Stability of Multi-Agent Learning in Competitive Networks: Delaying the Onset of Chaos

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Abstract

The behaviour of multi agent learning in competitive network games is often studied within the context of zero sum games, in which convergence guarantees may be obtained. However, outside of this class the behaviour of learning is known to display complex behaviours and convergence cannot be always guaranteed. Nonetheless, in order to develop a complete picture of the behaviour of multi agent learning in competitive settings, the zero sum assumption must be lifted.

Motivated by this we study the Q Learning dynamics, a popular model of exploration and exploitation in multi agent learning, in competitive network games. We determine how the degree of competition, exploration rate and network connectivity impact the convergence of Q Learning. To study generic competitive games, we parameterise network games in terms of correlations between agent payoffs and study the average behaviour of the Q Learning dynamics across all games drawn from a choice of this parameter. This statistical approach establishes choices of parameters for which Q Learning dynamics converge to a stable fixed point. Differently to previous works, we find that the stability of Q Learning is explicitly dependent only on the network connectivity rather than the total number of agents. Our experiments validate these findings and show that, under certain network structures, the total number of agents can be increased without increasing the likelihood of unstable or chaotic behaviours.

Introduction

Multi-Agent Learning in competitive games requires agents to maximise their individual, competing rewards whilst simultaneously exploring their actions to find optimal strategies. This leads to a highly non-stationary problem where agents must react to the changing behaviour of adversarial agents. The study of Multi-Agent Learning in competitive settings has achieved a number of successes within the context of zero-sum games and their network variants These games model perfect competition between agents, yielding an underlying structure which make them amenable for studying multi-agent learning. In particular it is known that certain learning dynamics asymptotically converge to an equilibrium in network zero-sum games (Ewerhart and Valkanova 2020; Leonardos, Piliouras, and Spendlove 2021; Kadan and Fu 2021) whilst others converge in time average (Anagnostides et al. 2022; Hadikhanloo et al. 2022; Bailey and Piliouras 2019).

Yet in practice the requirement of an arbitrary competitive game to exactly satisfy the zero-sum condition is restrictive. It therefore becomes important to study the behaviour of learning agents in arbitrary competitive games.The challenge in taking this step is that there are infinitely many realisations of games which can be considered competitive, making a case by case analysis intractable. Furthermore, the non-stationarity of learning in competitive games often leads to complex behaviours such as cycles (Galla 2011; Mertikopoulos, Papadimitriou, and Piliouras 2018) and even chaos (Griffin, Semonsen, and Belmonte 2022; Sato, Akiyama, and Farmer 2002). In fact, recent work has shown that chaotic dynamics occur in games even slightly perturbed from the zero-sum setting (Galla and Farmer 2013; Sato, Akiyama, and Farmer 2002). In addition, recent work (Hussain, Belardinelli, and Piliouras 2023; Sanders, Farmer, and Galla 2018) has shown that the ability of learning dynamics to reach an equilibrium with low exploration rates diminishes as the number of agents increases. These technical challenges present a strong barrier towards ensuring the convergence of learning in competitive games with many players.

However, in both (Hussain, Belardinelli, and Piliouras 2023) and (Sanders, Farmer, and Galla 2018) it was assumed that all agents are directly influenced by all other agents in the environment. In practice, however, this does not hold. Many ML applications, including Generative Adversarial Networks (GANs) enforce structured interactions between models (Hoang et al. 2018; LI et al. 2017). Furthermore, real world problems such as robotic systems (Hamann 2018; Shokri and Kebriaei 2020) and competitive game playing (Perolat et al. 2022) impose a communication network between agents. In economic settings, agents interact through social networks either online or in communities.

Model and Contributions Motivated by this, we study multi-agent learning in *network games*, in which interactions between agents are modelled by an underlying communication network. In this setting, we study the *Q-Learning* dynamic (Sato and Crutchfield 2003; Tuyls, Hoen, and Vanschoenwinkel 2006), a foundational model for studying the behaviour of agents who explore their state space, whilst simultaneously exploiting their rewards.

To address the issue of studying generic competitive games, we take a statistical approach towards our analysis which is inspired by the study of ecological systems (Opper and Diederich 1992; Galla 2006) and statistical mechanics (Hertz, Roudi, and Sollich 2016; De Giuli and Scalliet 2022). Rather than engaging in a case-by-case analysis, we parameterise competitive network games by the strength of anti-correlation between agent payoffs. Then, we perform a kind of *average case* analysis over all games which are drawn from this parameter. This process has shown a number of success in the analysis of learning in games (Galla and Farmer 2013; Sanders, Farmer, and Galla 2018; Coolen 2005) and neural networks (Coolen 2001; Kadmon and Sompolinsky 2015; Sompolinsky, Crisanti, and Sommers 1988).

