

Algorithmic Bayesian Persuasion with Combinatorial Actions

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Abstract

Bayesian persuasion is a model for understanding strategic information revelation: an agent with an informational advantage, called a sender, strategically discloses information by sending signals to another agent, called a receiver. In algorithmic Bayesian persuasion, we are interested in efficiently designing the sender's signaling schemes that lead the receiver to take action in favor of the sender. This paper studies algorithmic Bayesian-persuasion settings where the receiver's feasible actions are specified by combinatorial constraints, e.g., matroids or paths in graphs. We first show that constant-factor approximation is NP-hard even in some special cases of matroids or paths. We then propose a polynomial-time algorithm for general matroids by assuming the number of states of nature to be a constant. We finally consider a relaxed notion of persuasiveness, called CCE-persuasiveness, and present a sufficient condition for polynomial-time approximability.

Introduction

Information asymmetry is ubiquitous. For example, a seller has more information about a product than a customer, and a client sometimes knows more about a delegated task than a worker. Such an agent with an informational advantage often strategically discloses information to influence the receiver's decisions. A fundamental question in information economics asks how much influence an informational advantage has or what kind of information disclosure strategy the sender should use.

Bayesian persuasion (Kamenica and Gentzkow 2011) is a model for understanding strategic information disclosure. In the standard setting, there are two agents called the *sender* and the *receiver*. The sender, who has an informational advantage, strategically reveals information by sending a signal to the receiver. Then, according to the revealed information, the receiver selects an action that maximizes his expected utility. The receiver's action also affects the sender's utility, which is usually different from the receiver's utility. The sender's goal is to design a strategy for information disclosure, called a *signaling scheme*, to lead the receiver to select an action that is favorable for the sender. In this paper, we are interested in the algorithmic aspect of Bayesian per-

suation: when can we efficiently compute (approximately) optimal signaling schemes?

In many practical scenarios, the receiver's actions are specified by some combinatorial constraints. For example, a receiver's action may be a path in a network or a portfolio of a limited number of stocks. In such cases, lists of the receiver's actions can be prohibitively large, making standard methods for computing the sender's signaling schemes impractical. To reveal when we can efficiently compute optimal signaling schemes in the presence of such combinatorial actions is an important question, but it remains unexplored as mentioned in (Dughmi 2017, Open Question 2.8). This paper addresses this open question and elucidates several classes of problems for which we can/cannot efficiently compute (approximately) optimal signaling schemes.

To motivate our work, we below give concrete situations where the receiver's actions are combinatorial. An important application of Bayesian persuasion is financial advice. Suppose the sender and the receiver to be a financial adviser and an investor, respectively. The sender knows accurate predictions on stock returns, which are unknown to the receiver. Since the sender's returns are not always aligned with those of the receiver, the sender strategically reveals information to increase her returns. After receiving a signal (advice) from the sender, the receiver decides which stocks to buy. In practice, there are a huge number of stocks, and it is unrealistic to assume that the receiver can hold a portfolio of arbitrarily many stocks. Thus, a cardinality constraint is often imposed when optimizing portfolios (Ito et al. 2018; Zhu et al. 2020). In this situation, each receiver's action is a combination of a limited number of stocks.

More complicated combinatorial constraints can appear in other applications. For example, let us imagine that a boss (sender) asks a subordinate (receiver) to create a committee by choosing a representative for each of several groups. An action of the receiver is a set of people that contains a single person from each group. In another example, the receiver is a company that constructs an electrical grid connecting all electricity consumers, and the sender is a government that requests the construction. An action of the receiver is a tree in a network that covers all electricity consumers.

Our Results

We study Bayesian persuasion with combinatorial actions, each of which is a subset of a finite set, denoted by E , and satisfies some combinatorial constraints. We call each component of E an *element*, i.e., an action is a combination of some elements. Utility functions of the sender and the receiver are set functions defined on E . The main issue of this setting is that there may exist exponentially many actions in $|E|$. We aim to clarify under what conditions we can/cannot compute (approximately) optimal signaling schemes efficiently with respect to input sizes, including $|E|$.

First, we prove that it is NP-hard to achieve constant-factor approximation even in some special cases of matroid or path constraints. For partition matroid constraints, we utilize a hardness result studied in public Bayesian persuasion with no externalities (Dughmi and Xu 2017). For other combinatorial constraints, including uniform matroid constraints, graphic matroid constraints, and path constraints, we construct reductions from an existing hard problem called LINEQ-MA (Guruswami and Raghavendra 2009), which asks to determine whether a large fraction of linear equations can hold or even a small fraction cannot hold. These reductions have connections to the NP-hardness proof for the OPT-SIGNAL problem (Castiglioni et al. 2020) and, as a by-product of our result on partition matroids, we obtain a hardness result for the OPT-SIGNAL problem.

Next, we develop a polynomial-time algorithm for general matroid constraints by assuming the number of states of nature to be a constant (we explain what states and nature are in the next section). We formulate the problem as an exponentially large linear program (LP) and then show that its size can be reduced by enumerating only relevant variables and constraints. To enumerate them efficiently, we utilize a computational-geometric algorithm. For some special matroids, we show that the LP size can be further reduced.

Finally, we consider a relaxed persuasiveness condition called *CCE-persuasiveness* (Xu 2020; Celli, Coniglio, and Gatti 2020), which is based on the concept of Bayes coarse correlated equilibria. We provide a sufficient condition under which a polynomial-time approximation algorithm exists. Specifically, if we can approximately maximize a sum of the sender’s and receiver’s utilities so that only the sender’s has an approximation factor, we can compute approximately optimal CCE-persuasive schemes in polynomial time. This result is applicable to important cases where the sender’s utility is a monotone submodular function.

