# Inconsistent Planning: When in Doubt, Toss a Coin!

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**Abstract**

One of the most widespread human behavioral biases is the present bias—the tendency to overestimate current costs by a bias factor. Kleinberg and Oren (2014) introduced an elegant graph-theoretical model of inconsistent planning capturing the behavior of a present-biased agent accomplishing a set of actions. The essential measure of the system introduced by Kleinberg and Oren is the *cost of irrationality*—the ratio of the total cost of the actions performed by the present-biased agent to the optimal cost. This measure is vital for a task designer to estimate the aftermaths of human behavior related to time-inconsistent planning, including procrastination and abandonment.

As we prove in this paper, the cost of irrationality is highly susceptible to the agent's choices when faced with a few possible actions of equal estimated costs. To address this issue, we propose a modification of Kleinberg-Oren's model of inconsistent planning. In our model, when an agent selects from several options of minimum prescribed cost, he uses a randomized procedure. We explore the algorithmic complexity of computing and estimating the cost of irrationality in the new model.

## 1 Introduction

*Time-inconsistent behavior* is the term in behavioral economics and psychology describing the behavior of an agent optimizing a course of future actions but changing his optimal plans in the short run without new circumstances (Thaler 2016). For example, why do we buy a year swim membership and not go to the swimming pool after that? Why do we procrastinate when it comes to paying off credit card debt? Why do we want to eat healthier but have little incentive to do so? As Socrates in Plato's Protagoras asks, if one judges a certain behavior to be the best course of action, why would one do anything else?

A standard assumption in behavioral economics used to explain the gap between long-term intention and short-term decision-making is the notion of *present bias*. According to (O'Donoghue and Rabin 1999), when considering trade-offs between two future moments, present-biased preferences give stronger relative weight to the earlier moment as it gets closer.



The mathematical idea of present bias goes back to 1937 when (Samuelson 1937) introduced the discounted-utility model. It has developed into the hyperbolic discounting model, one of the cornerstones of behavioral economics (Laibson 1994; McClure et al. 2004). A simple mathematical model of present bias was suggested in (Akerlof 1991). In Akerlof's model, the salience factor causes the agent to put more weight on immediate events than on the future. Thus the cost of an action that will be perceived in the future is assumed to be $\beta$ times smaller than its actual cost, for some present-bias parameter $\beta < 1$. It appears that even a tiny salience factor could yield high extra costs for the agent.

Kleinberg and Oren (Kleinberg and Oren 2014, 2018) introduced an elegant graph-theoretic model encapsulating the salience factor and scenarios of Akerlof. The approach is based on analyzing how an agent traverses from a source $s$ to a target $t$ in a directed edge-weighted graph $G$. Before defining this model formally, we provide an illustrating example. The example, up to small modifications, is borrowed from (Kleinberg and Oren 2014) and originally due to Akerlof (Akerlof 1991).

**Example.** One of the authors of this paper, we call him Bob, is planning to write reviews for AAAI. He estimates the cost (say, estimated time) of this task as $c$. While the definite deadline is on Friday, Bob's initial plan is to write reviews on Monday. However, on Monday, Bob realizes that he also needs to check Google Scholar to find out who cited his paper. He estimates the cost of the other task as $x$. Now Bob meets the dilemma. He can either (a) write reviews today or (b) check Google Scholar today and write reviews tomorrow. While estimating the costs, Bob uses the present-bias parameter $\beta$. Thus he estimates the cost of (b) as $x + \beta c$. It appears that $x+\beta c < c$, therefore Bob decides to pursue (b). On Tuesday, the story repeats by procrastinating with Instagram, and on Wednesday with Facebook, see Fig. 1. At the end, Bob sends the reviews on Friday, spending on this job totally $4x + c$ instead of $c$.

**Kleinberg-Oren's Model.** An instance of the *time-inconsistent planning model* is a 5-tuple $M = (G, w, s, t, \beta)$ where:

- $G = (V(G), E(G))$ is a directed acyclic $n$-vertex graph called a *task graph*. $V(G)$ is a set of elements called *ver-*

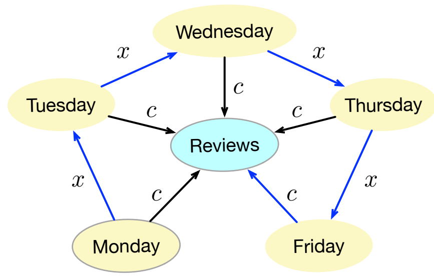


Figure 1: Task graph $G$ with the source point $s = \texttt{Monday}$ and the target point $t = \texttt{Reviews}$. Assuming $x + \beta c < c$, Bob will follow the blue path rather just writing the reviews on Monday. The ratio between the costs of the path followed by the agent and the optimal $s$-$t$ path can be made arbitrarily large by adding more days of procrastination.

*tices*, and $E(G) \subseteq V(G) \times V(G)$ is a set of *arcs* (directed edges). The graph is *acyclic*, which means that there exists an ordering of the vertices called a *topological order* such that, for each edge, its first endpoint comes strictly before its second endpoint in the ordering. Informally speaking, vertices represent states of intermediate progress, whereas edges represent possible actions that transitions an agent between states.