Our analysis allows us to determine how the stability of the Q-Learning dynamics is influenced by the competitiveness of the game and on the exploration rate. In particular, we are able to define a *stability boundary* in terms of these parameters. We find that stable behaviours occur with low exploration rates in highly competitive games, such as zero-sum games. However, as the game deviates further from perfect competition, higher exploration rates are required to ensure convergent behaviours. We also analyse how the network itself influences Q-learning dynamics. We find that complex dynamics occurs frequently in strongly connected networks as the number of agents increases. By contrast, there are networks for which the total number of agents has no influence on the asymptotic convergence of learning.

The statistical approach requires taking the limit of large action spaces. As a result, our theoretical stability boundary holds exactly in this limit. However, we evaluate its predictions through rigorous numerical experiments in finite games, including representative examples from the literature. We find that the experiments agree with the theoretical results and show that the likelihood of complex learning dynamics depends explicitly on the network structure rather than the total number of agents. In fact it is found that, as long as the network is chosen appropriately, an arbitrarily large number of agents can be added to the multi-agent system without compromising convergence of learning.

Related Work A number of recent advances in the theory of learning in games have drawn from tools in evolutionary game theory (Hofbauer and Sigmund 1998, 2003; Tuyls 2023). Here, popular learning algorithms such as Q-Learning (Sutton and Barto 2018), Follow-the-Regularised-Leader (Shalev-Shwartz 2011) and Fictitious Play (Brown P 1949) can be approximated by continuous time models (Tuyls, Hoen, and Vanschoenwinkel 2006; Mertikopoulos and Sandholm 2016). Then, tools from the study of continuous dynamical systems (Strogatz 2015; Meiss 2007) can be used to analyse the asymptotic behaviour of the learning dynamic. In this manner, strong predictions can be made regarding convergence of learning in games (Krichene 2016; Abe, Sakamoto, and Iwasaki 2022; Bloembergen et al. 2015; Perolat et al. 2020). Notable successes of this method lie in *network zero sum games* (Cai et al. 2016; Abernethy, Lai, and Wibisono 2021) which models perfect competition between agents. In this setting, it is known that a number of learning dynamics converge asymptotically to an equilibrium (Leonardos, Piliouras, and Spendlove 2021; Ewerhart and Valkanova 2020; Kadan and Fu 2021).

By contrast, few guarantees can be provided outside of this class (Anagnostides et al. 2022). In fact, complex behaviour such as limit cycles (Imhof, Fudenberg, and Nowak 2005; Galla 2011; Mertikopoulos, Papadimitriou, and Piliouras 2018) and chaos (Sanders, Farmer, and Galla 2018; van Strien and Sparrow 2011) are known to be prevalent in generic games. To make progress on this front, learning in games has benefited from tools derived from the study of *disordered systems* (Hertz, Roudi, and Sollich 2016). The premise is that the exact choice of rewards in generic games has infinitely many possible realisations. Therefore, it becomes necessary to parameterise the game and then analyse the *average* behaviour of the learning dynamic under all games which share the same parameter. This analysis has been successful in the analysis of ecological systems (Opper and Diederich 1992; De Giuli and Scalliet 2022), Recurrent Neural Networks, (Coolen 2001) and evolutionary game theory (Coolen 2005; Chowdhury et al. 2021).

Most similar to our work are (Galla and Farmer 2013; Sanders, Farmer, and Galla 2018). In the former, the authors analysed two player competitive games and studied Experience Weighted Attraction (EWA) (Camerer and Ho 1999), a learning algorithm closely related to Q-Learning (Leonardos, Piliouras, and Spendlove 2021). They were able to derive a boundary in terms of game competitiveness and exploration rate between stable learning dynamics and complex dynamics. In (Sanders, Farmer, and Galla 2018), the authors extended this work towards multi-player games in which each agent interacts with all others. In this setting, it was shown the stability boundary depends on the total number of agents in the game. In particular, the region in which learning converges to fixed point seems to vanish as the number of agents increases. This result is supported by that of (Hussain, Belardinelli, and Piliouras 2023) in which a lower bound on exploration rates was determined so that Q-Learning dynamics converge to a unique equilibrium. Again it was shown that this lower bound increases with the number of agents.

In this work, we refine the result of (Sanders, Farmer, and Galla 2018) towards the setting of generic *network games*. Importantly, we find that the stability boundary is independent of the total number of agents in the game, but rather explicitly dependent on the connectivity of the network.