Most of the proofs are provided in the appendices of the full version (Fujii and Sakaue 2021).

Related Work

Kamenica and Gentzkow (2011) proposed the original Bayesian-persuasion model. Utilizing the model, researchers analyzed various social situations, including voting (Schnakenberg 2015; Alonso and Câmara 2016), financial sector stress tests (Goldstein and Leitner 2018), financial markets (Duffie, Dworzak, and Zhu 2017), routing (Bhaskar et al. 2016), auctions (Dughmi, Immorlica, and Roth 2014), and information spread (Arieli and Babichenko

2019). On the other hand, various algorithms have been developed for Bayesian persuasion with additional settings: exponentially many states of nature (Dughmi and Xu 2016), multiple receivers (Arieli and Babichenko 2019), secretary problem (Hahn, Hoefer, and Smorodinsky 2020b), prophet inequalities (Hahn, Hoefer, and Smorodinsky 2020a), and payment (Dughmi et al. 2019).

The closest setting to ours is Bayesian persuasion with no externalities (Babichenko and Barman 2017; Dughmi and Xu 2017; Arieli and Babichenko 2019; Xu 2020; Castiglioni et al. 2021). In this setting, there are multiple receivers, and each one selects a binary action from $\{0, 1\}$ to maximize his own utility, which does not depend on the other receivers’ actions. This setting can be decomposed into two subclasses: private and public. The former supposes that the sender can send a signal to each receiver privately, while the latter supposes that the sender sends a public signal to all the receivers. As we prove in the hardness section, public Bayesian persuasion with no externalities is equivalent to our setting with a special partition matroid constraint. Our study provides hardness results for other types of constraints and algorithms for general matroid constraints, thus going beyond the existing results on public Bayesian persuasion.

Some existing studies introduced constraints on Bayesian persuasion. A series of studies (Dughmi, Immorlica, and Roth 2014; Dughmi, Kempe, and Qiang 2016; Gradwohl et al. 2021) considered a setting where the sender can use only a limited number of bits for signaling. Babichenko, Talgam-Cohen, and Zabarnyi (2021) considered ex ante and ex post constraints, which restrict structures of posterior distributions induced by the sender’s signals. Note that the constraints considered in those studies are imposed on the sender’s signaling scheme, while we consider combinatorial constraints on the receiver’s possible actions.

Problem Settings and Preliminaries

This section provides problem settings of Bayesian persuasion with combinatorial actions. We first describe the basic setting, where sender and receiver maximize their utility functions and the receiver’s actions are represented by a matroid. We then present some basics and examples of matroids. We finally describe the setting with path constraints, where sender and receiver minimize their cost functions.

Notation. We denote the set of non-negative reals by $\mathbb{R}_{\geq 0}$. For any set X , let $\Delta_X := \{p: X \rightarrow \mathbb{R}_{\geq 0} \mid \sum_{i \in X} p(i) = 1\}$ be the probability simplex over X .

Basic Setting

Let Θ be the family of states of nature. A state $\theta \in \Theta$ is drawn from a distribution $\mu \in \Delta_{\Theta}$, which is common knowledge shared by the sender and the receiver. In the remainder of this paper, we assume $\mu(\theta) > 0$ for all $\theta \in \Theta$. If this does not hold, we can remove such θ from the problem. The sender has the informational advantage of knowing the realized state θ , while the receiver cannot access θ directly.

After observing the state of nature θ , the sender sends a signal $\sigma \in \Sigma$ to the receiver. The signals can be randomized, that is, the sender can select any distribution $\phi_{\theta} \in \Delta_{\Sigma}$ for

each $\theta \in \Theta$ and send signal σ randomly sampled from ϕ_θ depending on the observed state θ . A tuple $(\phi_\theta)_{\theta \in \Theta}$ of probability distributions over Σ is called a *signaling scheme*. The core assumption of Bayesian persuasion is the *commitment assumption*, which compels the sender to publicly commit to a signaling scheme before observing the state of nature θ . Under this assumption, the process goes on as follows: the sender publicly declares a signaling scheme $(\phi_\theta)_{\theta \in \Theta}$, then observes the state of nature θ sampled from μ , and then sends signal σ to the receiver following the distribution ϕ_θ .

Let E be a finite set of n elements. A combinatorial action is a combination of elements in E . Let $\mathcal{I} \subseteq 2^E$ be the set of all possible combinatorial actions the receiver can take. After observing signal σ , the receiver takes a combinatorial action $S \in \mathcal{I}$. As a result of this action, the sender and the receiver obtain utility values, which are specified by non-negative set functions $s_\theta: 2^E \rightarrow \mathbb{R}_{\geq 0}$ and $r_\theta: 2^E \rightarrow \mathbb{R}_{\geq 0}$, respectively, for each state of nature $\theta \in \Theta$. When receiving signal $\sigma \in \Sigma$, the receiver's belief on the state of nature is represented by the posterior distribution $\xi_\sigma \in \Delta_\Theta$ such that

$$\xi_\sigma(\theta) = \frac{\mu(\theta)\phi_\theta(\sigma)}{\sum_{\theta' \in \Theta} \mu(\theta')\phi_{\theta'}(\sigma)}.$$

The receiver takes the best response according to this posterior distribution, that is, he selects $S_\sigma^* \in \mathcal{I}$ such that

$$S_\sigma^* \in \operatorname{argmax}_{S \in \mathcal{I}} \sum_{\theta \in \Theta} \xi_\sigma(\theta) r_\theta(S).$$

If there are multiple best responses, we assume that ties are broken in favor of the sender. Consequently, the sender, who commits to the signaling scheme $(\phi_\theta)_{\theta \in \Theta}$, obtains the following payoff in expectation: $\sum_{\sigma \in \Sigma} \sum_{\theta \in \Theta} \mu(\theta)\phi_\theta(\sigma) s_\theta(S_\sigma^*)$.

