Enhancing the Efficiency of Altruism and Taxes in Affine Congestion Games through Signalling

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Abstract

We address the problem of improving the worst-case efficiency of pure Nash equilibria (aka, the price of anarchy) in affine congestion games, through a novel use of signalling. We assume that, for each player in the game, a most preferred strategy is publicly signalled. This can be done either distributedly by the players themselves, or be the outcome of some centralized algorithm. We apply this signalling scheme to two well-studied scenarios: games with partially altruistic players and games with resource taxation. We show a significant improvement in the price of anarchy of these games, whenever the aggregate signalled strategy profile is a good approximation of the game social optimum.

Introduction

The analysis of the inefficiencies caused by selfish uncoordinated behaviour in multi-agent systems has been a fervent research topic during the last twenty-five years. The celebrated notions of *price of anarchy* (Koutsoupias and Papadimitriou 1999) and *price of stability* (Anshelevich et al. 2008) are the state-of-the-art measures for these degraded performances. The former compares the social value of the worst stable outcome, usually a Nash equilibrium (Nash 1950), against that of a socially optimal solution, while the latter follows a best-case approach.

One of the most studied classes of games in the literature is that of (atomic) *congestion games* (Rosenthal 1973). These are cost-minimization games defined by a set of n players competing for the usage of a set of resources. The cost that a player incurs when using a resource is given by a resource-specific function, called *latency function*, which only depends on the number of players using it. A particularly meaningful subclass of congestion games, called *affine congestion games* (Fotakis, Kontogiannis, and Spirakis 2005; Harks and Klimm 2012; Panagopoulou and Spirakis 2006), focuses on the case of affine latency functions. As congestion games are known to always possess pure Nash equilibria (Rosenthal 1973), their price of anarchy and price of stability are usually expressed with respect to this solution concept. For affine congestion games, in particular, the price of anarchy equals $5/2$ (Awerbuch, Azar, and Epstein 2005; Christodoulou and Koutsoupias 2005b) and the price of stability equals $1 + 1/\sqrt{3} \approx 1.577$ (Caragiannis et al. 2011; Christodoulou and Koutsoupias 2005a).

A fundamental question in this domain is whether better performances can be obtained when applying plausible changes to the game model. Among these, we consider two well-studied scenarios: partially altruistic players (Bilò 2014; Caragiannis et al. 2010; Hoefer and Skopalik 2013) and resource taxation (Bilò and Vinci 2019; Caragiannis, Kaklamanis, and Kanellopoulos 2010; Paccagnan et al. 2021; Paccagnan and Gairing 2021; Vijayalakshmi and Skopalik 2020).

For a parameter $\theta \in [0, 1]$, a θ-altruistic player seeks to minimize $1-\theta$ times her personal cost, plus θ times the sum of the personal costs of all other players. Thus, $\theta = 0$ recovers the model of purely selfish players and, the more θ increases, the more players become altruistic. A surprising result by Caragiannis, Kaklamanis, and Kanellopoulos (2010) shows that altruism has a negative impact on the worst-case efficiency of pure Nash equilibria. In fact, the price of anarchy in games with θ-altruistic players is never smaller than $5/2$ and equals $5/2$ only when $\theta = 0$. Although Caragiannis, Kaklamanis, and Kanellopoulos (2010) proved that a better price of anarchy holds in symmetric singleton games (i.e., games in which all players share the same strategy space which is made of single resources only) as long as $\theta < 0.7$, and Bilò (2014) showed that the price of stability improves as long as $\theta < \frac{17+\sqrt{3}}{26} \approx 0.72$, the question of whether other forms of altruism can yield a better price of anarchy in general (i.e., non-singleton and asymmetric) affine congestion games remained open.

A taxation mechanism in congestion games adds an extra cost for using a resource, in the form of either a tax or a toll, which is equal for all players. Taxes are usually distinguished in two types: *refundable* and *non-refundable*. In the first case, players consider the presence of taxes when determining their strategy, but, when considering the social cost of a realized profile, the cost yielded by taxes is left out. That is, the efficiency of a strategy profile is still measured by the sum of the personal tax-free costs of all players. When assuming non-refundable taxes, instead, the amount of taxes paid by every player is considered in the equilibrium social cost, but left out from the social optimum. Bilò and Vinci (2019) and Caragiannis, Kaklamanis, and Kanellopoulos (2010) show that, through refundable taxation, the price of anarchy of affine congestion games drops to 2 and Paccagnan and Gairing (2021) further prove that this is the best-possible bound that can be achieved in polynomial time, unless P = NP. For non-refundable taxes, instead, no improvements have been achieved so far.

Our Contribution

We reconsider the impact of altruism and taxation in affine congestion games enhanced by a novel use of *signalling*. Signalling (Kamenica and Gentzkow 2011) has been adopted in a variety of stochastic scenarios, as a mean for a centralized authority to persuade selfish agents to pursue a desired behaviour. Recently, it has also been used in congestion games with incomplete information to induce selfish agents to play optimally (Bhaskar et al. 2016; Castiglioni et al. 2021; Griesbach et al. 2022). In such a setting, the resource latency functions are assumed to depend on possible outcomes of an uncertain state of nature, which follows a publicly known probability distribution. A benevolent authority publicly advertises a signalling scheme, which is a distribution of possible signals for each state of nature. When the state of nature is realized, the authority sends a signal realized according to the chosen signalling scheme. The players update their believes on the resource latency functions according to the signal and use this information to determine their strategy.

In this work, we look at signalling schemes in *games with complete information*, where the resource latency functions are publicly known. We assume that, for each player in the game, a most preferred strategy is publicly signalled. This can be done either distributedly by the players themselves, or be the outcome of some centralized algorithm simulating a benevolent authority. Under this assumption, we consider affine congestion games in which players seek to minimize $1-\theta$ times her personal cost plus θ times the increase she causes in the cost of the signalled strategies of the other players. For this reason, and for other technical reasons we better explain in the next section, this type of players can be reinterpreted as θ-altruistic players in presence of signalled strategies. We show that, in this new model, if the socially optimal solution is signalled, the price of anarchy becomes equal to $\frac{5-7\theta}{2-\theta}$ when $\theta \in [0, 1/2]$, to $\frac{8\theta-5+2\sqrt{3}(2-3\theta)}{2\sqrt{3}(1-\theta)-1}$ when $\theta \in [1/2, (3-\sqrt{3})/2]$, to $\frac{5\theta^2-6\theta+2}{(1-\theta)^2}$ when $\theta \in [(3-\sqrt{3})/2, 2/3]$, and to $\frac{\theta}{1-\theta}$ when $\theta \in [2/3, 1]$ (see Figure 1 for a pictorial representation of these bounds). The most important consequence of this result is that, for $\theta = 1/2$, the price of anarchy drops to one, implying that every pure Nash equilibrium is socially optimal. Moreover, the price of anarchy improves upon the $5/2$ value holding for selfish players, as long as $\theta < 5/7 \approx 0.714$. More generally, whenever a ξ-approximation of the socially optimal solution is signalled, these bounds degrade by a factor ξ.