- $w : E(G) \to \mathbb{N}$ is a function representing the costs of transitions between states. The transition of the agent from state $u$ to state $v$ along arc $uv \in E(G)$ is of cost $w(uv)$.
- The agent starts from the start vertex $s \in V(G)$.
- $t \in V(G)$ is the target vertex.
- $\beta \leq 1$ is the agent's present-bias parameter.

An agent is initially at vertex $s$ and can move in the graph along arcs in their designated direction. The agent's task is to reach the target $t$. The agent moves according to the following rule. When standing at a vertex $v$, the agent evaluates (with a present bias) all possible paths from $v$ to $t$. In particular, a $v$-$t$ path $P \subseteq G$ with edges $e_1, e_2, \ldots, e_p$ is evaluated by the agent standing at $v$ to cost

$$\zeta_M(P) = w(e_1) + \beta \cdot \sum_{i=2}^{p} w(e_i).$$

We refer to this as the *perceived* cost of the path $P$. For a vertex $v$, its *perceived cost to the target* is the minimum perceived cost of any path to $t$,

$$\zeta_M(v) = \min\{\zeta_M(P) \mid P \text{ is a } v\text{-}t \text{ path}\}.$$

We refer to an $v$-$t$ path $P$ with perceived cost $\zeta_M(v)$ as to a *perceived path*. Thus when in vertex $v$, the agent picks one of the perceived paths and traverses its first edge, say $vu$. After arriving to the new vertex $u$, the agent computes the perceived cost to the target $\zeta_M(u)$, selects a perceived $u$-$t$ path , and traverse its first edge. This repeats until the agent reaches $t$.

Let $P_\beta(s,t)$ be a $s$-$t$ path followed by an agent with present-bias $\beta$ and let $c_\beta(s,t)$ be the cost of this path. Let $d(s,t)$ be the distance, that is the cost of a shortest $s$-$t$ path. Then Kleinberg and Oren defined the measure describing the "*price of irrationality*" of the system.

**Definition 1** (Cost of irrationality (Kleinberg and Oren 2014))**.** *The* cost of the irrationality *of the time-inconsistent planning model $M = (G, w, s, t, \beta)$ is*

$$\frac{c_\beta(s,t)}{d(s,t)}.$$

Thus in our example in Fig. 1, the cost of irrationality is $\frac{4x+c}{c}$. One omitted detail in the definition makes the meaning of the cost of irrationality ambiguous. It could be that several paths with minimum perceived cost $\zeta_M(v)$ lead from $v$. In this situation an agent in the state $v$ might be indifferent between several arcs leaving $v$—they both evaluate to equal perceived costs. Hence there could be several different *feasible* paths $P_\beta(s,t)$ which the agent could follow. While for the agent standing in a vertex $v$ the perceived costs of all perceived paths are the same, the actual costs of feasible paths could be different.

Two approaches to address this ambiguity could be found in the literature. First, one can assume that an agent uses a consistent tie-breaking rule. For example, Kleinberg and Oren in (Kleinberg and Oren 2014) suggest selecting the node that is earlier in a fixed topological ordering of $G$. Kleinberg, Oren, and Raghavan (Kleinberg, Oren, and Raghavan 2016) consider the situation when the arcs are ordered, and an agent selects the largest available arc. The disadvantage of this approach is that it is hard to imagine someone building their plans based on a topological ordering of tasks or arbitrarily labeled arcs in a real-life scenario. Another approach taken by Albers and Kraft (Albers and Kraft 2019) and by Fomin and Strømme (Fomin and Strømme 2020) is to break ties arbitrarily. As we will see, depending on how an agent breaks the ties, the cost of irrationality could change exponentially in the number of vertices. Therefore, with the second approach, the value of the cost of irrationality is not well-defined.

Because of that, we revisit the model of Kleinberg and Oren in (Kleinberg and Oren 2014) and redefine the cost of irrationality. Our approach is natural —*when in doubt, toss a coin!* When several paths of minimum prescribed cost lead from $v$, the agent selects one of them with some probability and traverses the first arc of this path.

More precisely, we view the graph as a Markov decision process. Thus the instance of the *time-inconsistent planning model* is a 6-tuple $M = (G, w, s, t, p, \beta)$, where for each edge $uv$ of the task graph, we assign the probability $p(u,v)$ of transition $u \to v$. Here for every $u \in V(G)$, $\sum_{uv \in E(G)} p(u,v) = 1$.

Moreover, the probability can be positive only for edges that could serve for transitions of the agent. In other words, $p(u,v) > 0$ only if there is a $u$-$t$ path $P$ of perceived cost $\zeta_M(u)$ whose first edge is $uv$. The selection of probability $p$ corresponds to some predictions or future preferences in breaking the ties. For example, when the agent at stage $u$ faces $\ell$ $u$-$t$ paths of minimum perceived cost and has no preferences over any of them, it would be natural to assign each transition from $u$ the probability $1/\ell$. On the other hand, if the agent has preferences in selecting from paths of equal costs, this can be controlled by a different selection of $p$. With these settings, we call an $s$-$t$ path $P$ *feasible*, if with a non-zero probability the present-biased agent will follow $P$.