The revelation principle (Kamenica and Gentzkow 2011) claims that to consider only direct and persuasive signaling schemes is sufficient. A direct signaling scheme associates each signal with an action, i.e., $\Sigma = \mathcal{I}$, and recommends action $S \in \mathcal{I}$ by sending the corresponding signal $S \in \Sigma$. A direct signaling scheme is *persuasive* if the receiver has no incentive to deviate from recommendation $S \in \mathcal{I}$ when receiving signal $S \in \Sigma$. The persuasiveness constraint requires signaling scheme $(\phi_\theta)_{\theta \in \Theta}$ to satisfy $\sum_{\theta \in \Theta} \xi_S(\theta) r_\theta(S) \geq \sum_{\theta \in \Theta} \xi_S(\theta) r_\theta(S')$ for every pair $S, S' \in \mathcal{I}$, where ξ_S is the posterior distribution when the receiver observes $S \in \Sigma$.

By considering the revelation principle, we can formulate the problem of computing an optimal signaling scheme as

$$\begin{aligned} & \text{maximize} && \sum_{\theta \in \Theta} \mu(\theta) \sum_{S \in \mathcal{I}} \phi_\theta(S) s_\theta(S) \\ & \text{subject to} && \sum_{\theta \in \Theta} \mu(\theta) \phi_\theta(S) (r_\theta(S) - r_\theta(S')) \geq 0 \quad (S, S' \in \mathcal{I}) \quad (1) \\ & && \phi_\theta \in \Delta_{\mathcal{I}} \quad (\theta \in \Theta). \end{aligned}$$

The first constraint is the persuasiveness constraint, under which S must be one of the receiver's best responses when the sender sends signal S . The second constraint requires ϕ_θ to be a probability distribution in $\Delta_{\mathcal{I}}$ for each $\theta \in \Theta$.

In general, the utility set functions $(s_\theta)_{\theta \in \Theta}$ and $(r_\theta)_{\theta \in \Theta}$ do not have a polynomial-size representation in $|E|$ and $|\Theta|$.

In such cases, we assume that we have value oracles, i.e., $s_\theta(S)$ and $r_\theta(S)$ values are available for any given $\theta \in \Theta$ and $S \subseteq E$. Similarly, if the set \mathcal{I} of the receiver's actions does not have a polynomial-size representation, we assume access to an independence oracle, which returns whether $S \in \mathcal{I}$ or not for any given $S \subseteq E$. These assumptions are common in combinatorial optimization.

For the hardness results, we let $(s_\theta)_{\theta \in \Theta}$ and $(r_\theta)_{\theta \in \Theta}$ be linear functions, i.e., $s_\theta(S) = \sum_{i \in S} s_\theta(\{i\})$ and $r_\theta(S) = \sum_{i \in S} r_\theta(\{i\})$ for all $S \subseteq E$ and $\theta \in \Theta$. Such $(s_\theta)_{\theta \in \Theta}$ and $(r_\theta)_{\theta \in \Theta}$ have polynomial-size representations. We also use sets \mathcal{I} of actions that have polynomial-size representations. Thus, our hardness results indeed come from computational hardness, not from the hardness of representing problems. For the polynomial-time algorithms for a constant number of states, we assume $(r_\theta)_{\theta \in \Theta}$ to be linear but allow $(s_\theta)_{\theta \in \Theta}$ to be a general set function. For the CCE-persuasiveness result, we allow both $(s_\theta)_{\theta \in \Theta}$ and $(r_\theta)_{\theta \in \Theta}$ to be general set functions, while we impose an approximability assumption as detailed later.

Matroids

Many of our results consider combinatorial constraints represented by matroids, which are useful to model various combinatorial actions that appear in practice.

A pair (E, \mathcal{I}) of a finite set E and a non-empty set family $\mathcal{I} \subseteq 2^E$ is called a matroid if the following conditions hold:

- $S \subseteq T \in \mathcal{I}$ implies $S \in \mathcal{I}$.
- For any $S, T \in \mathcal{I}$ with $|S| < |T|$, there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{I}$.

A set $S \subseteq E$ is called independent if $S \in \mathcal{I}$ holds. Given a matroid (E, \mathcal{I}) and a non-negative weight $w(i) \in \mathbb{R}_{\geq 0}$ for each element $i \in E$, a maximum weight independent set $S^* \in \operatorname{argmax}_{S \in \mathcal{I}} \sum_{i \in S} w(i)$ can be found by using the greedy algorithm, which starts with the empty set and, in descending order of weights, adds elements that maintain independence. Below, we introduce some special matroids that are useful in practical scenarios.

Uniform matroids. A uniform matroid is a matroid with $\mathcal{I} = \{S \subseteq E \mid |S| \leq k\}$ for some integer $k > 0$. Under a uniform matroid constraint, the receiver selects at most k elements that yield the largest expected utility value. This constraint fits the portfolio optimization scenario, where the receiver selects k stocks expected to yield the largest return.

Partition matroids. Let $E_1, \dots, E_P \subseteq E$ be a partition, i.e., $E_1 \cup \dots \cup E_P = E$ and $E_i \cap E_j = \emptyset$ for every distinct $i, j \in [P]$. Assign some positive integer k_i for each $i \in [P]$. A matroid with $\mathcal{I} = \{S \subseteq E \mid \forall i \in [P], |S \cap E_i| \leq k_i\}$ is called a partition matroid. Under a partition matroid constraint, the receiver selects at most k_i elements from the i th partition. This can model a scenario where the receiver creates a committee by choosing one person from each group.