The upper bounds on the price of anarchy are established through the primal-dual method (Bilò 2018; Bilò and Vinci 2023). This method relies on a pair of primal-dual formulations and, in particular, exploits the fact that, usually, the dual program possesses a very simple structure. This eases the analysis and still provides good, and often tight, upper bounds on the price of anarchy. These formulations, however, are not fixed and need to be suitably (re)defined in each setting of application, so as to incorporate the specific features of the domain. Moreover, in order to obtain the best-possible upper bound achievable through the defined pair of programs, one has to determine the optimal solution of the dual. However, the dual program has always an infinite number of constraints and so, in order to solve it optimally, one has to fully and deeply understand the combinatorial properties of its constraints. So, with this respect, every different pair of primal-dual formulations usually requires a different analysis. For what concern the particular pair considered in this work, we can highlight the fact that, differently from previous applications of the method, a constraint modelling some properties of the social optimum (inequality (4) derived in Lemma 1) has been used for the first time in the literature.

We then observe that the model of affine congestion games with θ-altruistic players under signalling is equivalent to affine congestion games with certain *personalized* (i.e., player-dependent) refundable taxes. As a consequence, the same bounds on the price of anarchy hold under this extended form of taxation. This is in sharp contrast with non-personalized taxes, for which the best known bound is equal to 2. We must stress, however, that our results are of existential nature only, and so the lower bound given in (Paccagnan and Gairing 2021), which requires P $\neq$ NP, does not apply. We also consider the case in which our taxes are non-refundable and provide the first improved bound on the price of anarchy achievable by this type of taxes in affine congestion games. Specifically, we show that, choosing $\theta = 1/2$, the price of anarchy drops to 2.

Related Work

The application of signalling, as a mean for improving the efficiency of congestion games with stochastic affine latency functions, has been almost entirely considered in the model of non-atomic routing games, where resources are links in a traffic network and each player controls a negligible amount of traffic. Bhaskar et al. (2016) show that the optimal signalling scheme cannot be approximated in polynomial time up to a factor $4/3-\epsilon$, for any $\epsilon > 0$, unless P = NP. As this negative result requires a huge degree of freedom in the network structure and a very large number of states of nature, Griesbach et al. (2022) provide a set of conditions for which an optimal scheme can be devised. Vasserman, Feldman, and Hassidim (2015) define the notion of mediation ratio, as the ratio between the expected social cost of the best mediated equilibrium and the social optimum. Castiglioni et al. (2021) consider a variant in which the players commit to following the recommendation before receiving the signal and show that symmetry is necessary and sufficient for polynomial time computation of optimal schemes.

The setting of atomic congestion games, as considered in our work, is examined by Zhou, Nguyen, and Xu (2022) under the hypothesis of singleton player strategies. They show that, under both public and private signalling, optimal schemes can be computed in polynomial time when the number of resources is constant.

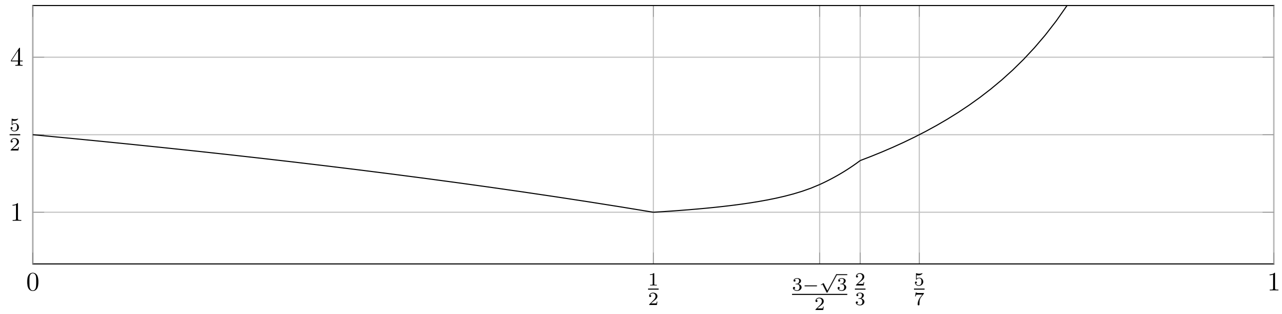

Figure 1: The price of anarchy as a function of θ.

Paper Organization

The paper is organized in four sections as follows. The first explains in full details our model, its motivations and connections with previous approaches. In the second, we derive the main result of the paper, that is, tight bounds on the price of anarchy of altruistic games with signalling. The third section discusses how to apply our framework to obtain personalized taxes with provably good performance. Finally, in the last one, we conclude and discuss possible research directions. Because of space limitation, some proofs are omitted.

Cost Model and Motivations

A congestion game $\mathsf{CG} = (N, R, (S_i)_{i\in N}, (\ell_r)_{r\in R})$ is a quadruple such that $N = \{1, \dots, n\}$ is a set of n players, R is a set of resources, $S_i \subseteq 2^R \setminus \{\emptyset\}$ is the strategy space of player i and $\ell_r : N \mapsto \mathbb{R}_{\geq 0}$ is the latency function of resource r. In *affine congestion games*, each latency function is of the form $\ell_r(x) = \alpha_r x + \beta_r$, with $\alpha_r, \beta_r \in \mathbb{R}_{\geq 0}$. A *strategy profile* is a vector $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ such that σ_i is the strategy chosen by player i. We denote by $\mathcal{S} = S_1 \times \dots \times S_n$ the set of strategy profiles of CG. The cost that player i pays in strategy profile $\boldsymbol{\sigma}$ is defined as $cost_i(\boldsymbol{\sigma}) = \sum_{r\in\sigma_i} \ell_r(n_r(\boldsymbol{\sigma}))$, where $n_r(\boldsymbol{\sigma}) = |\{i \in N : r \in \sigma_i\}|$ denotes the number of players using resource r in $\boldsymbol{\sigma}$, also called the *congestion* of r in $\boldsymbol{\sigma}$.