Now we can define the cost of the agent with present-bias $\beta$ as discrete random variable $C_\beta$ with $\Pr(C_\beta = W)$ being the probability that the path traversed by the agent is of cost $W$. Then we can redefine the cost of irrationality as follows.

**Definition 2** (Revised cost of irrationality)**.** *The* cost of the irrationality *of the time-inconsistent planning model* $M = (G, w, s, t, p, \beta)$ *is*

$$X_\beta = \frac{C_\beta}{d(s,t)}.$$

Let us note that when no ties occur, our definition coincides with the definition of Kleinberg and Oren. Estimating the cost of irrationality $X_\beta$ could help the task-designer to evaluate the chances of *abandonment*, the situation when an agent realizes that accomplishing the task takes much more effort than he presumed initially, and thus ultimately gives up.

**Example cont.** In the example in Fig. 1, let us put $c = 6$, $x = 3$, and $\beta = \frac{1}{2}$. We also assume that Bob does not have preferences between two actions of minimum perceived costs and thus pursue one of the actions with probability $p = 1/2$. On Monday, Bob selects between two options of perceived costs 6: either to write reviews that costs $c = 6$, or to check Google Scholar and write reviews tomorrow, which costs $x + \beta c = 6$. The probability that Bob will finish reviews on Monday, and thus will spend $c = 6$ hours, is $1/2$. The optimal cost $d(s,t) = c = 6$. Hence $\Pr(C_\beta \leq 6) = \frac{1}{2}$ and thus $\Pr(X_\beta \leq 1) = \frac{1}{2}$ . The probability that Bob finishes the job on Tuesday and thus will spend $x + c = 9$, is $\left(\frac{1}{2}\right)^2$. Therefore, $\Pr(X_\beta \leq 9/6 = 3/2) = \frac{1}{2} + \left(\frac{1}{2}\right)^2$. The situation repeats up till Thursday, and we have that for $1 \leq i \leq 4$, $\Pr(X_\beta \leq 1 + (i-1)/2) = \sum_{j=1}^{i} \left(\frac{1}{2}\right)^j$. Bob has to submit by Friday, so $\Pr(X_\beta \leq 3) = 1$.

**Our contribution.** We introduce the randomized version of the cost of irrationality and initiate its study from the computational perspective. To support our point of view on the cost of irrationality, we start from the combinatorial result (Theorem 1), showing that there are time-inconsistent planning models with exponentially (in $n$) many feasible paths of different costs. It yields that in the deterministic model of Kleinberg and Oren (Definition 1) there could be exponentially many different costs of irrationality.

To study the cost of irrationality $X_\beta$, we define the following computational problem.

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| ESTIMATING THE COST OF IRRATIONALITY (ECI)<br>**Input:** A time-inconsistent planning model $M = (G, w, s, t, p, \beta)$, and $W \geq 0$.<br>**Task:** Compute $\Pr(X_\beta \leq W)$. |

We show in Theorem 2 that ECI is #P-hard. Thus computationally, ECI is not easier than counting Hamiltonian cycles, counting perfect matching, satidying assignments, and all other #P-complete problems. Our hardness proof strongly exploits the fact that the edge weight $w$ of the model are exponential in the $n$, the number of vertices of $G$. We show that when the edge weights are bounded by some polynomial of $n$, then ECI is solvable in polynomial time. We also obtain polynomial time algorithms, even for exponential weights, for the important "border" cases: minimum, maximum, and average. More precisely, we prove that each of the following tasks

(a) finding the minimum value $W$ such that $\Pr(X_\beta \leq W)$ is positive and computing $\Pr(X_\beta \leq W)$ (Theorem 3),

(b) finding the minimum value $W$ such that $\Pr(X_\beta \leq W) = 1$ (Theorem 3), and

(c) computing $\mathbf{E}(X_\beta)$ (Theorem 5),

can be done in polynomial time.

We also take a look at ECI from the perspective of structural parameterized complexity. Structural parameterized complexity is the common tool in graph algorithms for analyzing intractable problems. Thus we are interested how the structure of the graph $G$ in the time-inconsistent model could be used to design efficient algorithms. For example, the problem of finding a maximum weight set of independent vertices is an NP-hard problem. However, it becomes tractable when the treewidth of the input graph is bounded. We will have more discussions about parameterized complexity in Section 5. The most popular graph parameters in parameterized complexity are the treewidth of a graph, the size of a minimum feedback vertex set and vertex cover (Cygan et al. 2015). For a directed graph $G$, let $\mathrm{tw}(G)$, $\mathrm{fvs}(G)$, and $\mathrm{vc}(G)$ be the treewidth, the minimum size of a feedback vertex set and the minimum size of a vertex cover of the underlying undirected graph of $G$, correspondingly. We prove the following

- ECI is #P-hard even when in the time-inconsistent planning model $M = (G, w, s, t, p, \beta)$, we have $\mathrm{tw}(G) = 2$. (This result actually follows directly from the reduction of Theorem 2.)
- ECI is W[1]-hard parameterized by $\mathrm{fvs}(G)$ and by $\mathrm{vc}(G)$ (Theorem 6). On the other hand, ECI is solvable in times $n^{\mathcal{O}(\mathrm{fvs}(G))}$ and $n^{\mathcal{O}(\mathrm{vc}(G))}$.