Graphic matroids. Given an undirected graph (V, E) , a graphic matroid (E, \mathcal{I}) is a matroid with $\mathcal{I} = \{S \subseteq E \mid S \text{ does not contain a cycle}\}$. Any maximal independent set of a graphic matroid forms a spanning tree (or a spanning

forest if graph (V, E) is not connected). If E is a set of all edges that can be used for constructing an electrical grid and V is a set of all electricity consumers, a graphic matroid constraint models the situation where the receiver constructs an electrical grid covering all consumers.

Minimization Setting with Path Constraints

Given a directed graph $G = (V, E)$ with origin $v_s \in V$ and destination $v_t \in V$, we consider a setting where the receiver selects a v_s-v_t path $S \subseteq E$ according to a signal from the sender. This setting appears, for example, when the sender is an association that manages traffic by recommending a route, and the receiver is a taxi driver. In this setting, the sender and the receiver usually aim to minimize their costs rather than maximize utility functions. Thus, we focus on the minimization setting regarding path constraints (we distinguish maximization and minimization settings since some of our results concerning approximability cannot be translated from one to the other). The problem of computing an optimal signaling scheme can be formulated as a minimization version of (1), where \mathcal{I} is the set of all v_s-v_t paths. The receiver selects a path that minimizes his expected cost based on the posterior distribution; therefore, we reverse the inequality sign of the persuasiveness constraint. The sender's goal is to minimize her expected cost.

Hardness Results

We show the NP-hardness of obtaining a constant-factor approximation for Bayesian persuasion with partition matroid, uniform matroid, graphic matroid, and path constraints. Throughout this section, we assume the utility functions of the sender and the receiver to be linear.

Partition Matroid Constraints

We show that Bayesian persuasion with a partition matroid constraint is intractable even in a special case where each partition has two elements. We prove this via a reduction from public Bayesian persuasion with no externalities. Dughmi and Xu (2017) proved that constant-factor approximation for public Bayesian persuasion with no externalities is NP-hard even if we restrict the sender's utility function to $s_\theta(S) = |S|$ for each $\theta \in \Theta$. By associating each partition with a receiver of public Bayesian persuasion, we obtain the reduction, thus proving the following hardness result.

Theorem 1. *For any constant $\alpha \in (0, 1]$, it is NP-hard to compute an α -approximate solution for Bayesian persuasion with a partition matroid constraint whose partitions are of size two.*

Remark 2. We can also construct a reduction for the opposite direction. Therefore, we establish an equivalence in terms of the approximability between public Bayesian persuasion with no externalities and Bayesian persuasion with a partition matroid constraint whose partitions are of size two.

As a by-product of Theorem 1, we can obtain a hardness result on the OPT-SIGNAL problem (Castiglioni et al. 2020), which is a variant of Bayesian persuasion where the receiver's utility depends on a random type that is unknown to the sender.

Uniform Matroid Constraints

We prove the NP-hardness of constant-factor approximation for Bayesian persuasion with a uniform matroid constraint. Our proof is inspired by that of the OPT-SIGNAL problem (Castiglioni et al. 2020), which is equivalent to the special case of partition matroid constraints, as mentioned above. Note that a uniform matroid cannot be represented by the special partition matroid with size-two partitions. Nevertheless, we can use a similar proof strategy to the one used for the OPT-SIGNAL problem. Specifically, we construct a reduction from a variant of an existing hard problem called LIENQ-MA (Guruswami and Raghavendra 2009), which asks us to distinguish whether a linear system $Ax = c$ has a solution satisfying most of the equations or no solution satisfies even a small fraction of the equations.

Theorem 3. *For any constant $\alpha \in (0, 1]$, it is NP-hard to compute an α -approximate solution for Bayesian persuasion with a uniform matroid constraint.*

Graphic Matroid Constraints

The hardness result for Bayesian persuasion with graphic matroid constraints can be proved in a similar way to that of uniform matroid constraints, i.e., we construct a reduction from the variant of the LINEQ-MA problem.

Theorem 4. *For any constant $\alpha \in (0, 1]$, it is NP-hard to compute an α -approximate solution for Bayesian persuasion with a graphic matroid constraint.*

Path Constraints

Recall that in this setting, the sender and the receiver aim to minimize their expected costs, while the above settings consider maximizing utilities. Thus, the approximation ratio for this problem is lower-bounded by 1, and a smaller value implies a better approximation. The proof is similar to those for uniform or graphic matroids.

Theorem 5. *For any constant $\alpha \in [1, \infty)$, it is NP-hard to compute an α -approximate solution for Bayesian persuasion with a path constraint.*

Polynomial-time Algorithm for Constant Number of States

We present a polynomial-time algorithm for Bayesian persuasion with general matroid constraints by assuming the number of states of nature to be a constant, i.e., $|\Theta| = O(1)$. We denote by d the dimension of Δ_Θ , i.e., $d := |\Theta| - 1$. Throughout this section, we assume the receiver's utility function to be linear, while the sender's is a general set function, and we regard computation costs that depend only on d as constants.