We consider a variant of affine congestion games in which the cost function of each player is redefined as follows. Let $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$ be a given strategy profile, which we shall call the *signalled strategy profile*. This strategy profile can be obtained either distributedly, by assuming that every player i announces the strategy she would like to play (independently of the choices of the others), or centralizedly, as the output of some algorithm. Given a resource r and a player i, let $\mathbb{I}_r(\tilde{\sigma}_i)$ be the indicator function which equals 1 if $r \in \tilde{\sigma}_i$ and 0 otherwise. For a real value $\theta \in [0, 1]$, the θ-cost of player i in strategy profile $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ is defined as:

$$\theta\text{-}cost_i(\boldsymbol{\sigma}) = (1-\theta)\sum_{r\in\sigma_i}\left(\alpha_r\left(n_r(\boldsymbol{\sigma})\right) + \beta_r\right) + \theta\sum_{r\in\sigma_i}\alpha_r\left(n_r(\tilde{\boldsymbol{\sigma}}) - \mathbb{I}_r(\tilde{\sigma}_i)\right). \quad (1)$$

The intuition behind this definition is based on a combination of signalling with players' partially altruistic behaviour. To see this, observe that the first term of the cost function is player i's cost in CG, namely $cost_i(\boldsymbol{\sigma})$, to which she assigns a weight of $1 - \theta$. The second term, weighted θ, expresses the increase in the cost of the signalled strategies of the other players caused by player i. In fact, when using resource r, as player i contributes for a unit to the congestion of r, she causes an extra cost of α_r to all players using r in their signalled strategy, with respect to the case in which i does not choose this resource. By summing this extra cost over all players using r in their signalled strategies, and then over all resources used by i, we get

$$\sum_{r\in\sigma_i}\sum_{j\neq i: r\in\tilde{\sigma}_j}\alpha_r = \sum_{r\in\sigma_i}\alpha_r\left(n_r(\tilde{\boldsymbol{\sigma}}) - \mathbb{I}_r(\tilde{\sigma}_i)\right).$$

So, the smaller θ, the more players are selfish, the larger θ, the more players will try to not interfere with the signalled strategies of the others. For such a reason, we call an affine congestion game coupled with the θ-cost function defined in (1), a *θ-altruistic affine congestion game with signalling*. For the case of $\theta = 0$, in which players are purely selfish, we reobtain $\theta\text{-}cost_i(\boldsymbol{\sigma}) = cost_i(\boldsymbol{\sigma})$. When $\theta = 1$, we say that players are purely altruistic and, for $\theta = 1/2$, we say that altruism is perfectly balanced.

Given a strategy profile $\boldsymbol{\sigma}$, a player i and a strategy $s \in S_i$, let $(\boldsymbol{\sigma}_{-i}, s)$ denote the strategy profile obtained from $\boldsymbol{\sigma}$ when i changes her strategy to s. A strategy profile $\boldsymbol{\sigma}$ is a pure Nash equilibrium for a θ-altruistic affine congestion game with signalling if, for each $i \in N$ and $s \in S_i$, $\theta\text{-}cost_i(\boldsymbol{\sigma}) \leq \theta\text{-}cost_i(\boldsymbol{\sigma}_{-i}, s)$. We denote by $\mathsf{PNE}(\mathsf{CG})$ the set of pure Nash equilibria of a θ-altruistic affine congestion game with signalling CG.

As usual in this setting, we measure the efficiency of a strategy profile by the utilitarian social cost

$$\mathsf{USC}(\boldsymbol{\sigma}) = \sum_{i\in N} cost_i(\boldsymbol{\sigma}) = \sum_{r\in R}\left(\alpha_r n_r(\boldsymbol{\sigma})^2 + \beta_r n_r(\boldsymbol{\sigma})\right),$$

which sums the cost of all players in the game. Observe that we use the personal cost of a player, instead of her θ-cost, in this definition. The reason is that we want to evaluate whether, and by how much, altruism with signalling may improve the efficiency of equilibria in affine congestion games. Thus, we use the θ-cost of a player as the cost function modelling her behaviour, but then evaluate the final outcome of the game with respect to the classical cost function. Denote by $\boldsymbol{\sigma}^*$, called *social optimum*, any strategy profile minimizing the utilitarian social cost. The *price of anarchy* of a θ-altruistic affine congestion game with signalling CG is defined as $\mathsf{PoA}(\mathsf{CG}) = \max_{\boldsymbol{\sigma}\in\mathsf{NE}(\mathsf{CG})} \frac{\mathsf{USC}(\boldsymbol{\sigma})}{\mathsf{USC}(\boldsymbol{\sigma}^*)}$.

Let us now consider the inequality obtained when, starting from a generic profile $\boldsymbol{\sigma}$, player i deviates to her signalled strategy $\tilde{\sigma}_i$. After observing that $n_r(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i) = n_r(\boldsymbol{\sigma}) + 1$ for each $r \in \tilde{\sigma}_i \setminus \sigma_i$, $n_r(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i) = n_r(\boldsymbol{\sigma}) - 1$ for each $r \in \sigma_i \setminus \tilde{\sigma}_i$, and $n_r(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i) = n_r(\boldsymbol{\sigma})$ in any other situation, we obtain:

$$\begin{aligned}
& \theta\text{-}cost_i(\boldsymbol{\sigma}) - \theta\text{-}cost_i(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i) \\
= & (1-\theta) \sum_{r\in\sigma_i\setminus\tilde{\sigma}_i} (\alpha_r n_r(\boldsymbol{\sigma}) + \beta_r) \\
& -(1-\theta) \sum_{r\in\tilde{\sigma}_i\setminus\sigma_i} (\alpha_r(n_r(\boldsymbol{\sigma}) + 1) + \beta_r) \\
& +\theta \left(\sum_{r\in\sigma_i\setminus\tilde{\sigma}_i} \alpha_r n_r(\tilde{\boldsymbol{\sigma}}) - \sum_{r\in\tilde{\sigma}_i\setminus\sigma_i} \alpha_r(n_r(\tilde{\boldsymbol{\sigma}}) - 1) \right).
\end{aligned}$$

For a pure Nash equilibrium $\boldsymbol{\sigma}$ and a signalled strategy profile $\tilde{\boldsymbol{\sigma}}$, define $k_r := n_r(\boldsymbol{\sigma})$, $\tilde{k}_r := n_r(\tilde{\boldsymbol{\sigma}})$ and $s_r := |\{i \in N : r \in \sigma_i \cap \tilde{\sigma}_i\}|$. By summing inequality $\theta\text{-}cost_i(\boldsymbol{\sigma}) - \theta\text{-}cost_i(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i) \leq 0$ for each $i \in N$, we obtain the following result.