On the other hand, when parameterized by the minimum size of the feedback edge set of the underlying graph, $\mathrm{fes}(G)$, that is the set of edges whose removal makes the graph acyclic, the problem becomes fixed-parameter tractable.

Our results demonstrate that while computing the cost of irrationality is intractable in the worst-case, in many interesting situations this parameter could be computed efficiently.

**Related Work.** The graph-theoretical model we use in this paper for time-inconsistent planning is due to (Kleinberg and Oren 2014, 2018). We refer to these papers for a survey of earlier work on time-inconsistent planning, with connections to procrastination, abandonment, and choice reduction. There is a significant amount of the follow-up work on the

the model of Kleinberg and Oren. Albers and Kraft (Albers and Kraft 2019) studied the ability to place rewards at nodes for motivating and guiding the agent. They show hardness and inapproximability results and provide an approximation algorithm whose performances match the inapproximability bound. The same authors considered another approach in (Albers and Kraft 2017) for overcoming these hardness issues by allowing not to remove edges but to increase their weight. They were able to design a 2-approximation algorithm in this context. Tang et al. (Tang et al. 2017) also proved hardness results related to the placement of rewards and showed that finding a motivating subgraph is NP-hard. Fomin and Strømme (Fomin and Strømme 2020) studied the parameterized complexity of computing a motivating subgraph in the model of Kleinberg and Oren.

## 2 Exponential Number of Different Cost of Irrationality

In this section we provide an example supporting our definition of the cost of irrationality (Definition 2). In our construction, the agent following from $s$ to $t$ could follow one of exponentially many feasible paths of different final costs. It implies that the cost of irrationality in deterministic (Definition 1) could vary exponentially depending on how the agent selects between paths of equal perceived costs. The construction we use to prove Theorem 1 is also used to obtain the complexity result.

**Theorem 1.** *There is a graph with an exponential (in the number of vertices) number of feasible paths of different costs.*

For the omitted proof please consult the supplementary material.

**Remark 1.** *By Theorem 1, the difference between the costs of the minimum and maximum feasible paths in the graph can be exponential from the number of vertices.*

## 3 Estimating the Cost of Irrationality

In this section, we will evaluate the complexity of estimating a random value $X_\beta$. We give a parsimonious reduction of the following problem to ECI.

> COUNTING PARTITIONS
> **Input:** Set of positive integers $S = \{s_1, \dots, s_n\}$.
> **Task:** Count the number of partitions of $S$ into sets $S_1$ and $S_2$ such that the sums of numbers in both sets are equal.

COUNTING PARTITIONS is known to be $\#$P-hard (Dyer et al. 1993).

**Theorem 2.** *The* ECI *problem is* $\#$P*-hard.*

*Proof.* Let's reduce the COUNTING PARTITIONS to our problem. For an instance $S = \{s_1, \dots, s_n\}$ we construct a time-inconsistent planning model $M = (G, w, s, t, p, \beta)$. Every $s$-$t$ paths in $G$ will be feasible and there will be a bijection between feasible paths of certain cost in $G$ and partitions of the set $S$ into two parts. Thus the number of feasible paths will be the solution to COUNTING PARTITIONS.

Our construction works for any present bias $\beta < 1$. Consider a graph consisting of "diamond" gadgets. The diamond consists of 2 vertices connected by 2 paths of length 2, see Fig. 2. The graph consists of $n$ diamonds $D_1, \dots, D_n$ concatenated together. The weights of the edges are defined as follows. Let $W$ be an integer that is greater than all $s_i$. For every $i \in \{1, \dots, n\}$, the edges of the first path of the diamond $D_i$ obtain weights $s_i$ and $\frac{W-s_i}{\beta}$. The edges of the second path of the diamond $D_i$ obtain weights $-s_i$ and $\frac{W+s_i}{\beta}$. This completes the construction of $G$.

We also add that we can get rid of the negative weights of the edges by adding the same additive to all the edges, it is easy to understand that the agent's solution will not change from this additive.

Let us note that for the agent standing in the first vertex $v$ of a diamond $D_i$ there are exactly two perceived paths, the first of which starts with the upper $(s_i)$-path of the diamond $D_i$, the second with the bottom $(-s_i)$-path, and both of them continue with the upper (shortest) paths of all the remaining diamonds. In the Markov decision process, we assume that the agent select one of these edges with probability $p = 1/2$. Since each of the $s$-$t$ paths in model $M$ is feasible, each of these paths will be used with probability $(1/2)^n$.

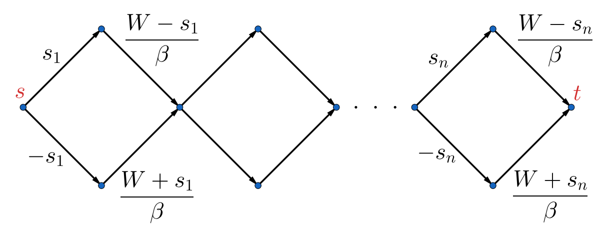


Figure 2: Gadget used in the proof of the theorem.

Now let us show the bijection between feasible paths of cost $\frac{n \cdot W}{\beta}$ and equal partitions of $S$.