The main obstacle is the fact that the number of variables and constraints in LP (1) is exponential in $n := |E|$. In the case of matroid constraints, however, we can show that only a small number of variables can take non-zero values and that a large fraction of constraints is unnecessary. Thus, if we can efficiently enumerate all relevant variables and constraints, we can obtain a smaller LP that is equivalent to the original LP (1) and can be solved more cheaply. This

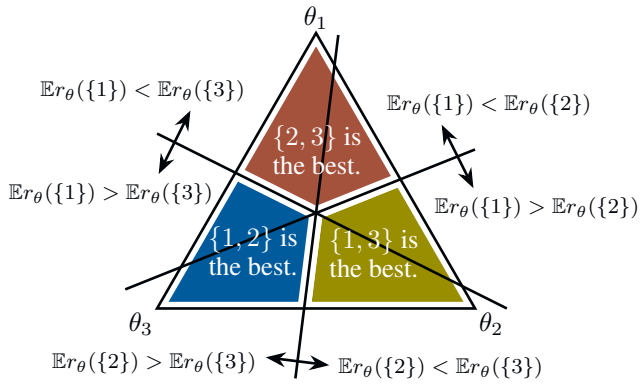


Figure 1: An illustration of cells for Bayesian persuasion with a uniform matroid constraint, where the receiver selects the top-2 elements from $E = \{1, 2, 3\}$. For each pair of distinct $i, j \in E$, a hyperplane (a line) divides the probability simplex Δ_Θ into the region where $\mathbb{E}_{\theta \sim \xi}[r_\theta(\{i\})] > \mathbb{E}_{\theta \sim \xi}[r_\theta(\{j\})]$ holds and the complement. The receiver's best response is identical within each cell.

approach is inspired by the one studied in public Bayesian persuasion with no externalities (Xu 2020), which is equivalent to Bayesian persuasion with a special partition matroid constraint, as mentioned the hardness section. Our algorithm can be seen as an extension of (Xu 2020) to the case of general matroids. Below we present the technical details.

A key observation is that if d is small, the number of combinatorial actions that the receiver can take is also small. We define the set \mathcal{I}^* of all combinatorial actions that can be the best response for some posterior distribution as

$$\mathcal{I}^* = \left\{ S \in \mathcal{I} \mid \exists \xi \in \Delta_\Theta : S \in \operatorname{argmax}_{S \in \mathcal{I}} \sum_{\theta \in \Theta} \xi(\theta) r_\theta(S) \right\}.$$

By using this reduced set of combinatorial actions, we formulate a smaller LP as follows:

$$\begin{aligned} & \text{maximize} && \sum_{\theta \in \Theta} \mu(\theta) \sum_{S \in \mathcal{I}^*} \phi_\theta(S) s_\theta(S) \\ & \text{subject to} && \sum_{\theta \in \Theta} \mu(\theta) \phi_\theta(S) (r_\theta(S) - r_\theta(S')) \geq 0 \quad (S, S' \in \mathcal{I}^*) \quad (2) \\ & && \phi_\theta \in \Delta_{\mathcal{I}} \quad (\theta \in \Theta). \end{aligned}$$

This LP formulation has variables $\phi_\theta(S)$ only for $S \in \mathcal{I}^*$, and the first constraint exists only for $S, S' \in \mathcal{I}^*$. We can show that this reduced LP formulation (2) is equivalent to the original one (1).

Proposition 6. *There is a bijection between the feasible regions of (1) and (2) that does not change the objective value.*

As illustrated in (Xu 2020, Example 1), if the receiver's expected utility has certain degeneracy, the receiver may have exponentially many best responses, i.e., $|\mathcal{I}^*| = \Omega(2^n)$. To exclude such troubling corner cases, we make a mild non-degeneracy assumption regarding the receiver's utility.

Assumption 7. For each $i \in E$, let $\psi_i = (r_\theta(\{i\}))_{\theta \in \Theta} \in \mathbb{R}^\Theta$ be a vector representing the utilities of the i th element. We assume that, for any permutation $\pi: \{1, \dots, n\} \rightarrow E$

Algorithm 1: Algorithm for Bayesian persuasion with a general matroid constraint

Input: value oracles of s_θ and r_θ , independence oracle of \mathcal{I} , μ .

Output: signaling scheme $(\phi_\theta)_{\theta \in \Theta}$.

- 1: Let $\mathcal{H} = \{h_{ij} \mid i, j \in E, i \neq j\}$ be the set of hyperplanes, where $h_{ij} = \{\xi \in \operatorname{aff}(\Delta_\Theta) \mid \sum_{\theta \in \Theta} \xi(\theta) (r_\theta(\{i\}) - r_\theta(\{j\})) = 0\}$.
- 2: Obtain the set of all cells \mathcal{C} of the arrangement of \mathcal{H} by using cell enumeration algorithm (Edelsbrunner 1987).
- 3: $\mathcal{I}_C \leftarrow \emptyset$.
- 4: **for** each cell $C \in \mathcal{C}$ **do**
- 5: Let $\xi \in C$ be any interior point of C .
- 6: Apply the greedy algorithm to matroid (E, \mathcal{I}) and weights $\{\mathbb{E}_{\theta \sim \xi}[r_\theta(\{i\})]\}_{i \in E}$ to obtain the maximum weight independent set and add it to \mathcal{I}_C .
- 7: Solve LP (2) with $\mathcal{I}^* = \mathcal{I}_C$ and obtain solution $(\phi_\theta)_{\theta \in \Theta}$.
- 8: **return** signaling scheme $(\phi_\theta)_{\theta \in \Theta}$.

and any subset $S \subseteq \{1, \dots, n-1\}$ with $|S| = |\Theta|$, vectors $(\psi_{\pi(i)} - \psi_{\pi(i+1)})_{i \in S}$ are linearly independent.