Proposition 1 *For a pure Nash equilibrium $\boldsymbol{\sigma}$ and a signalled strategy profile $\tilde{\boldsymbol{\sigma}}$, it holds that*

$$\begin{aligned}
\sum_{r\in R} \alpha_r \left((1-\theta)k_r^2 - \theta\tilde{k}_r^2 + (1-2\theta)(s_r - \tilde{k}_r - \tilde{k}_r k_r) \right) \\
+(1-\theta)\sum_{r\in R} \beta_r \left(k_r - \tilde{k}_r \right) \leq 0.
\end{aligned} \tag{2}$$

Proof: We have

$$\begin{aligned}
0 \geq & \sum_{i\in N} (\theta\text{-}cost_i(\boldsymbol{\sigma}) - \theta\text{-}cost_i(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i)) \\
= & (1-\theta) \sum_{i\in N} \sum_{r\in\sigma_i\setminus\tilde{\sigma}_i} (\alpha_r k_r + \beta_r) \\
& -(1-\theta) \sum_{i\in N} \sum_{r\in\tilde{\sigma}_i\setminus\sigma_i} (\alpha_r(k_r + 1) + \beta_r) \\
& +\theta \sum_{i\in N} \sum_{r\in\sigma_i\setminus\tilde{\sigma}_i} \alpha_r\tilde{k}_r - \theta \sum_{i\in N} \sum_{r\in\tilde{\sigma}_i\setminus\sigma_i} \alpha_r(\tilde{k}_r - 1) \\
= & (1-\theta) \sum_{r\in R} \sum_{i\in N: r\in\sigma_i\setminus\tilde{\sigma}_i} (\alpha_r k_r + \beta_r) \\
& -(1-\theta) \sum_{r\in R} \sum_{i\in N: r\in\tilde{\sigma}_i\setminus\sigma_i} (\alpha_r(k_r + 1) + \beta_r) \\
& +\theta \sum_{r\in R} \sum_{i\in N: r\in\sigma_i\setminus\tilde{\sigma}_i} \alpha_r\tilde{k}_r \\
& -\theta \sum_{r\in R} \sum_{i\in N: r\in\tilde{\sigma}_i\setminus\sigma_i} \alpha_r(\tilde{k}_r - 1) \\
= & (1-\theta) \sum_{r\in R} (k_r - s_r)(\alpha_r k_r + \beta_r) \\
& -(1-\theta) \sum_{r\in R} (\tilde{k}_r - s_r)(\alpha_r(k_r + 1) + \beta_r) \\
& +\theta \sum_{r\in R} (k_r - s_r)\alpha_r\tilde{k}_r - \theta \sum_{r\in R} (\tilde{k}_r - s_r)\alpha_r(\tilde{k}_r - 1)
\end{aligned}$$

and the claim follows by summing all terms. □

The model of altruism for congestion games previously considered in the literature assumes that player i minimizes $1-\theta$ times her personal cost, plus θ times the total personal cost of the other players, i.e., the function $(1-\theta)cost_i(\boldsymbol{\sigma}) + \theta\sum_{j\neq i} cost_j(\boldsymbol{\sigma})$. By incorporating the notion of signalling within the classical definition of altruism, we may replace the second term with θ times the total cost of the other players in strategy profile $(\tilde{\boldsymbol{\sigma}}_{-i}, \sigma_i)$. This means that, when evaluating the cost of the other players, player i assumes that they are all playing according to the signal and she is the only one to deviate.

Denoting with $\theta\text{-}\widehat{cost}_i(\cdot)$ this cost function, we have

$$\begin{aligned}
\theta\text{-}\widehat{cost}_i(\boldsymbol{\sigma}) = (1-\theta) \sum_{r\in\sigma_i} (\alpha_r n_r(\boldsymbol{\sigma}) + \beta_r) \\
+\theta \sum_{j\neq i} \sum_{r\in\tilde{\sigma}_j} (\alpha_r n_r(\tilde{\boldsymbol{\sigma}}_{-i}, \sigma_i) + \beta_r).
\end{aligned} \tag{3}$$

Although this cost function differs from (1), when computing the difference $\theta\text{-}\widehat{cost}_i(\boldsymbol{\sigma}) - \theta\text{-}\widehat{cost}_i(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i)$, we obtain

$$\begin{aligned}
& \theta\text{-}\widehat{cost}_i(\boldsymbol{\sigma}) - \theta\text{-}\widehat{cost}_i(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i) \\
= & (1-\theta) \sum_{r\in\sigma_i\setminus\tilde{\sigma}_i} (\alpha_r n_r(\boldsymbol{\sigma}) + \beta_r) \\
& +\theta \sum_{j\neq i} \sum_{r\in\tilde{\sigma}_j\cap(\sigma_i\setminus\tilde{\sigma}_i)} (\alpha_r(n_r(\tilde{\boldsymbol{\sigma}}) + 1) + \beta_r) \\
& +\theta \sum_{j\neq i} \sum_{r\in\tilde{\sigma}_j\cap(\tilde{\sigma}_i\setminus\sigma_i)} (\alpha_r(n_r(\tilde{\boldsymbol{\sigma}}) - 1) + \beta_r) \\
& -(1-\theta) \sum_{r\in\tilde{\sigma}_i\setminus\sigma_i} (\alpha_r(n_r(\boldsymbol{\sigma}) + 1) + \beta_r) \\
& -\theta \sum_{j\neq i} \sum_{r\in\tilde{\sigma}_j\cap(\sigma_i\setminus\tilde{\sigma}_i)} (\alpha_r n_r(\tilde{\boldsymbol{\sigma}}) + \beta_r) \\
& -\theta \sum_{j\neq i} \sum_{r\in\tilde{\sigma}_j\cap(\tilde{\sigma}_i\setminus\sigma_i)} (\alpha_r n_r(\tilde{\boldsymbol{\sigma}}) + \beta_r) \\
= & (1-\theta) \sum_{r\in\sigma_i\setminus\tilde{\sigma}_i} (\alpha_r n_r(\boldsymbol{\sigma}) + \beta_r) \\
& +\theta \sum_{r\in\sigma_i\setminus\tilde{\sigma}_i} n_r(\tilde{\boldsymbol{\sigma}})(\alpha_r(n_r(\tilde{\boldsymbol{\sigma}}) + 1) + \beta_r) \\
& +\theta \sum_{r\in\tilde{\sigma}_i\setminus\sigma_i} (n_r(\tilde{\boldsymbol{\sigma}}) - 1)(\alpha_r(n_r(\tilde{\boldsymbol{\sigma}}) - 1) + \beta_r) \\
& -(1-\theta) \sum_{r\in\tilde{\sigma}_i\setminus\sigma_i} (\alpha_r(n_r(\boldsymbol{\sigma}) + 1) + \beta_r)
\end{aligned}$$