Preliminaries

Game Model

A network polymatrix game (henceforth network game) is described by a tuple $\mathcal{G} = (\mathcal{N}, \mathcal{E}, (A^{kl}, A^{lk})_{(k,l)\in\mathcal{E}})$. Here, $\mathcal{N}$ denotes a set of agents indexed by $k = 1, \ldots, N$, and $\mathcal{E}$ denotes the set of edges in an underlying network. In particular, $(k, l) \in \mathcal{E}$ if agents k and l are connected in the network. The set of *neighbours* of an agent k is denoted by

$\mathcal{N}_k = \{l \in \mathcal{N} : (k,l) \in \mathcal{E}\}$. Associated with each edge $(k,l) \in \mathcal{E}$ are the payoff matrices $A^{kl}, A^{lk} \in M_n(\mathbb{R})$, where n denotes the number of actions an agent can play. The set of actions playable by agent k is indexed by $i = 1, \ldots, n$. The strategy $\mathbf{x}_k$ of agent k is a probability distribution over their set of actions and so is chosen from the n-simplex $\Delta_n = \{\mathbf{x} \in \mathbb{R}^n : \sum_i x_{ki} = 1, x_{ki} \geq 0\}$. Then for any agent k, given the joint strategy $\mathbf{x}_{-k}$ of their opponents, their total reward r_{ki} is given by

$$r_{ki}(\mathbf{x}_{-k}) = \sum_{(k,l)\in\mathcal{E}} \left(A^{kl}\mathbf{x}_l\right)_i \tag{1}$$

In words, it is the sum of the payoff they receive in each of the games played along their edges. With this in place we can define an equilibrium for the game as follows.

Definition 1 (Quantal Response Equilibrium (QRE))**.** A joint mixed strategy $\bar{\mathbf{x}} \in \Delta$ is a *Quantal Response Equilibrium* (QRE) if, for all agents k and all actions $i \in S_k$

$$\bar{\mathbf{x}}_{ki} = \frac{\exp(r_{ki}(\bar{\mathbf{x}}_{-k})/T)}{\sum_{j\in S_k}\exp(r_{kj}(\bar{\mathbf{x}}_{-k})/T)}.$$

where $T \in [0, \infty)$ denotes the *exploration rate* of all agents.

The QRE (Camerer, Ho, and Chong 2004) is the prototypical extension of the Nash Equilibrium to the case of agents with bounded rationality, parameterised by the *exploration rate* T. In particular, the limit $T \to 0$ corresponds exactly to the Nash Equilibrium, whereas the limit $T \to \infty$ corresponds to a purely irrational case, where the QRE is unique and lies at the uniform distribution (McKelvey and Palfrey 1995).

Payoff Correlations As mentioned in the introduction, the entries of A^{kl}, A^{lk} can take any value in $\mathbb{R}$, making a case-by-case analysis intractable. We therefore move towards a kind of *average case* analysis. In particular, we construct ensembles of games at random which are parameterised by the strength of anti-correlation between opponent payoffs. Then we can analyse the *expected* behaviour of Q-Learning dynamics for different choices of this parameter. This approach has yielded a number of successes in analysing replicator dynamics (Opper and Diederich 1992) and learning in games (Galla and Farmer 2013; Sanders, Farmer, and Galla 2018).

Averaging over the infinite possibilities of payoff matrices which could arise in a game theoretic setting has the immediate effect of reducing the information available regarding the effect of the payoffs on stability. However, the primary concern of this work is to understand the effect of the network structure on stability. As such, it makes sense to average over other factors. Indeed, the relevance of the payoff matrices on the learning dynamics is an open and important topic for research and we point the interested reader to (Pangallo, Heinrich, and Farmer 2019; Pangallo et al. 2022) for rigorous treatments on the matter.

We must then ask how best to parameterise the payoffs. We do this by invoking the *maximum entropy* principle which is foundational to statistical mechanics (Galla and Farmer 2013; Hertz, Roudi, and Sollich 2016). This states that the natural choice for the payoff matrices is that which maximises entropy subject to given conditions. In particular, these conditions are

$$\begin{aligned}\mathbb{E}[A^{kl}_{ij}] &= 0, && \forall k \in \mathcal{N}, \forall i, j \in S_k \\ \mathbb{E}[\left(A^{kl}_{ij}\right)^2] &= 1, && \forall k \in \mathcal{N}, \forall i, j \in S_k \\ \mathbb{E}[\left(A^{kl}_{ij}\right)\left(A^{lk}_{ji}\right)] &= \Gamma, && \forall l \in N_k, \forall i \in S_k, j \in S_l\end{aligned} \tag{2}$$

Intuitively, these conditions enforce that payoffs have zero mean and positive variance and, crucially, enforces a correlation between the payoffs between two connected agents k, l parameterised by $\Gamma \in [-1, 0]$. In the Supplementary Material [1] we discuss, as an example, games which are drawn with $\Gamma = -1$. Here, the payoffs to each agent are exactly negatively correlated, corresponding to a zero-sum game. By contrast, when $\Gamma = 0$, the payoffs are completely uncorrelated. As such, Γ controls the *competitiveness* of the game. In line with the maximum entropy argument and previous literature (Galla and Farmer 2013; Sanders, Farmer, and Galla 2018), we draw the payoff matrices from a multivariate Gaussian distribution with mean and covariance defined as in (2). A careful treatment on why the multivariate distribution satisfies the maximum entropy argument can be found in (Galla and Farmer 2013). Furthermore, as we show in our experiments(and in line with previous studies on the analysis of random games (Galla and Farmer 2013; Sanders, Farmer, and Galla 2018)), the predictions made in the average case analysis of random games carries over strongly in experiments.