Under this assumption, we develop a polynomial-time algorithm, which first enumerates all combinatorial actions in \mathcal{I}^* and then solves the reduced LP (2). We below describe our algorithm for general matroids and then present faster algorithms for some special matroids.

General Matroids

We explain how to enumerate $\mathcal{I}^* \subseteq \mathcal{I}$ in polynomial time when (E, \mathcal{I}) is a matroid. Let $\operatorname{aff}(\Delta_\Theta) = \{\xi: \Theta \rightarrow \mathbb{R} \mid \sum_{\theta \in \Theta} \xi(\theta) = 1\}$ be the affine hull of Δ_Θ , i.e., the smallest affine space containing Δ_Θ . We consider hyperplane $h_{ij} = \{\xi \in \operatorname{aff}(\Delta_\Theta) \mid \sum_{\theta \in \Theta} \xi(\theta) (r_\theta(\{i\}) - r_\theta(\{j\})) = 0\}$ for each pair of distinct $i, j \in E$. Each hyperplane h_{ij} divides $\operatorname{aff}(\Delta_\Theta)$ into two halfspaces: the region where the i 's expected utility $\mathbb{E}_{\theta \sim \xi}[r_\theta(\{i\})] = \sum_{\theta \in \Theta} \xi(\theta) r_\theta(\{i\})$ is at least that of j and the complement (see Figure 1). Thus, the set of hyperplanes $\mathcal{H} = \{h_{ij} \mid i, j \in E, i \neq j\}$ divides $\operatorname{aff}(\Delta_\Theta)$ into small pieces, which we call *cells*. Within each cell, the descending order of expected utility values is identical.

Algorithm 1 presents the details of our algorithm. We first enumerate all cells and then use the greedy algorithm to obtain a maximum weight independent set for an interior point of each cell. Although \mathcal{H} has degeneracy, by using an algorithm for constructing arrangements of hyperplanes (Edelsbrunner 1987), we can enumerate all cells. Finally, we solve the reduced LP (2) where \mathcal{I}^* is replaced with the family of obtained independent sets, denoted by \mathcal{I}_C .

Now, we show that \mathcal{I}_C obtained by the algorithm recovers \mathcal{I}^* , the family of possible best responses. The following lemma guarantees that considering only interior points of cells is sufficient for enumerating \mathcal{I}^* under Assumption 7, i.e., we do not need to care about the boundaries of cells.

Lemma 8. *Let $\psi_1, \dots, \psi_n \in \mathbb{R}^\Theta$ be vectors that satisfy Assumption 7. Then, for any permutation π , $\{\xi \in \mathbb{R}^\Theta \mid \psi_{\pi(1)}^\top \xi \geq \dots \geq \psi_{\pi(n)}^\top \xi \text{ and } \mathbf{1}^\top \xi = 1\} \neq \emptyset$ implies $\{\xi \in \mathbb{R}^\Theta \mid \psi_{\pi(1)}^\top \xi > \dots > \psi_{\pi(n)}^\top \xi \text{ and } \mathbf{1}^\top \xi = 1\} \neq \emptyset$.*

Therefore, we can enumerate all possible best responses \mathcal{I}^* by collecting maximum weight independent sets returned by the greedy algorithm, whose behavior depends only on the descending order of the weights.

Theorem 9. *Under Assumption 7, $\mathcal{I}^* \subseteq \mathcal{I}_C$ holds. Moreover, $|\mathcal{I}_C| = O(n^{2d})$.*

Since $|\mathcal{H}| = O(n^2)$, the cells can be enumerated in $O(n^{2d})$ time (Edelsbrunner 1987). The resulting LP (2) has $O(n^{2d})$ variables and $O(n^{4d})$ constraints, which can be solved in $\text{poly}(n^d)$ time.

Faster Algorithms for Special Matroids

As described above, our algorithm first enumerates every combinatorial action that can be the receiver's best response for some $\xi \in \Delta_\Theta$, and then solves the reduced LP. For several special cases of matroids, we can use faster enumeration algorithms and obtain better upper bounds on $|\mathcal{I}^*|$.

Uniform matroids. In this case, a receiver's action consists of top- k elements, and thus we can enumerate all possible best responses \mathcal{I}^* in a simpler manner. We consider n hyperplanes on Δ_Θ associated with n expected utility values, and enumerate all combinations of k hyperplanes that correspond to top- k elements for some $\xi \in \Delta_\Theta$. The enumeration problem is closely related to a discrete-geometric subject, called the k -level in an arrangement of hyperplanes. By using an algorithm of (Mulmuley 1991) that enumerates all faces of level at most k , we can enumerate all k -level faces corresponding to \mathcal{I}^* in $O(k^{\lceil (d+1)/2 \rceil} n^{\lfloor (d+1)/2 \rfloor})$ time for $d \geq 3$, and in $O(k^{\lceil (d+1)/2 \rceil} n^{\lfloor (d+1)/2 \rfloor} \log(n/k))$ time for $d \leq 2$. Moreover, Clarkson and Shor (1989) presented an upper bound on the number of faces of level at most k , which implies $|\mathcal{I}^*| = O(k^{\lceil (d+1)/2 \rceil} n^{\lfloor (d+1)/2 \rfloor})$.

Partition matroids. As with the case of general matroids, we enumerate cells of the hyperplane arrangement, but the number of hyperplanes can be reduced. A key observation is that the maximum weight independent set is determined by the order of weights of elements in each partition, not by the order of weights of all elements. Therefore, it is sufficient to enumerate all possible orders of weights in each partition. Let $n_1 := |E_1|, n_2 := |E_2|, \dots, n_P := |E_P|$ be the size of each partition. For each $p \in [P]$, we consider hyperplanes for all pair of distinct $i, j \in E_p$. As with the case of general matroids, we can obtain $\mathcal{I}_C \supseteq \mathcal{I}^*$ by enumerating the cells of the arrangement of those hyperplanes. If $n_1 = \dots = n_P$, the number of hyperplanes is $O(n^2/P)$. Therefore, in this case, we have $|\mathcal{I}^*| = O(n^{2d}/P^d)$.