In one direction, let $A$ and $B$ be a partition of $S$ such that $\sum_{s \in A} s = \sum_{s \in B} s$. We take the path corresponding to this partition. When passing through diamond $D_i$, the path goes through edge of weight $s_i$ is $s_i \in A$ and $-s_i$ otherwise. We define

$$\begin{cases} \delta = 0, \text{ if } s_i \in A \\ \delta = 1, \text{ if } s_i \in B. \end{cases}$$

Then the total cost of such a path is equal to

$$\sum_{i=1}^{n} (-1)^{\delta} \cdot s_i + \sum_{i=1}^{n} \frac{W + (-1)^{\delta+1} \cdot s_i}{\beta} = \frac{n \cdot W}{\beta}.$$

For the opposite direction. Let $P$ be a path of cost $\frac{n \cdot W}{\beta}$. The way $P$ traverses through each of the diamonds, specify a partition of $S$ into two sets $A$ and $B$. We claim that $\sum_{s \in A} s = \sum_{s \in B} s$. Targeting towards a contradiction, assume that $Q = \sum_{s \in A} s - \sum_{s \in B} s > 0$. (The arguments for $Q = \sum_{s \in A} s - \sum_{s \in B} s < 0$ are similar.)

Then the cost of $P$ is equal to

$$\begin{aligned}\sum_{i=1}^{n}(-1)^{\delta}\cdot s_i \quad + \quad \sum_{i=1}^{n}\frac{W+(-1)^{\delta+1}\cdot s_i}{\beta} \\ = \quad Q+\frac{n\cdot W-Q}{\beta}<\frac{n\cdot W}{\beta}.\end{aligned}$$

But this is a contradiction to our assumption that the cost of $P$ is $\frac{n\cdot W}{\beta}$.

We have constructed a parsimonious reduction of the partitioning problem to the problem of counting the number of feasible paths of cost $\frac{n\cdot W}{\beta}$. Thus, by counting the number of different feasible paths of cost $\frac{n\cdot W}{\beta}$, we can count the number of different solutions for the partition problem.

Let $T=\frac{n\cdot W}{\beta}$. We already established that counting the number of $s$-$t$ paths of cost $T$ in $G$ is #P-hard. Now we show that computing $\Pr(C_\beta \leq T)$ is #P-hard. Note that in our graph all paths are feasible to the agent. Thus each of the paths will be traversed by the agent with the same probability $(\frac{1}{2})^n$. Let $P_{\leq T}$ be the number of paths of length at most $T$ and $P_{=T}$ be the number of paths of length exactly $T$. Then

$$\begin{aligned}\Pr(C_\beta \leq T) \quad &= \quad \frac{P_{\leq T}}{2^n}=\frac{P_{=T}+P_{\leq T-1}}{2^n} \\ &= \quad \frac{P_T}{2^n}+\Pr(C_\beta \leq T-1).\end{aligned}$$

Therefore, the existence of a polynomial time algorithm computing $\Pr(C_\beta \leq T)$ would allow us to count in polynomial time the number of paths of cost $T$.

Finally, let us remind that $X_\beta=\frac{C_\beta}{d(s,t)}$. Since the minimum cost $d(s,t)$ is computable in polynomial time by making use of the Bellman-Ford algorithm, a polynomial time algorithm computing $\Pr(X_\beta \leq T/d(s,t))$ would allow to compute in polynomial time $Pr(C_\beta \leq T)$, which is #P-hard. □

Although in general ECI appears to be a difficult problem, in some interesting cases described below, it can be solved in polynomial time. We define the following two "extremal" cases of the problem. In the first one we estimate the probability that the agent will follow one of the feasible paths of minimum cost. The second is to compute the maximum cost of a feasible path.

> MINIMUM COST OF IRRATIONALITY (MCI)
> **Input:** A time-inconsistent planning model $M=(G,w,s,t,p,\beta)$.
> **Task:** Compute the minimum value $W$ such that $\Pr(X_\beta \leq W)>0$ and compute $\Pr(X_\beta \leq W)$.

> MAXIMUM COST OF IRRATIONALITY
> **Input:** A time-inconsistent planning model $M=(G,w,s,t,p,\beta)$.
> **Task:** Compute the minimum value $W$ such that $\Pr(X_\beta \leq W)=1$.

**Algorithm 1**: Dynamic programming for ECI

**Input**: $M=(G,w,s,t,p,\beta)$, $W\geq 0$
**Output**: $\Pr(C_\beta \leq \lfloor W\cdot d(s,t)\rfloor)$

1: Let $A$ – an array of topologically sorted vertices, $s=A[0],\ t=A[n]$
2: $P_s=[1,0,0,\dots,0]$
3: $P_v=[0,0,0,\dots,0]\ \forall v\neq s$
4: **for** $v\in A$ **do**
5:     $U$ - the set of neighbors of the vertex $v$, such that from $u$ the agent can go to $v$
6:     **for** $k:=0$ to $\lfloor W\cdot d(s,t)\rfloor$ **do**
7:         $P_v[k]:=\sum_{U'}P_u[k-w(u,v)]\cdot\Pr(u\to v)$, where $U'=\{u\in U \mid w(uv)\leq k\}$
8:     **end for**
9: **end for**
10: **return** $\sum_{k=0}^{\lfloor W\cdot d(s,t)\rfloor}P_t[k]$

**Theorem 3.** MINIMUM COST OF IRRATIONALITY *and* MAXIMUM COST OF IRRATIONALITY *admits an algorithm with running time* $\mathcal{O}(n^3)$.