Learning Model

In this work, we analyse the *Q-Learning dynamic*, a prototypical model for determining optimal policies by balancing exploration and exploitation. In this model, each agent $k \in \mathcal{N}$ maintains a history of the past performance of each of their actions. This history is updated via the Q-update:

$$Q_{ki}(\tau+1) = (1-\alpha_k)Q_{ki}(\tau) + \alpha_k r_{ki}(\mathbf{x}_{-k}(\tau)),$$

where τ denotes the current time step, and $Q_{ki}(\tau)$ denotes the *Q-value* maintained by agent k about the performance of action $i \in S_k$. In effect Q_{ki} gives a discounted history of the rewards received when i is played, with $1 - \alpha_k$ as the discount factor.

Given these Q-values, each agent updates their mixed strategies according to the Boltzmann distribution, given by

$$x_{ki}(\tau) = \frac{\exp(Q_{ki}(\tau)/T)}{\sum_j \exp(Q_{kj}(\tau)/T)},$$

in which $T \in [0, \infty)$ is the *exploration rate* of all agents.

It was shown in (Tuyls, Hoen, and Vanschoenwinkel 2006; Sato and Crutchfield 2003) that a continuous time approximation of the Q-Learning algorithm could be written as

$$\frac{\dot{x}_{ki}}{x_{ki}} = r_{ki}(\mathbf{x}_{-k}) - \langle \mathbf{x}_k, r_k(\mathbf{x}_{-k})\rangle + T\sum_{j\in S_k} x_{kj}\ln\frac{x_{kj}}{x_{ki}}, \tag{QLD}$$

[1] https://arxiv.org/abs/2312.11943

which we call the *Q-Learning dynamics* (QLD). The fixed points of this dynamic coincide with the QRE of the game (Leonardos, Piliouras, and Spendlove 2021). We can rewrite the dynamic in the following form

$$\frac{\dot{x}_{ki}}{x_{ki}} = r_{ki}(\mathbf{x}_{-k}) - T \ln x_{kj} - \rho_k \tag{3}$$

$$\rho_k = \langle \mathbf{x}_k, r_k(\mathbf{x}_{-k}) \rangle - T \langle \mathbf{x}_k, \ln \mathbf{x}_k \rangle$$

$$r_{ki}(\mathbf{x}_{-k}) = \sum_{l \in N_k} \left(A^{kl} \mathbf{x}_l \right)_i . \tag{4}$$

In this light, $\rho_k = \rho_k(\mathbf{x}(t))$ can be seen as a normalisation factor which ensures that $\sum_{i \in S_k} x_{ki} = 1$. In addition, it is clear that the behaviour of the Q-Learning dynamics depend strongly on the choice of T, the structure of the edge set $\mathcal{E}$, and the payoffs A^{kl}. In (Sanders, Farmer, and Galla 2018), a variant of (QLD) was analysed in which the concept of the interaction network was not introduced. Rather, each agent would be required to play a single p-player game against all other agents in the environment. In their work, it was found that complex dynamics (cycles and chaos) becomes more prominent as the number of players in the game increases. This is similarly true of (Hussain, Belardinelli, and Piliouras 2023) in which (QLD) was analysed in arbitrary games, without imposing any structure on interactions between agents. The authors similarly concluded that, as the number of agents increases, higher exploration rates T are required to ensure convergence to a QRE.

The introduction of a communication network allows for the interactions between agents to be included in the study. In particular, we determine how the number of neighbours for each agent affects the stability of (QLD) and compare this with the total number of players. In this light, we make the following assumption.

Assumption 1. The agents all share the same number of neighbours N_0, i.e., $N_0 = |\mathcal{N}_1|, \ldots, |\mathcal{N}_N|$. In graph theoretic terms, we require that the network is *regular* (Ji and Egerstedt 2007).

Assumption 1 allows us to parameterise the connectivity of the network using only the number of neighbours N_0. Additionally it allows us to make a direct comparison of our results with that of (Sanders, Farmer, and Galla 2018) who study the case where all agents are connected, in our case this is $N_0 = N - 1$. Whilst the study of regular networks is certainly well motivated in the literature on multi-agent systems (Ji and Egerstedt 2007; Rahmani et al. 2009; Olfati-Saber, Fax, and Murray 2007), a fruitful direction for extending the work in this paper would be to introduce graph theoretic parameterisations (e.g. norms on the adjacency matrix) to include heterogeneously coupled networks in the analysis.