Graphic matroids. In this case, we need to enumerate all spanning trees $S \in \mathcal{I}$ that can attain the maximum weight for some $\xi \in \Delta_\Theta$. To this end, we can use existing results on the parametric spanning tree problem. When $d = 1$, by using an algorithm of (Fernández-Baca, Slutzki, and Eppstein 1996), we can enumerate all the spanning trees \mathcal{I}^* in $O(|V||E| \log |V|)$ time. Moreover, when $d = 1$, it is known that $|\mathcal{I}^*| = O(|E||V|^{1/3})$ holds (Dey 1998).

Remark 10. For path constraints, even if $d = 1$, the number of paths that can be the shortest path for some $\xi \in \Delta_\Theta$ is

known to be $n^{\Omega(\log n)}$ in general (Carstensen 1983). Thus, our approach, which enumerates every path that can be the shortest path for some $\xi \in \Delta_\Theta$, fails to run in polynomial time even if d is a constant. We need a different approach for obtaining an efficient algorithm for path constraints.

Polynomial-time Algorithm for CCE-Persuasiveness

We present how to compute approximately optimal CCE-persuasive signaling schemes for Bayesian persuasion with combinatorial actions. To motivate us to study CCE-persuasiveness, a relaxed notion of persuasiveness, in the combinatorial setting, let us consider the following investment example. Let the receiver be an investor wondering whether to buy an investment trust or build a portfolio by himself. If the investor decides to buy an investment trust, stocks are bought following a policy of a portfolio manager, who is the sender. The sender wants to sell the investment trust, and a signaling scheme corresponds to her portfolio-management policy. The receiver buys the investment trust if it is at least as beneficial as the portfolio built following his prior belief. In this example, in contrast to the standard setting, the sender's recommendation is regarded as persuasive if it is better than the best action taken based on the receiver's prior belief. This idea of persuasiveness can be modeled by CCE-persuasiveness, as detailed below.

As pointed out by Bergemann and Morris (2016), Bayesian persuasion is the problem of finding a Bayes correlated equilibrium that is optimal for the sender. By relaxing the condition of Bayes correlated equilibria, we can obtain a broader class of equilibria called Bayes coarse correlated equilibria, as mentioned in several studies (Cai and Papadimitriou 2014; Hartline, Syrgkanis, and Tardos 2015). In a Bayes coarse correlated equilibrium, the receiver has no incentive to ignore the signal, i.e., following the signal is at least as beneficial as the best action selected based on his prior distribution. Let $C = \max_{S \in \mathcal{I}} \sum_{\theta \in \Theta} \mu(\theta) r_\theta(S)$ be the receiver's utility when he selects the best action based on his prior distribution. The LP for computing an optimal CCE-persuasive signaling scheme can be written as follows:

$$\begin{aligned} & \text{maximize} && \sum_{\theta \in \Theta} \sum_{S \in \mathcal{I}} \mu(\theta) \phi_\theta(S) s_\theta(S) \\ & \text{subject to} && \sum_{\theta \in \Theta} \sum_{S \in \mathcal{I}} \mu(\theta) \phi_\theta(S) r_\theta(S) \geq C \\ & && \phi_\theta \in \Delta_{\mathcal{I}} \quad (\theta \in \Theta). \end{aligned} \tag{3}$$

We present a sufficient condition for achieving a constant-factor approximation for the above LP. For public Bayesian persuasion, the equivalence between exact maximization of certain set functions and computation of CCE-persuasive schemes has been established in (Xu 2020, Theorem 5.1). Compared with the existing result, ours is restricted to one direction (maximization implies persuasion). However, it is applicable to important cases where maximization can be done only approximately. For example, we can allow the sender's utility to be a monotone submodular function, as explained later. To prove the result, we carefully combine an

Algorithm 2: Algorithm for CCE-persuasiveness

Input: value oracles of s_θ and r_θ , independence oracle of \mathcal{I} ,

α -approximation oracle, μ , v_{\min} , v_{\max} , ϵ .

Output: signaling scheme $(\phi_\theta)_{\theta \in \Theta}$.

- 1: $v_l \leftarrow 0$, $v_u \leftarrow v_{\max}$, $v \leftarrow \frac{v_l + v_u}{2}$.
 - 2: $\epsilon' \leftarrow \epsilon v_{\min}$.
 - 3: **for** $t = 1, 2, \dots, \lceil \log(v_{\max}/\epsilon') \rceil + 1$ **do**
 - 4: Apply the ellipsoid method with an α -approximate separation oracle to dual LP (4) with additional constraint $-C \cdot y + \sum_{\theta \in \Theta} x_\theta \leq v$.
 - 5: **if** v is approximately feasible **then**
 - 6: $v_u \leftarrow v$ and $v \leftarrow \frac{v_l + v_u}{2}$.
 - 7: **else** /* v is infeasible */
 - 8: $v_l \leftarrow v$ and $v \leftarrow \frac{v_l + v_u}{2}$.
 - 9: **if** $v_l = 0$ **then**
 - 10: **return** any signaling scheme.
 - 11: Solve restricted primal LP to obtain solution $(\phi_\theta)_{\theta \in \Theta}$.
 - 12: **return** signaling scheme $(\phi_\theta)_{\theta \in \Theta}$.
-

existing binary search framework for approximate separation oracles (Jansen 2003; Jain, Mahdian, and Salavatipour 2003) and the proof of (Xu 2020, Theorem 5.1). Throughout this section, we do not make any assumption on the sender's and the receiver's utility functions; instead, we assume that an approximate maximization oracle, together with upper and lower bounds on the optimal value, are available.