$$
\begin{aligned}
&-\theta \sum_{r\in\sigma_i\setminus\tilde{\sigma}_i} n_r(\tilde{\boldsymbol{\sigma}})(\alpha_r n_r(\tilde{\boldsymbol{\sigma}})+\beta_r)\\
&-\theta \sum_{r\in\tilde{\sigma}_i\setminus\sigma_i} (n_r(\tilde{\boldsymbol{\sigma}})-1)(\alpha_r n_r(\tilde{\boldsymbol{\sigma}})+\beta_r)\\
=\;& \theta\text{-}cost_i(\boldsymbol{\sigma})-\theta\text{-}cost_i(\boldsymbol{\sigma}_{-i},\tilde{\sigma}_i).
\end{aligned}
$$

The first equality comes from the fact that all contributions coming from resources in $\sigma_i \cap \tilde{\sigma}_i$ and not in $\sigma_i \cup \tilde{\sigma}_i$ cancel out in the difference; the second equality is obtained by exchanging the order of the summations and observing that the set of players other than i using a resource $r \in \sigma_i \setminus \tilde{\sigma}_i$ in both $(\tilde{\boldsymbol{\sigma}}_{-i}, \sigma_i)$ and $\tilde{\boldsymbol{\sigma}}$ has cardinality $n_r(\tilde{\boldsymbol{\sigma}})$, while the set of players other than i using a resource $r \in \tilde{\sigma}_i \setminus \sigma_i$ in both $(\tilde{\boldsymbol{\sigma}}_{-i}, \sigma_i)$ and $\tilde{\boldsymbol{\sigma}}$ has cardinality $n_r(\tilde{\boldsymbol{\sigma}}) - 1$; the last equality comes by summing all terms.

As we shall formally see in the next section, since the evaluation of the price of anarchy of our games only exploits the inequality stating that, in a pure Nash equilibrium, no player improves by deviating to any other strategy, it follows that, from the perspective of the worst-case efficiency of pure Nash equilibria, the two cost functions defined in (1) and (3) generate games with the same price of anarchy. This evidence provides another altruistic interpretation for the cost function studied in this work.

Price of Anarchy

In this section, we characterize the efficiency of pure Nash equilibria of θ-altruistic affine congestion games with signalling, by deriving tight bounds on the price of anarchy as a function of θ. In particular, we shall restrict our analysis to the case in which a social optimum is signalled. We stress, however, that all our bounds, degraded by a factor ξ, extend directly to the case in which a ξ-approximation of the social optimum is signalled.

To obtain the upper bounds, we resort to the primal-dual method developed in (Bilò 2018; Bilò and Vinci 2023). Before presenting our main result, we need the following technical lemma. Recall that, for a pure Nash equilibrium $\boldsymbol{\sigma}$ and a signalled strategy profile $\tilde{\boldsymbol{\sigma}}$, we set $k_r := n_r(\boldsymbol{\sigma})$, $\tilde{k}_r := n_r(\tilde{\boldsymbol{\sigma}})$ and $s_r := |\{i \in N : r \in \sigma_i \cap \tilde{\sigma}_i\}|$.

Lemma 1 *If $\tilde{\boldsymbol{\sigma}}$ is a social optimum, we have*

$$
\begin{aligned}
&\sum_{r\in R}\alpha_r(2\tilde{k}_r^2-2\tilde{k}_r k_r-\tilde{k}_r-k_r+2s_r)\\
&\qquad\qquad+\sum_{r\in R}\beta_r(\tilde{k}_r-k_r)\le 0.
\end{aligned}\tag{4}
$$

Theorem 1 *The price of anarchy of θ-altruistic affine congestion games when signalling a social optimum is at most*

- $\frac{5-7\theta}{2-\theta}$ *for* $\theta \in [0, \frac{1}{2}]$,
- $\frac{8\theta-5+2\sqrt{3}(2-3\theta)}{2\sqrt{3}(1-\theta)-1}$ *for* $\theta \in [\frac{1}{2}, \frac{3-\sqrt{3}}{2}]$,
- $\frac{5\theta^2-6\theta+2}{(1-\theta)^2}$ *for* $\theta \in [\frac{3-\sqrt{3}}{2}, \frac{2}{3}]$
- $\frac{\theta}{1-\theta}$ *for* $\theta \in [\frac{2}{3}, 1]$.[1]

[1]For $\theta = 1$, we interpret the ratio $\frac{\theta}{1-\theta}$ as unbounded.

Proof: According to the primal-dual method, we need to formulate, as a linear program on variables $(\alpha_r, \beta_r)_{r\in R}$, the problem of maximizing the utilitarian social cost of a pure Nash equilibrium under the assumption that the utilitarian social cost of a social optimum equals 1 and inequalities (2) and (4) hold true. We, thus, have:

$$
\begin{aligned}
\max\quad & \sum_{r\in R}\left(\alpha_r k_r^2+\beta_r k_r\right)\\
s.t.\quad & \sum_{r\in R}\alpha_r(1-\theta)k_r^2-\theta\tilde{k}_r^2\\
& \quad+(1-2\theta)\sum_{r\in R}\alpha_r(s_r-\tilde{k}_r-\tilde{k}_r k_r)\\
& \quad\;+(1-\theta)\sum_{r\in R}\beta_r\left(k_r-\tilde{k}_r\right)\le 0\\
& \sum_{r\in R}\alpha_r(2\tilde{k}_r^2-2\tilde{k}_r k_r-\tilde{k}_r-k_r+2s_r)\\
& \quad+\sum_{r\in R}\beta_r(\tilde{k}_r-k_r)\le 0\\
& \sum_{r\in R}\left(\alpha_r\tilde{k}_r^2+\beta_r\tilde{k}_r\right)=1\\
& \alpha_r,\beta_r\ge 0 \qquad r\in R.
\end{aligned}
$$