Statistical Analysis of Learning in Networks

In this section, we derive a necessary condition for (QLD) to converge in generic network games. The process for doing this is as follows. First, we average over the ensemble of games drawn using a particular choice of Γ. This allows us to define an *effective dynamic*, which reflects the expected behaviour of (QLD) over all games within the ensemble. Then, we determine a necessary condition for a fixed point of this dynamic to be stable. Similar calculations have been used to analyse replicator dynamics (Galla 2006; Opper and Diederich 1992) and variants of Q-Learning (Sanders, Farmer, and Galla 2018; Galla and Farmer 2013), as well as minority games (Coolen 2005) and recurrent neural networks (Coolen 2001). By following this process, we derive an estimate for the boundary between stable fixed points and other behaviours, such as limit cycles and chaos. This boundary is determined with respect to the payoff correlations Γ and the exploration rate T.

The analytic result requires applying techniques from statistical mechanics, where the theory holds exactly only in the limit of large payoff matrices, i.e., the number of actions $n \to \infty$. As shown in this Section the method allows us to isolate precisely how the number of neighbours N_0 and the total number of players N affects the stability of learning in arbitrary competitive games. The limitation is that the analytic result will overestimate the stability boundary for finite games. However, our numerical experiments show that the predictions made in the limit hold in practice for finite games.

Effective Dynamics

In our first step, we derive an *effective dynamics* which describes the expected behaviour of the Q-Learning dynamics averaged over all possible assignments of payoff matrices. This calculation is lengthy, so we report the full details in the Supplementary Material. The idea is to define a probability measure (Generating Functional described in the Appendix) over trajectories generated by Q-Learning dynamics, given some payoff matrices. Next, we invoke the assumption that the payoff matrices are drawn from a multivariate Gaussian with correlations parameterised by Γ. Using this, we find the average form of the probability measure over all games generated with a choice of Γ, which we call its *effective* form. It is here that we use the limit of large payoff matrices (number of actions $n \to \infty$). Finally, we identify the effective probability measure with an associated dynamical system, which we call the *effective* form for the Q-Learning dynamics. Intuitively, the *effective Q-Learning dynamics* (5) describes the average trajectories of Q-Learning over all, possibly infinite, games which are generated from a choice of Γ.

$$\frac{\dot{x}(t)}{x(t)} = N_0 \Gamma \int dt' G(t,t') x(t') - T \ln x(t) - \rho(t) + \sqrt{N_0} \eta(t), \tag{5}$$

Here, $\eta(t)$ is a Gaussian random variable which satisfies $\langle \eta(t), \eta(t') \rangle_* = \langle x(t), x(t') \rangle_*$ and $\langle \eta(t) \rangle_* = 1$. Following (Opper and Diederich 1992), we use $\langle \cdot \rangle_*$ to denote an average taken over all possible realisations of payoffs drawn using a choice of Γ. Similarly, $G(t,t')$ is a random variable satisfying $G(t,t') = \left\langle \frac{\delta x(t)}{\delta \eta(t')} \right\rangle_*$ As such, G and η capture the time correlations between the strategy at times t and t'. We assume, as part of our derivation, that the initial conditions for all agent strategies are independently and identically distributed (i.i.d) and so we drop the distiction between agents k and actions i in (5).

Stability Analysis

Next, we determine the stability of fixed points for the effective dynamics. To do this, we write $x(t) = \bar{x} + \tilde{x}(t)$ where $\bar{x}$ denotes a fixed point of (5) and $\tilde{x}(t)$ denotes perturbations due to an additive white noise term $\xi(t)$ which is drawn from a Gaussian of zero mean and unit variance. Similarly we write $\eta(t) = \bar{\eta} + \tilde{\eta}(t)$ where $\tilde{\eta}$ gives perturbations in the time correlation variable due to the random noise. The problem of determining stability of fixed points is now a question of the growth or decay of $\tilde{x}(t)$ close to the fixed point $\bar{x}$. To do this, we only need to keep terms in (5) which are linear in $\tilde{x}(t), \tilde{\eta}(t), \xi(t)$. This yields

$$\frac{d}{dt}\tilde{x}(t) = -T\tilde{x}(t) + \bar{x}\left(N_0\Gamma\int dt' G(t-t')\tilde{x}(t') + \sqrt{N_0}\tilde{\eta}(t) + \xi(t)\right) \tag{6}$$

where we also account for the fact that, at the fixed point time correlations are constant so that $G(t,t') = G(t-t',0)$, which we rewrite as $G(t-t')$.