Theorem 11. *Let $\alpha \in (0, 1]$. For any $y \geq 0$ and $\theta \in \Theta$, assume that a polynomial-time α -approximation oracle that returns $S \in \mathcal{I}$ with the following guarantee is available: $s_\theta(S) + y \cdot r_\theta(S) \geq \alpha \cdot s_\theta(S') + y \cdot r_\theta(S')$ for any $S' \in \mathcal{I}$. Moreover, assume v_{\min} and v_{\max} to be given as inputs such that (i) $\text{OPT} > 0$ implies $\text{OPT} > v_{\min}$ and (ii) $\text{OPT} < v_{\max}$, where OPT is the optimal value. Then, for any $\epsilon \in (0, \alpha)$, there is a polynomial-time algorithm that computes an $(\alpha - \epsilon)$ -approximate CCE-persuasive signaling scheme.*

Proof. The dual of the LP (3) can be written with variables $\{x_\theta\}_{\theta \in \Theta}$ and $y \in \mathbb{R}$ as follows:

$$\begin{aligned}
 & \text{minimize} && -C \cdot y + \sum_{\theta \in \Theta} x_\theta \\
 & \text{subject to} && x_\theta - \mu(\theta)r_\theta(S) \cdot y \geq \mu(\theta)s_\theta(S) \quad (\theta \in \Theta, S \in \mathcal{I}) \\
 & && y \geq 0.
 \end{aligned} \tag{4}$$

We consider applying the ellipsoid method to this LP. Here, since we cannot directly use the α -approximation oracle as a separation oracle, we employ a binary-search framework of (Jansen 2003; Jain, Mahdian, and Salavatipour 2003), which enables us to combine the ellipsoid method with the following α -approximate separation oracle.

Lemma 12. *Under the assumption of Theorem 11, there is a polynomial-time α -approximate separation oracle such that for any $\{x_\theta\}_{\theta \in \Theta}$ and y , it either returns a separating hyperplane or guarantees the feasibility of $\{x_\theta/\alpha\}_{\theta \in \Theta}$ and y/α .*

We perform a binary search on $[0, v_{\max}]$ to estimate the optimal value of the dual LP (see Algorithm 2). Given an

estimated value $v \in \mathbb{R}$, we add $-C \cdot y + \sum_{\theta \in \Theta} x_\theta \leq v$ to the constraints in (4) and check the feasibility of the resulting inequality system using the ellipsoid method with the α -approximate separation oracle. Note that since we can use only an α -approximate separation oracle, the ellipsoid method guarantees that the estimated value v is either approximately feasible or infeasible. The binary search continues until the interval becomes smaller than $\epsilon' = \epsilon v_{\min}$. Let v^* be the smallest approximately feasible value found by this binary search, and let $(\{x_\theta^*\}_{\theta \in \Theta}, y^*)$ be its corresponding solution. From Lemma 12, $(\{x_\theta^*/\alpha\}_{\theta \in \Theta}, y^*/\alpha)$ satisfies the constraints of the dual LP (4), and thus the optimal value is at most v^*/α . Furthermore, since $v^* - \epsilon'$ is infeasible, the optimal value of the dual LP (4) is in $(v^* - \epsilon', v^*/\alpha]$.

If the binary search does not find any infeasible v , then $v^* \in [0, \epsilon v_{\min}]$ and the optimal value is less than v_{\min} . From the assumption on v_{\min} , the optimal value is 0. Thus, any signaling scheme is optimal. Otherwise, to obtain a solution, we construct a restricted version of the primal LP (3) as follows. When executing the ellipsoid method for the smallest v such that $v \geq v^* - \epsilon'$, we check polynomially many constraints, which are enough to conclude that the optimal value of the dual LP is larger than $v^* - \epsilon'$. This process yields a restricted dual LP with the polynomially many constraints, each of which corresponds to a primal variable. By allowing only those primal variables to be non-zero, we can obtain a restricted primal LP. From the strong duality, the optimal value of the restricted primal LP is also larger than $v^* - \epsilon'$. On the other hand, the weak duality implies that the optimal value of the (non-restricted) primal LP (3) is at most v^*/α . Thus, by solving the restricted primal LP, we can compute an $(\alpha - \epsilon)$ -approximate solution in polynomial time. \square

For several classes of utility functions, we can implement approximate separation oracles that run in polynomial time. For example, we can exactly solve $\max_{S \in \mathcal{I}} s_\theta(S) + y \cdot r_\theta(S)$ if both s_θ and r_θ are linear and \mathcal{I} is an independence system of a matroid. When s_θ is a monotone submodular function, r_θ is linear, and \mathcal{I} is an independence system of a matroid, the continuous greedy algorithm can be used as a $(1 - 1/e)$ -approximation oracle (Sviridenko, Vondrák, and Ward 2017, Theorem 1). Furthermore, when s_θ is a monotone submodular function, r_θ is a gross substitute function, and \mathcal{I} is an independence system of a uniform matroid, a variant of the continuous greedy algorithm serves as a $(1 - 1/e)$ -approximation oracle (Soma and Yoshida 2018). By considering the minimization version, we can use a shortest-path algorithm as an exact separation oracle if both s_θ and r_θ are linear and \mathcal{I} is the set of paths on some graph.

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