For the omitted proof please consult the supplementary material.

Finally we prove that if the weights of edges are polynomial in $n$, then ECI is solvable in polynomial time.

**Theorem 4.** ECI *admits an algorithm with running time* $\mathcal{O}(\lfloor W\cdot d(s,t)\rfloor\cdot n^2+n^3)$.

*Proof.* We will traverse the vertices in the order of their topological sorting. For each vertex $v$, we will calculate the array $P_v$, numbered $0,\dots,\lfloor W\cdot d(s,t)\rfloor$, where cell $P_v[k]$ will store the probability that the agent arrived at the vertex $v$ along the path of cost k. See Algorithm 1.

It is possible to make a topological sorting of the vertices of the graph $G$ and obtain an array $A$ in time $\mathcal{O}(n^2)$. Note that having first counted the shortest paths in the graph between any pair of vertices in time $\mathcal{O}(n^3)$, for each vertex $v$ we can find the set $U$ from line 5 of the Algorithm 1 in time $\mathcal{O}(n^2)$, going through all ancestors of $v$, and for each of them by modeling the agent's estimate in linear time.

Note that in line 7 of our algorithm, for each vertex $v$ and each possible cost of the path $k$, the probability that the agent will arrive at the vertex $v$ along the path of cost $k$ is correctly calculated, since the events of arrival at the vertex from different neighbors are inconsistent and the total probability of getting to the vertex $v$ is calculated as the sum of the probabilities for all available neighbors.

So the total running time of the algorithm is $\mathcal{O}(n^3)+\mathcal{O}(\lfloor W\cdot d(s,t)\rfloor\cdot n^2)$.

□

## 4 Computing Expected Cost

We proved that the ECI problem is $\#P$-hard. In this section we show that computing the expectation $\mathbf{E}(X_\beta)$ and the variance $\mathbf{Var}(X_\beta)$ of random variable $X_\beta$ can be done in polynomial time.

**Theorem 5.** *For the input $M = (G, w, s, t, p, \beta)$ of* ECI*, the values $\mathbf{E}(X_\beta)$ and $\mathbf{Var}(X_\beta)$ are computable in time $\mathcal{O}(n^3)$.*

For the omitted proof please consult the supplementary material.

The algorithms for the mean and the variance could be useful to *motivate* the agent. Let us consider the situation when for time-inconsistent planning model $M = (G, w, s, t, p, \beta)$, we can choose a reward to motivate an agent to achieve a goal (target vertex). At every step, the agent decides whether he wants to proceed further based on the following estimations. The agent compares the perceived cost of the remaining tasks, taking into account the present bias, and the reward: if the reward is greater than the estimate of the remaining path, then the agent moves further, otherwise he *abandon* his attempts to reach the goal. Then the natural algorithmic question in time-inconsistent planning (Kleinberg and Oren 2014), is how to identify the minimum reward that will allow the agent to reach his goal?

With the mean and variance, we can estimate the minimum award that can help to avoid abandonment. We need the Chebyshev's inequality:

$$\Pr(|C_\beta - \mathbf{E}(C_\beta)| \geq a) \leq \frac{\mathbf{Var}(C_\beta)}{a^2}.$$

For $a = 2\sqrt{\mathbf{Var}(C_\beta)}$, we have

$$\Pr(|C_\beta - \mathbf{E}(C_\beta)| \leq 2\sqrt{\mathbf{Var}(C_\beta)}) \geq \frac{3}{4}.$$

Then, as a reward, we take $\mathbf{E}(C_\beta)+2\sqrt{\mathbf{Var}(C_\beta)}$. With such reward the probability that the agent will reach his goal is at least $3/4$.

One can also consider a model in which the costs that the agent already has spent are deducted from the reward. (Imagine the situation when the agent has some resources and will not reach the goal when the resources are exhausted). In this case, the reward must be at least the cost of the perceived path. In this situation, the reward provided by the Chebyshev's inequality is optimal for a probability $\frac{3}{4}$ of reaching the goal.

## 5 Parameterized Complexity of ECI

In this section we investigate parameterized complexity of ESTIMATING THE COST OF IRRATIONALITY. A *parameterized problem* is a language $Q \subseteq \Sigma^* \times \mathbb{N}$ where $\Sigma^*$ is the set of strings over a finite alphabet $\Sigma$. Respectively, an input of $Q$ is a pair $(I, k)$ where $I \subseteq \Sigma^*$ and $k \in \mathbb{N}$; $k$ is the *parameter* of the problem. A parameterized problem $Q$ is *fixed-parameter tractable* ($\mathsf{FPT}$) if it can be decided whether $(I, k) \in Q$ in time $f(k) \cdot |I|^{\mathcal{O}(1)}$ for some function $f$ that depends of the parameter $k$ only. Respectively, the parameterized complexity class $\mathsf{FPT}$ is composed by fixed-parameter tractable problems. The W-hierarchy is a collection of computational complexity classes: we omit the technical definitions here. The following relation is known amongst the classes in the W-hierarchy: $\mathsf{FPT} = \mathsf{W}[0] \subseteq \mathsf{W}[1] \subseteq \mathsf{W}[2] \subseteq \ldots \subseteq \mathsf{W}[P]$. It is widely believed that $\mathsf{FPT} \neq \mathsf{W}[1]$, and hence if a problem is hard for the class $\mathsf{W}[i]$ (for any $i \geq 1$) then it is considered to be fixed-parameter intractable. For our purposes, to prove that a problem is $\mathsf{W}[1]$-hard it is sufficient to show that an $\mathsf{FPT}$ algorithm for this problem yields an $\mathsf{FPT}$ algorithm for some $\mathsf{W}[1]$-complete problem. We refer to (Cygan et al. 2015) for the detailed introduction to parameterized complexity.