The dual program, obtained associating variable x with the first constraint, variable y with the second constraint and variable γ with the third one of the primal, is the following:

$$
\begin{aligned}
\min\quad & \gamma\\
s.t.\quad & x((1-\theta)k_r^2-\theta\tilde{k}_r^2+(1-2\theta)(s_r-\tilde{k}_r-\tilde{k}_r k_r))\\
& \quad+y(2\tilde{k}_r^2-2\tilde{k}_r k_r-\tilde{k}_r-k_r+2s_r)\\
& \quad+\gamma\tilde{k}_r^2\ge k_r^2 \qquad\qquad r\in R\\
& x(1-\theta)(k_r-\tilde{k}_r)+y(\tilde{k}_r-k_r)\\
& \quad+\gamma\tilde{k}_r\ge k_r \qquad\qquad r\in R\\
& x,y\ge 0.
\end{aligned}
$$

To prove the claim it suffices to provide feasible dual solutions with γ equal to the desired upper bounds.

For $\theta \in [0, \frac{1}{2}]$, set $x = \frac{3}{2-\theta} \geq 0$, $y = 0$, and $\gamma = \frac{5-7\theta}{2-\theta}$. By substituting these values in the second dual constraint, we obtain the inequality $\frac{(1-2\theta)}{2-\theta}(k_r + 2\tilde{k}_r) \geq 0$, which holds true as $\theta \in [0, \frac{1}{2}]$ and both k_r and $\tilde{k}_r$ are non-negative integers. Substituting in the first constraint, we obtain the inequality $\frac{(1-2\theta)}{2-\theta}(k_r^2 - 3k_r\tilde{k}_r + 5\tilde{k}_r^2 - 3\tilde{k}_r + 3s_r) \geq 0$, which can be easily shown to hold true. In fact, as $\theta \in [0, \frac{1}{2}]$ implies $\frac{(1-2\theta)}{2-\theta} \geq 0$, we only need to care of the validity of inequality $k_r^2 - 3k_r\tilde{k}_r + 5\tilde{k}_r^2 - 3\tilde{k}_r + 3s_r \geq 0$, which can be rewritten as $(k_r - \frac{3}{2}\tilde{k}_r)^2 + \frac{11}{4}\tilde{k}_r^2 - 3\tilde{k}_r + 3s_r \geq 0$. This inequality holds true for any integer $\tilde{k}_r \neq 1$. For the leftover case of $\tilde{k}_r = 1$, the inequality becomes $(k_r - 1)(k_r - 2) + 3s_r \geq 0$, which holds true as both k_r and s_r are non-negative integers.

For $\theta \in [\frac{1}{2}, \frac{3-\sqrt{3}}{2}]$, set $x = \frac{2(\sqrt{3}(2\theta-1)+5-6\theta)}{12\theta^2-24\theta+11}$, $y = \frac{(2\theta-1)(2\sqrt{3}(1-\theta)+1)}{12\theta^2-24\theta+11}$, and $\gamma = \frac{8\theta-5+2\sqrt{3}(2-3\theta)}{2\sqrt{3}(1-\theta)+1}$. Observe that, $\theta \in [\frac{1}{2}, \frac{3-\sqrt{3}}{2}]$, implies $x \geq 0$ and $y \geq 0$, as required. By substituting these values in the second dual constraint, we obtain the inequality $\frac{\tilde{k}_r(2\theta-1)}{2\sqrt{3}(1-\theta)-1} \geq 0$, which holds true as $\theta \in [\frac{1}{2}, \frac{3-\sqrt{3}}{2}]$ and $\tilde{k}_r$ is non-negative. Substituting in the first constraint, we obtain the inequality $\frac{2\theta-1}{2\sqrt{3}(1-\theta)-1}(k_r^2 - (4-2\sqrt{3})\tilde{k}_r k_r - k_r + (7-4\sqrt{3})\tilde{k}_r^2 - (3-2\sqrt{3})\tilde{k}_r + (4-2\sqrt{3})s_r) \geq 0$, which can be shown to hold true. In fact, as $\theta \in [\frac{1}{2}, \frac{3-\sqrt{3}}{2}]$ implies $\frac{2\theta-1}{2\sqrt{3}(1-\theta)-1} \geq 0$, we only need to care of the validity of inequality $k_r^2 - (4-2\sqrt{3})\tilde{k}_r k_r - k_r + (7-4\sqrt{3})\tilde{k}_r^2 - (3-2\sqrt{3})\tilde{k}_r + (4-2\sqrt{3})s_r \geq 0$. First, we observe that we can get rid of the term $(4-2\sqrt{3})s_r$, as it is always non-negative. If we solve for k_r the homogeneous equality associated with the remaining part, we get that the discriminant is negative for $\tilde{k}_r \geq 2$. Thus, it remains to show that $k_r^2 - (4-2\sqrt{3})\tilde{k}_r k_r - k_r + (7-4\sqrt{3})\tilde{k}_r^2 - (3-2\sqrt{3})\tilde{k}_r \geq 0$ holds whenever $\tilde{k}_r \in \{0,1\}$. In one case, we get inequality $k_r(k_r-1) \geq 0$, in the other, we get inequality $k_r^2 - (5-2\sqrt{3})k_r + 4 - 2\sqrt{3} \geq 0$ and both of them hold true as k_r is a non-negative integer.