Since (6) contains a convolution term $\int dt' G(t-t')\tilde{x}(t')$, a classical eigenvalue analysis is intractable as an approach to determining stability. Instead, we adopt the procedure presented in (Opper and Diederich 1992) which we now outline, with full details given in the Supplementary Material. We first take the Fourier transform of (6). Doing so yields an equation in terms of frequency ω rather than time t and reduces the convolution term into a product. In particular we obtain

$$A(\omega, N_0)x(\omega) = \sqrt{N_0}\eta(\omega) + \xi(\omega). \tag{7}$$

$$A(\omega, N_0) = \left[\frac{i\omega + T}{\bar{x}} - N_0\Gamma G(\omega)\right]$$

where we overload notation by identifying each variable with its Fourier Transform, e.g. $\eta(\omega) = \mathcal{F}(\tilde{\eta})(t)$. This leads to the relation

$$\langle |x(\omega)|^2\rangle_* = N_0\Big(\langle|\eta(\omega)|^2|\rangle_* + 1\Big)\left\langle\frac{1}{|A(\omega, N_0)|}^2\right\rangle_*$$

where we recall again that $\langle\cdot\rangle_*$ denotes an expectation over all realisations of the effective dynamics from an ensemble of games drawn with the same choice of Γ. In order to analyse asymptotic stability, we focus on the limit $\omega \to 0$, since this corresponds to longer timescales in t. Finally, we apply the relation $\langle\eta(t)\eta(t')\rangle_* = \langle x(t)x(t')\rangle_*$ to write the dynamic solely in terms of x. This gives

$$\langle |x(\omega = 0)|^2\rangle_* = \left(\frac{1}{\langle\frac{1}{|A(\omega=0,N_0)|}^2\rangle_*} - N_0\right)^{-1} \tag{8}$$

By definition, the left hand side of (8) is positive, so a contradiction is reached if

$$N_0^{-1} < \left\langle\frac{1}{\left|\frac{T}{\bar{x}} - N_0\Gamma\chi\right|^2}\right\rangle_*. \tag{9}$$

where $\chi = \int_0^\infty G(s)ds$. As a result, (9) defines a sufficient condition for the onset of instability in the effective dynamics.

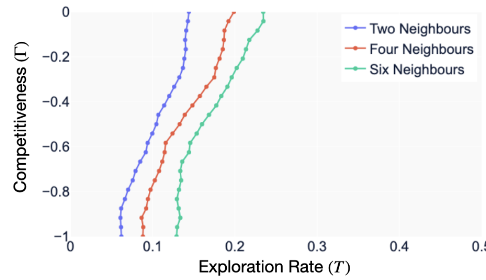


Figure 1: Stability boundary in terms of payoff correlation (Γ) and exploration rate (T) as given by the condition (9). Parameter choices on the right of the line satisfy the necessary condition for stability whilst those on the left fail.

Discussion

In general, the stability condition (9) is not straightforward to parse and cannot be solved in closed form. This is mostly due to the dependence on the fixed point $\bar{x}$, whose form can be complicated. Nevertheless, it is possible to numerically estimate the location of the stability boundary, i.e., the boundary between choices of (Γ, T) which satisfy (9) and those which do not. To do this, we fix a choice of Γ, T, iteratively solve for $\bar{x}$ and subsequently evaluate (9). Repeating this procedure for many choices of Γ, T yields the stability boundary depicted in Figure 1, in which each line depicts the transition from satisfying the condition (9) on the right to its violation on the left. By examining this we can assess how each of the parameters influence the stability of Q-Learning Dynamics.

The most notable feature of the necessary condition for stability (9) is the dependence on the number of neighbours N_0. Even from (9) itself we can discern the explicit independence on the total number of agents N, which does not appear anywhere in the condition. Therefore, in competitive games, the stability of learning is not influenced by the total number of agents, so long as the number of neighbours per agent is kept constant. By contrast, as the number of neighbours increases the unstable region occupies more of the parameter space.

This result refines that of (Sanders, Farmer, and Galla 2018) in which a stability boundary was determined for multi-player games without any underlying communication structure. In this setting the authors showed that increasing number of players leads to a larger unstable region. In our setting, this corresponds to a fully connected network in which $N_0 = N - 1$. From Figure 1 it is clear that our result yields the same prediction. Similarly, (9) aligns exactly with the result of (Galla and Farmer 2013) which derives a stability boundary in two-player games. In our case this corresponds to any network game in which $N_0 = 1$.

Another feature shown by Figure 1 is that the stability boundary increases in T as Γ decreases from -1 to 0. That is, as the strength of anticorrelation between agent payoffs decreases, higher exploration rates are required for Q-Learning dynamics to settle to an equilbrium. Recall that $\Gamma = -1$ corresponds to case where games along each edge are exactly negatively correlated, i.e., a zero-sum game. From (Leonardos, Piliouras, and Spendlove 2021) it is known that that Q-Learning dynamics asymptotically converge in network games which are exactly zero-sum for any positive value of T. Figure 1 shows that, as the competitiveness of the game decreases (i.e., as $\Gamma \to 0$), higher exploration rates are required to guarantee convergence.

Experiments

In this section we test the validity of the predictions made in our theoretical analysis in the case of games with finite action sets by running the Q-Learning algorithm, outlined in the Preliminaries.