In graph algorithms, one of the most popular parameter is the treewidth of (an undirected) graph. Many $\mathsf{NP}$-hard problems are $\mathsf{FPT}$ parameterized by the treewidth of the input graph. We refer (Cygan et al. 2015) for the definition of treewidth. For directed graph $G$, let $\mathrm{tw}(G)$ be the treewidth of its underlying undiricted graph. In Theorem 2, we have proved that ECI is $\#P$-hard. The underlying undirected graph used in the reduction in Theorem 2, has treewidth at most 2. (See Fig. 2). Thus we immediately obtain the following corollary.

**Corollary 1.** *The* ECI *problem remains $\#P$-hard even when the graph $G$ in the time-inconsistent model has* $\mathrm{tw}(G) \leq 2$.

Besides the treewidth of the graph, another popular in the literature graph parameters are vertex cover and feedback vertex set. Let us remain, that for an undirected graph $G$, a *vertex cover* of $G$ is a set of vertices $S \subseteq V(G)$, such that every edge of $G$ has at least one endpoint in $S$. In other words, the graph $G - S$ has no edges. For directed graph $G$, we use $\mathrm{vc}(G)$ to denote the minimum size of a vertex cover in the underlying undirected graph of $G$. A *feedback vertex set* of an undirected graph $G$ is the set of vertices $S$ such that every cycle in $G$ contains at least one vertex from $S$. In other words, graph $G - S$ has no cycles and thus is a forest. For directed graph $G$, we use $\mathrm{fvs}(G)$ to denote the minimum size of a feedback vertex cover in the underlying undirected graph of $G$. Let us note that

$$\mathrm{tw}(G) \leq \mathrm{fvs}(G) \leq \mathrm{vc}(G).$$

In what follows, we prove that ECI is $\mathsf{W}[1]$-hard parameterized by $\mathrm{vc}(G)$. Since $\mathrm{fvs}(G) \leq \mathrm{vc}(G)$, it also yields that ECI is $\mathsf{W}[1]$-hard parameterized by $\mathrm{fvs}(G)$. On the other hand, we will give an algorithm solving ECI in time $n^{\mathcal{O}(\mathrm{fvs}(G))}$. Thus the problem is $\mathsf{XP}$ parameterized by $\mathrm{fvs}(G)$. Since $\mathrm{fvs}(G) \leq \mathrm{vc}(G)$ it also implies that the problem is $\mathsf{XP}$ parameterized by $\mathrm{vc}(G)$.

We start from the lower bound. We reduce the following $\mathsf{W}[1]$-hard problem to ECI.

> MODIFIED $k$-SUM
> **Input:** Sets of integers $X_1, X_2, \ldots, X_k$ and integer $T$
> **Parameter:** $k$
> **Task:** Decide whether there is $x_1 \in X_1$, $x_2 \in X_2$, $\ldots$, $x_k \in X_k$ such that $x_1 + \ldots + x_k = T$.

The following lemma is a folklore. We could not find its proof in the literature and provide a sketch in the supplementary materials.

**Lemma 1.** *The* MODIFIED K-SUM *problem is* $\mathsf{W}[1]$*-hard with respect to the parameter $k$.*

**Theorem 6.** *The* ECI *problem is* W[1]*-hard parameterized by* $\mathrm{vc}(G)$ *and by* $\mathrm{fvs}(G)$*.*

*Proof.* We construct a parameterized reduction of the MODIFIED K-SUM problem to the ECI problem.

Let's give an instance of the MODIFIED K-SUM problem: $X_1, X_2, \ldots, X_k$ and $T$. Similarly to the proof of #P hardness, we construct for each set $X_i$ a gadget in which all paths will be perceived for the agent and will be evaluated in $W$ (for an edge $c_i$, an additional edge will have the weight $\frac{W-c_i}{\beta}$), where $W$ is an integer greater than all $x \in \cup_i X_i$. The graph $G$ consists of $k$ gadgets concatenated together. As $\beta$, we take any constant from 0 to 1, for example, $\beta = \frac{1}{2}$.

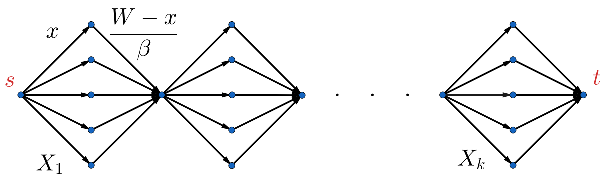


Figure 3: Gadget used in W[1]-hardness proof.

