide two finite adjustment sets that can be used to estimate the direct effect whenever it is identifiable. In an effort to be succinct we only give a sketch for the proof of the soundness of the second adjustment. For the full proof, please refer to the supplementary material.

For a given FTCCG  $\mathcal{G}_f$ , to estimate  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  from data, it is necessary and sufficient to adjust on a finite set  $\mathcal{Z}_f$  such that  $\mathcal{Z}_f \cap (\text{Desc}(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}, Y_t\}) = \emptyset$  which blocks every non-direct path from  $X_{t-\gamma_{xy}}$  to  $Y_t$ . Thus, given an SCG  $\mathcal{G}_s$ , one needs to find a set  $\mathcal{Z}_f$  such that  $\mathcal{Z}_f \cap (\text{Desc}(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}, Y_t\}) = \emptyset$  which blocks every non-direct path from  $X_{t-\gamma_{xy}}$  to  $Y_t$  in every FTCCG  $\mathcal{G}_f$  compatible with  $\mathcal{G}_s$  of maximal lag at most  $\gamma_{max}$ .

The following corollary formally indicates the soundness of the finite adjustment set defined in Definition 11.

**Corollary 2.** Let  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$  be an SCG,  $\gamma_{max} \geq 0$  a maximum lag. Consider two vertices  $X$  and  $Y$  such that  $X \in \text{Par}(Y, \mathcal{G}_s)$  and  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  is identifiable following Theorem 1. Then  $\mathcal{Z}_f^{\text{Def}11} = \mathcal{A}_{\leq t} \cup \mathcal{D}_{< t}$  as defined in Definition 11 is a valid adjustment set for  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ .

*Proof.* The proof of the forward implication of Theorem 1 proves this corollary.  $\square$

Note that many valid adjustment sets may exist. An estimator of direct effects based on any of these sets is unbiased, but the estimation variance may vary for different sets. Thus, it is interesting to search for many, and ideally all, such sets in order to optimize the estimation of direct effects. In the following, we give a smaller valid adjustment set.

**Definition 12** (A Second Finite Adjustment Set). Consider an SCG  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ , a maximal lag  $\gamma_{max}$ , two vertices  $X$  and  $Y$  with  $X \in \text{Par}(Y, \mathcal{G}_s)$  and a lag  $\gamma_{xy}$ .  $\mathcal{Z}_f^{\text{Def}12} = \mathcal{D}_{< t}^{\text{Anc}(Y)} \cup \mathcal{A}_{\leq t}^{\text{Anc}(Y)}$  is an adjustment set relative to  $(X_{t-\gamma_{xy}}, Y_t)$  such that:

$$\mathcal{D}_{< t}^{\text{Anc}(Y)} = \{V_{t'} \mid V \in \text{Anc}(Y, \mathcal{G}_s) \cap \text{Desc}(Y, \mathcal{G}_s), \\ t - \gamma_{max} \leq t' < t\} \setminus \{X_{t-\gamma_{xy}}\} \quad \text{and} \\ \mathcal{A}_{\leq t}^{\text{Anc}(Y)} = \{V_{t'} \mid V \in \text{Anc}(Y, \mathcal{G}_s) \setminus \text{Desc}(Y, \mathcal{G}_s), \\ t - \gamma_{max} \leq t' \leq t\} \setminus \{X_{t-\gamma_{xy}}\}.$$

The following proposition formally indicates the soundness of the finite adjustment set defined in Definition 12.

**Proposition 1.** Let  $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$  be an SCG,  $\gamma_{max} \geq 0$  a maximum lag. Consider two vertices  $X$  and  $Y$  such that  $X \in \text{Par}(Y, \mathcal{G}_s)$  and  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$  is identifiable following Theorem 1. Then  $\mathcal{Z}_f^{\text{Def}12}$  as defined in Definition 12 is a valid adjustment set for  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ .

*Proof Sketch.* Corollary 2 shows that the adjustment set  $\mathcal{Z}_f^{\text{Def}11}$  allows to estimate the direct effect  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ . Firstly, since  $\mathcal{Z}_f^{\text{Def}12} \subseteq \mathcal{Z}_f^{\text{Def}11}$  and  $\mathcal{Z}_f^{\text{Def}11} \cap (\text{Desc}(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$ , it is immediate that  $\mathcal{Z}_f^{\text{Def}12} \cap (\text{Desc}(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$ . Moreover, it is known that adjusting on ancestors of the cause  $Y_t$  is sufficient to estimate the direct effect in directed acyclic graphs. Therefore, it is intuitive that the restriction of  $\mathcal{Z}_f^{\text{Def}11}$  to ancestors of  $Y_t$ ,  $\mathcal{Z}_f^{\text{Def}11} \cap \{V_{t'} \mid V \in \text{Anc}(Y, \mathcal{G}_s), t' \leq t\} = \mathcal{Z}_f^{\text{Def}12}$ , allows to estimate the direct effect  $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ . The full proof is given in the supplementary materials.  $\square$

## Conclusion

In this paper, we developed a new graphical criteria for the identifiability of direct effects in linear dynamic structural causal models from summary causal graphs. Theorem 1 has important ramifications to the theory and practice of observational studies in dynamic systems. It implies that the key to graphical identifiability of the direct effect of  $X_{t-\gamma_{xy}}$  on  $Y_t$  from summary causal graphs lies not only in finding a set of non-descendants of  $Y$  in the summary causal graph that are able of blocking paths between  $X$  and  $Y$  but also in some descendants of  $Y$  in the case  $\gamma_{xy} > 0$ . Furthermore, in case of identifiability, we presented two adjustments sets that can be used to estimate the direct effects from data.

The finding of this paper should be useful for many applications such as root cause identification in dynamic systems and it should open new research questions. Namely, for future works, it would be interesting to have a criterion along with a completeness result describing every possible adjustment set. In addition, since in many real world applications causal relations can be nonlinear, it would be interesting to extend this work to nonlinear SCMs and consider non-parametric direct effects (Robins and Greenland 1992; Pearl 2001). Finally, as many other works, we assumed that the FTCCG is acyclic but we think that this assumption can be relaxed, so it would be interesting to formally check the validity of our results for cyclic FTCCGs.

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