Representative Examples of Networks In our experiments we analyse two examples of networks - the *ring network* and the *fully connected network*. These act as prototypical examples for regular networks, which satisfy Assumption 1. In the former, the payoff for any agent k is given by

$$u_k(\mathbf{x}_k, \mathbf{x}_{-k}) = \mathbf{x}_k^\top A^{k,k-1}\mathbf{x}_{k-1} + \mathbf{x}_k^\top A^{k,k+1}\mathbf{x}_{k+1}$$

where addition and subtraction are taken $\mod N$. In this case, each agent has only two neighbours, i.e. $N_0 = 2$ so the network connectivity is independent of the total number of agents N. In the fully connected network, the payoff is given by

$$u_k(\mathbf{x}_k, \mathbf{x}_{-k}) = \sum_{l \neq k} \mathbf{x}_k^\top A^{kl}\mathbf{x}_l$$

so that each agent has $N_0 = N - 1$ neighbours. This corresponds also to the case analysed by (Sanders, Farmer, and Galla 2018) and (Hussain, Belardinelli, and Piliouras 2023) in which it was predicted that the boundary between stable and unstable learning dynamics is impacted by the total number of agents.

Example: Network Sato Game We first illustrate the behaviour of Q-Learning on a representative example. This is an extension of the variant of Rock-Paper-Scissors first examined in (Sato, Akiyama, and Farmer 2002). The network extension is described in the Supplementary Material, alongside visualisations of chaotic trajectories generated by Q-Learning. In Figure 2, we simulate 50 agents playing the Network Sato Game. We record the agents' mixed strategies in the final 2500 iterations of Q-Learning and, for three representative agents, plot the probabilities with which they play their first action. As such, Figure 2 depicts the spread of the asymptotic trajectory over the simplex. It is clear that, in the fully connected network, a large value of T is required in order for the agents to converge to an equilibrium, whereas in the ring network $T \approx 0.3$ is sufficient.

Arbitrary Finite Games Next, we determine the correctness of the analytic result in arbitrary games. In Figure 3, we draw 50 games with 50 actions for each agent given choice of Γ using the formulation in 2. Once again, we simulate Q-Learning dynamics for $75,000$ time steps and record the final 10000 iterations. To characterise the limiting behaviour, we apply the following heuristic. We first determine whether, for each agent and each strategy component, the relative difference between the maximum and minimum value across all 10000 time-steps is less than 0.01. Formally, we determine whether

$$\frac{\max_{t'} x_{ki}(t') - \min_{t'} x_{ki}(t')}{\max_{t'} x_{ki}(t')} < 0.01$$

where t' is taken over the final 10000 iterations of learning. Next, we determine the variance across the final iterations as

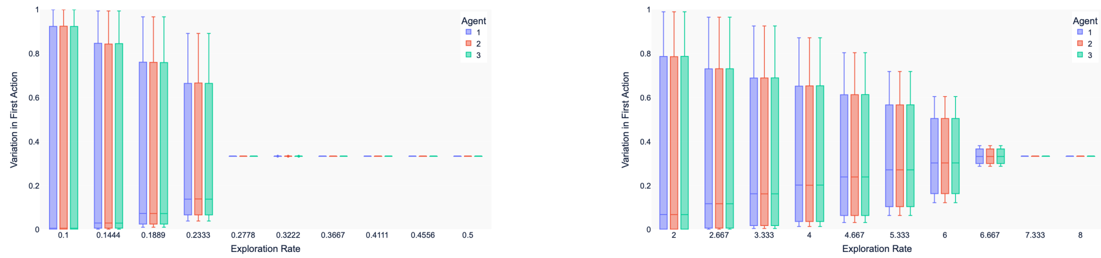
$$V = \frac{1}{Nn}\sum_{k,i}\frac{1}{10000}\sum_t x_{ki}(t)^2 - \left[\frac{1}{10000}\sum_t x_{ki}(t)\right]^2 \tag{10}$$

We check if the variance is less than 1×10^{-5}. If both of the above conditions are met, then the dynamic is considered to have converged. As an example, the convergent dynamics in Figure 2 satisfy both of these conditions. If, across all 10000 time steps, there is some τ such that, for all agents k and all $i \in S_k$, $x_{ki}(0)$, $x_{ki}(\tau)$, $x_{ki}(2\tau)$, $x_{ki}(3\tau)$ are all within 0.01 of each other, then it is considered that a stable periodic orbit has been reached.