For $\theta \in [\frac{3-\sqrt{3}}{2}, \frac{2}{3}]$, set $x = \frac{2}{1-\theta} > 0$, $y = 1$, and $\gamma = \frac{5\theta^2-6\theta+2}{(1-\theta)^2}$. By substituting these values in the second dual constraint, we obtain the inequality $(2\theta-1)^2\tilde{k}_r \geq 0$, which holds true as $\tilde{k}_r$ is non-negative. Substituting in the first constraint, we obtain the inequality $(1-\theta)^2 k_r^2 - (1-\theta)((4-6\theta)\tilde{k}_r + 1 - \theta)k_r + (2-3\theta)^2\tilde{k}_r^2 - (1-\theta)(3-5\theta)\tilde{k}_r + 2(1-\theta)(2-3\theta)s_r \geq 0$, which can be shown to hold true. First, we observe that we can get rid of the term $2(1-\theta)(2-3\theta)s_r$, as it is always non-negative. If we solve for k_r the homogeneous equality associated with the remaining part, we get that the discriminant is negative for $\tilde{k}_r \geq \frac{1-\theta}{4(8\theta-5)}$. As, for $\theta \in [\frac{3-\sqrt{3}}{2}, \frac{2}{3}]$, we have $\frac{1-\theta}{4(8\theta-5)} \leq 2$, it remains to show that $(1-\theta)^2 k_r^2 - (1-\theta)((4-6\theta)\tilde{k}_r + 1-\theta)k_r + (2-3\theta)^2\tilde{k}_r^2 - (1-\theta)(3-5\theta)\tilde{k}_r \geq 0$ holds whenever $\tilde{k}_r \in \{0,1\}$. For $\tilde{k}_r = 0$, we get inequality $k_r(k_r-1) \geq 0$, which holds true as k_r is a non-negative integer. For $\tilde{k}_r = 1$, we get inequality $(1-\theta)^2 k_r^2 - (1-\theta)(5-7\theta)k_r + (2t-1)^2 \geq 0$. To show that this inequality holds true, we proceed as follows. First, we analyse inequality $(1-\theta)^2 k_r^2 - (1-\theta)(5-7\theta)k_r \geq 0$. This holds true whenever $k_r \geq \frac{5-7\theta}{1-\theta}$. As, for $\theta \in [\frac{3-\sqrt{3}}{2}, \frac{2}{3}]$, we have $\frac{5-7\theta}{1-\theta} \leq 2$, it follows that $(1-\theta)^2 k_r^2 - (1-\theta)(5-7\theta)k_r \geq 0$, which implies $(1-\theta)^2 k_r^2 - (1-\theta)(5-7\theta)k_r + (2t-1)^2 \geq 0$, holds true when $k_r \geq 2$. Thus, we are left to show that $(1-\theta)^2 k_r^2 - (1-\theta)(5-7\theta)k_r + (2t-1)^2 \geq 0$ for $k_r \in \{0,1\}$. In one case, we get inequality $(2\theta-1)^2 \geq 0$ which is always true, in the other, we get inequality $2\theta^2 - 6\theta + 3 \leq 0$ which holds true as $\theta \in [\frac{3-\sqrt{3}}{2}, \frac{2}{3}]$.

For $\theta \in [\frac{2}{3}, 1)$, set $x = \frac{1}{1-\theta} > 0$, $y = 0$, and $\gamma = \frac{\theta}{1-\theta}$. By substituting the proposed values in the second dual constraint, we obtain the inequality $\frac{(2\theta-1)}{1-\theta}\tilde{k}_r \geq 0$, which holds true as $\theta \in [\frac{1}{2}, 1)$ and $\tilde{k}_r$ is non-negative. Substituting in the first constraint, we obtain the inequality $\frac{(2\theta-1)}{1-\theta}(\tilde{k}_r k_r + \tilde{k}_r - s_r) \geq 0$, which holds true as $\theta \in [\frac{1}{2}, 1)$, k_r, $\tilde{k}_r$ and s_r are non-negative, and, by definition, $s_r \leq \tilde{k}_r$.

Finally, as for $\theta = 1$ we claim an unbounded price of anarchy, this holds true trivially. □

The most important consequence of this result is that, for $\theta = 1/2$, social optimal performance can be achieved at any equilibrium. Also, we have that the price of anarchy improves whenever $\theta < 5/7$ (see Figure 1).

Corollary 1 *When altruism is perfectly balanced and a social optimum is signalled, any pure Nash equilibrium for a θ-altruistic congestion game with signalling is socially optimal.*

Next result provides matching lower bounds for any value of θ.

Theorem 2 *All upper bounds given in Theorem 1 are tight.*

Efficiency of Personalized Taxes

Another interesting way to interpret the θ-cost function defined in (1) is to look at its second term as a personalized tax (i.e., player-dependent) applied to each selected resource.

Bilò and Vinci (2019) define statical optimal-dependent taxes for affine congestion games as follows. In an affine congestion game with statical optimal-dependent taxes, given a social optimum $\boldsymbol{\sigma}^*$ and a real value $\tau > 0$, the cost that player i pays in a strategy profile $\boldsymbol{\sigma}$ is defined as $\sum_{r\in\sigma_i}(\alpha_r(n_r(\boldsymbol{\sigma}) + \tau n_r(\boldsymbol{\sigma}^*)) + \beta_r)$. The tax $\sum_{r\in\sigma_i}\alpha_r \tau n_r(\boldsymbol{\sigma}^*)$ is called statical because is does not depend on $\boldsymbol{\sigma}$ and optimal-dependent because it depends on $\boldsymbol{\sigma}^*$. However, the tax does not depend on the player identity, meaning that, on each resource, all users are charged the same tax. Now, for any $\theta \neq 1$, if we divide both terms of the θ-cost by $1-\theta$, we get a new cost function, which we call $\widehat{\theta}$-cost function, defined as follows:

$$\widehat{\theta}\text{-}cost_i(\boldsymbol{\sigma}) = \sum_{r\in\sigma_i}(\alpha_r(n_r(\boldsymbol{\sigma})) + \beta_r) + \frac{\theta}{1-\theta}\sum_{r\in\sigma_i}\alpha_r(n_r(\tilde{\boldsymbol{\sigma}}) - \mathbb{I}_r(\tilde{\sigma}_i)). \quad (5)$$

As it can be appreciated, when the signalled strategy profile $\tilde{\boldsymbol{\sigma}}$ is a social optimum, player i's personal cost in $\boldsymbol{\sigma}$, namely $cost_i(\boldsymbol{\sigma})$, is increased with a statical optimal-dependent tax, in which $\tau = \frac{\theta}{1-\theta}$, except for a possible discount equal to $\frac{\theta}{1-\theta}\alpha_r\mathbb{I}_r(\tilde{\sigma}_i)$. Thus, as this last quantity depends also on the identity of player i, we have that the $\widehat{\theta}$-cost function defined in (5) models the application of statical optimal-dependent *personalized* taxes.

It is important to observe that, if we derive the inequality $\sum_{i\in N}\left(\widehat{\theta}\text{-}cost_i(\boldsymbol{\sigma}) - \widehat{\theta}\text{-}cost_i(\boldsymbol{\sigma}_{-i}, \tilde{\sigma}_i)\right) \leq 0$ and then multiply both sides by $1-\theta$, we reobtain inequality (2). By exploiting the fact that, in presence of refundable taxes, the social cost of both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ are expressed in the same way in both θ-altruistic affine congestion games with signalling and affine congestion games with statical optimal-dependent personalized taxes, we derive the same price of anarchy obtained in the previous section. In particular, for $\theta = 1/2$, we obtain that statical optimal-dependent personalized taxes are able to force players to decentralizedly implement a social optimum. This is in contrast with what happens for statical optimal-dependent taxes, for which the price of anarchy is equal to 2 and no improvements are likely to be possible. This huge difference is due to the personalization introduced by our model.