Let's set the target weight of the path to the agent as follows: $Y = T \cdot (1 - \frac{1}{\beta}) + \frac{k \cdot W}{\beta}$. The task parameter is the vertex cover or feedback vertex set will be equal to $\mathcal{O}(k)$.

We now show that the answer to the MODIFIED K-SUM problem is positive if and only if the answer to the ECI problem is positive. Let there be a set $x_i \in X_i$, which in total gives $T$, then the agent, choosing a path in the $i$ gadget, the first edge of which has a weight of $x_i$, will get a path whose total weight is

$$\sum_i x_i + \frac{W - x_i}{\beta} = T + \frac{k \cdot W}{\beta} - \frac{T}{\beta}.$$

Conversely, let the agent find the path of the desired weight $Y$, denote the weight of the first edge in the $i$ gadget in this path for $x_i$. Let $\sum x_i = T'$. Then

$$T \cdot (1 - \frac{1}{\beta}) + \frac{k \cdot W}{\beta} = Y = \sum_i x_i + \frac{W - x_i}{\beta} =$$

$$= T' + \frac{k \cdot W}{\beta} - \frac{T'}{\beta} = T' \cdot (1 - \frac{1}{\beta}) + \frac{k \cdot W}{\beta}.$$

We get that $T' = T$.

We already established that existence of $s$-$t$ paths of cost $Y$ in $G$ is W[1]-hard. Now we show that computing $\Pr(C_\beta \leq Y)$ is W[1]-hard. Note that

$$\Pr(C_\beta = Y) = \Pr(C_\beta \leq Y) - \Pr(C_\beta \leq Y - 1).$$

Thus, if $\Pr(C_\beta = Y) > 0$, then there is a feasible path of cost $Y$. Therefore, the existence of a FPT algorithm computing $\Pr(C_\beta \leq Y)$ would allow us to check for existence the paths of cost $Y$ in FPT time.

$\square$

Theorem 6 rules out the existence of an algorithm solving ECI in time $f(\mathrm{vc}(G))n^{\mathcal{O}(1)}$ for any function $f$ of $\mathrm{vc}(G)$ only. (Unless FPT =W[1].) In what follows, we prove that when $\mathrm{fvs}(G)$ is a constant, then the problem is solvable in polynomial time, that is, is in XP parameterized by $\mathrm{fvs}(G)$ (and hence by $\mathrm{vc}(G)$).

For the omitted proofs of the last four results please consult the supplementary materials.

We start from the following combinatorial lemma.

**Lemma 2.** *Let* $\mathrm{fvs}(G) = k$*. Then the number of different* $s$*-*$t$ *paths in* $G$ *is at most* $k^k n^{\mathcal{O}(k)}$*.*

By making use of Lemma 2, we prove the following theorem.

**Theorem 7.** *The* ECI *problem admits an algorithm of running time* $n^{\mathcal{O}(\mathrm{fvs}(G))} \cdot \mathrm{fvs}(G)^{\mathrm{fvs}(G)}$*.*

Because of Theorem 6, the running time provided by Theorem 7 is basically the best we can hope for. However, the ECI problem is FPT being parameterized by the feedback edge set of the underlying undirected graph. Let us remind, that a *feedback edge set* of an undirected graph is a set of edges whose removal makes the graph acyclic. For directed graph $G$, we use $\mathrm{fes}(G)$ to denote the minimum size of a feedback edge set of its underlying undirected graph.

First we bound the number of paths in a graph by a function of $\mathrm{fes}(G)$.

**Lemma 3.** *Let* $\mathrm{fes}(G) = k$*. Then the number of different* $s$*-*$t$ *paths in* $G$ *is at most* $2^k$*.*

By Lemma 3, we obtain that ECI is FPT parameterized by $\mathrm{fes}(G)$.

**Theorem 8.** *The* ECI *problem is solvable in time* $2^{\mathrm{fes}(G)} \cdot$ *poly*$(n)$*.*

## 6 Conclusion

We introduced the new model of the cost of irrationality. For future research we present two open algorithmic questions related to our model.

The first question concerns the motivation and establishing rewards (Albers and Kraft 2019). Assume that by achieving the goal $t$ the agent hopes to receive a reward. If at some moment the perceived costs becomes larger than the reward, the agent abandons the mission. For a given probability $p$, how difficult is to compute (exactly or approximately) the minimum reward that would allow the agent not to abandon his mission with probability at least $p$?

The second question is related to the question of finding a motivating subgraph (Kleinberg and Oren 2014; Fomin and Strømme 2020). We gave a polynomial time algorithm computing the expected cost of irrationality $\mathbf{E}(X_\beta)$. Consider the following algorithmic task: delete at most $k$ edges (or vertices) such that in the resulting graph the expected cost of irrationality is less than $\mathbf{E}(X_\beta)$. Of course, there is a brute-force algorithm solving the problem in time $n^{\mathcal{O}(k)}$ by calling our polynomial-time algorithm for each of the $\binom{n}{k}$ possibilities of deleting $k$ edges (or vertices). But whether the problem is FPT parameterized by $k$, is an interesting open question.

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