If neither of these conditions are met, then the dynamic is deemed to be non-convergent. Note this does not conclude that the dynamics are formally chaotic, as limit cycle behaviours are also known to occur in learning (Mertikopoulos, Papadimitriou, and Piliouras 2018; Imhof, Fudenberg, and Nowak 2005). Examples of such behaviours are displayed in the Supplementary Material. However, in (Galla and Farmer 2013; Sanders, Farmer, and Galla 2018) Q-Learning dynamics were shown to exhibit chaotic behaviour for certain choices of Γ, T. This is also in line with the rich literature on chaos in multi-agent learning (Sato, Akiyama, and Farmer 2002; Mukhopadhyay and Chakraborty 2020; van Strien and Sparrow 2011).

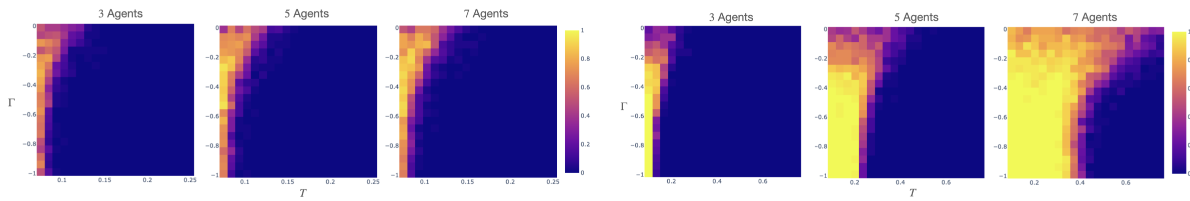
From Figure 3, it can be seen that the form of the analytic stability boundary holds in practice. The reason that the empirical boundary overestimates the theoretical result is that the latter considers asymptotic trajectories, i.e. infinite time-scales, whilst the former is evaluated over 75,000 steps. Therefore, in the experiments, slow convergence can be mistaken for non-convergence. Overall, however, we see a strong alignment between the theoretical predictions and experimental evaluation. Namely, lower exploration rates are required strictly competitive games, i.e. as $\Gamma \to -1$, whilst uncorrelated games require higher exploration rates in order to converge to an equilibrium. In addition, the stability boundary is unaffected by the number of players in the ring network.

The latter point is highlighted in Figure 4 in which we average over multiple choices of Γ to isolate the effect of the number of neighbours N_0 on the stability boundary in terms of T. It is evident that increasing the number of players in a ring network plays no impact on the stability boundary. As a result, it is possible to increase the number of agents in such games arbitrarily without compromising convergence of Q-Learning to a fixed point. By contrast fully connected networks do not scale well, as non-convergent behaviours remain prevalent for low values of T as the number of agents increase.



(a) Ring Network (b) Full Network

Figure 2: Boxplot depicting the final 2500 iterations of learning in a Network Sato Game with 50 agents for various values of T. The mixed strategies of three agents are plotted depicting the spread of the trajectories across the simplex.



(a) Ring Network (b) Full Network

Figure 3: Empirically determined probability of non-convergence for various choices of Γ, T. Hot colours denote that a higher fraction of randomly drawn games fail to reach an equilibrium whilst cool colours depict a higher probability of convergence.

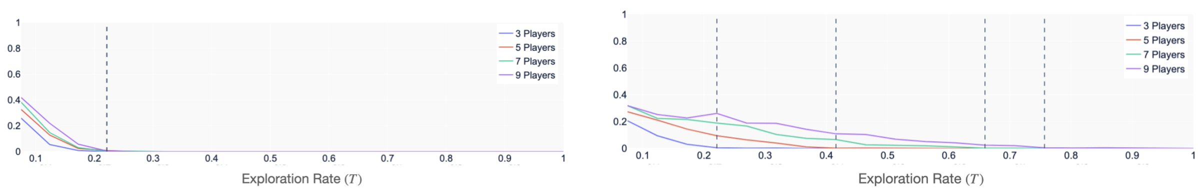


Figure 4: Probability of non-convergent dynamics with respect to T over 250 randomly generated games with random choice of Γ. The black line depicts the choice of T for which all games converge to a fixed point. In the ring network (Left), this occurs at a single value of T, whilst in the fully connected network (Right), the choice of T depends on the number of players.

Conclusion

In this study we have refined the previously held belief that chaotic behaviours are more prevalent in games with many players. In particular, we analyse network games and show that stability of the Q-Learning dynamic depends on the structure of the network, the competitiveness of the game and the exploration rates of agents. We show that in certain networks, such as the ring network, an arbitrary number of agents may be added to the system without compromising the propensity for learning to converge. By contrast, if agents are heavily connected in the network, non-convergent behaviours, such as limit cycles and chaos, become prevalent even with a small total number of agents.

The present work has isolated the effect of the number of neighbours and the number of players in the game. However, there are other factors which may affect the stability of Q-Learning. For instance, whilst we have required N_0 to be the same for all agents, it would be fruitful to analyse heterogeneously coupled networks. More generally, tools from graph theory may be applied within this framework to uncover the role of the network in convergence for arbitrary network games. Our work, therefore, presents a first step towards building a complete picture of the stability of multi-agent learning in network games.

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