One can argue that, if a tax function is either optimal- and player-dependent, then it is trivial to force the players to play according to a social optimum: simply charge players choosing a resource not belonging to their socially optimal strategy with an unbounded tax. This is clearly true, however, we argue that this is not what happens with our tax function for which, the difference in the amount of taxes charged to any two players for using a resource r is equal to $\frac{\theta}{1-\theta}\alpha_r$. Thus, whenever θ is sufficiently smaller than 1, the extra tax charged to players deviating from their socially optimal strategy remains well bounded (for $\theta = 1/2$, for instance, it is equal to α_r).

The most important consequence of our approach can be appreciated when considering non-refundable taxes. For this type of taxes, in fact, no improvements have been achieved so far when using, even dynamic, non-personalized taxes. As we shall show, the tax function defined in (5) is the first type of non-refundable taxes able to yield improved performance on the efficiency of pure Nash equilibria in affine congestion games.

To derive a tight upper bound on the price of anarchy for games with non-refundable taxes, we use the same approach of the previous section. The main difference is given by the social cost of a pure Nash equilibrium that now becomes $\sum_{r\in R}\left(\alpha_r\left(k_r^2+\frac{\theta}{1-\theta}\left(\tilde{k}_r k_r - s_r\right)\right)+\beta_r k_r\right)$.

Theorem 3 *For $\theta = 1/2$, the price of anarchy of affine congestion games with the non-refundable taxes defined in (5) is* 2.

Proof: According to the primal-dual method, in order to obtain an upper bound on the price of anarchy, we need to formulate, as a linear program on variables $(\alpha_r, \beta_r)_{r\in R}$, the problem of maximizing the utilitarian social cost of a pure Nash equilibrium, including the paid taxes, under the assumption that the utilitarian social cost of a social optimum equals 1 and inequality (2) holds true. We, thus, have:

$$\begin{aligned}
\max \quad & \sum_{r\in R}\left(\alpha_r\left(k_r^2+\tilde{k}_r k_r - s_r\right)+\beta_r k_r\right)\\
s.t. \quad & \sum_{r\in R}\left(\alpha_r\left(k_r^2-\tilde{k}_r^2\right)+\beta_r\left(k_r-\tilde{k}_r\right)\right)\le 0\\
& \sum_{r\in R}\left(\alpha_r\tilde{k}_r^2+\beta_r\tilde{k}_r\right)=1\\
& \alpha_r,\beta_r\ge 0 \qquad r\in R.
\end{aligned}$$

The dual program, obtained associating variable x with the first constraint and variable γ with the second constraint of the primal, is the following:

$$\begin{aligned}
\min \quad & \gamma\\
s.t. \quad & x\left(k_r^2-\tilde{k}_r^2\right)+\gamma\tilde{k}_r^2\ge k_r^2+\tilde{k}_r k_r - s_r && r\in R\\
& x(k_r-\tilde{k}_r)+\gamma\tilde{k}_r\ge k_r && r\in R\\
& x\ge 0.
\end{aligned}$$

To prove the claimed upper bound, it suffices to provide feasible dual solutions with γ equal to the desired upper bounds. Set $x = 3/2$ and $\gamma = 2$. By substituting these values in the second dual constraint, we obtain the inequality $k_r + \tilde{k}_r \ge 0$, which holds true as both k_r and $\tilde{k}_r$ are non-negative integers. Substituting in the first constraint, we obtain the inequality $k_r^2 - 2k_r\tilde{k}_r + \tilde{k}_r^2 + 2s_r = \left(k_r-\tilde{k}_r\right)^2 + 2s_r \ge 0$, which holds true as s_r is non-negative.

To show the lower bound, it suffices to consider a game with n players and n identical resources $r_1,\ldots,r_n$, each with latency function $\ell(x) = x$. Each player i can choose between two strategies: the *first strategy*, corresponding to $\{r_i\}$, and the *second strategy*, corresponding to $\{r_{i+1}\}$, where we set $r_{n+1} := r_1$. The strategy profile $\boldsymbol{\sigma}^*$ in which all players select their first strategy is a social optimum. The strategy profile $\boldsymbol{\sigma}$, in which all players select their second strategy instead, which is also a social optimum, can be shown to be a pure Nash equilibrium for the game with our taxation mechanism based on the signalled social optimum $\boldsymbol{\sigma}^*$. In fact, for every player i, we have $\widehat{\theta}\text{-}cost_i(\boldsymbol{\sigma}) = 2$ and, by deviating to her first strategy, player i pays a cost of $\widehat{\theta}\text{-}cost_i(\boldsymbol{\sigma}_{-i},\sigma_i^*) = 2 = \widehat{\theta}\text{-}cost_i(\boldsymbol{\sigma})$. By comparing $\mathsf{USC}(\boldsymbol{\sigma}^*) = n$ with $\mathsf{USC}(\boldsymbol{\sigma}) = 2n$, the desired lower bound follows. □

Conclusions

We have reconsidered affine congestion games under the assumption that, for every player, a most preferred strategy is signalled. We have shown that, when each signalled strategy comes from the social optimum, altruistic players perfectly balancing altruism and selfishness play optimally in any pure Nash equilibrium. As a byproduct of our approach, we were also able to design personalized taxation mechanisms achieving optimal performance in the refundable case and the first improved performance in the non-refundable one. A natural question is whether signalling can be exploited to improve also the price of stability. With this respect, it is worth observing that altruism without signalling already exhibit good performance and, in the case of perfectly balanced altruism, the price of stability is 1, and so it cannot be further improved.

Computing the social optimum in congestion games is a well-known hard problem (Meyers and Schulz 2012; Paccagnan and Gairing 2021) and, although our results extend to signalled strategies coming from an ξ-approximation of the social optimum by losing the same approximation guarantee, determining whether good performances can be achieved even by signalling simply computable strategies, as for instance the best *free-flow strategy* (Benita et al. 2020), is an intriguing research direction.

Finally, generalizing the results to the more general setting of congestion games with polynomial latency functions is also of interest.

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