Weighted Fairness Notions for Indivisible Items Revisited*

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Abstract

We revisit the setting of fairly allocating indivisible items when agents have different weights representing their entitlements. First, we propose a parameterized family of relaxations for weighted envy-freeness and the same for weighted proportionality; the parameters indicate whether smaller-weight or larger-weight agents should be given a higher priority. We show that each notion in these families can always be satisfied, but any two cannot necessarily be fulfilled simultaneously. We then introduce an intuitive weighted generalization of maximin share fairness and establish the optimal approximation of it that can be guaranteed. Furthermore, we characterize the implication relations between the various weighted fairness notions introduced in this and prior work, and relate them to the lower and upper quota axioms from apportionment.

1   Introduction

Research in fair division is quickly moving from the realm of theory to practical applications, ranging from the division of various assets between individuals (Goldman and Procaccia 2014) to the distribution of food to charities (Aleksandrov et al. 2015) and medical equipment among communities (Pathak et al. 2021). Two complicating factors in such applications are that items may be *indivisible*, and recipients may have *different entitlements*. For example, when dividing food packs or medical supplies among organizations or districts, it is reasonable to give larger shares to recipients that represent more individuals. This raises the need to define appropriate fairness notions taking these factors into account.

With divisible resources and equal entitlements, two well-established fairness benchmarks are *envy-freeness*—no agent prefers the bundle of another agent, and *proportionality*—every agent receives at least $1/n$ of her value for the set of all resources, where n denotes the number of agents. When the resources consist of indivisible items, neither of these benchmarks can always be met, for example if one item is extremely valuable in the eyes of all agents. This has motivated various relaxations, which can be broadly classified into two approaches. The first approach allows adding or removing a single item before applying the fairness notion.

In particular, *envy-freeness up to one item (EF1)* requires that if an agent envies another agent, the envy should disappear upon removing one item from the envied agent's bundle (Lipton et al. 2004; Budish 2011). *Proportionality up to one item (PROP1)* demands that if an agent falls short of the proportionality benchmark, this should be rectified after adding one item to her bundle (Conitzer, Freeman, and Shah 2017). The second approach modifies the fairness threshold itself: instead of $1/n$ of the total value, the threshold becomes the *maximin share (MMS)*, which is the maximum value that an agent can ensure herself by partitioning the items into n parts and receiving the worst part (Budish 2011). The existence guarantees with respect to these relaxations are quite well-understood by now—an allocation satisfying EF1 and PROP1 always exists, and even though the same is not true for MMS fairness, there is always an allocation that gives every agent a constant fraction of her MMS (Ghodsi et al. 2018; Kurokawa, Procaccia, and Wang 2018; Garg and Taki 2021).

Recently, there have been several attempts to extend these approximate notions to agents with different entitlements. However, the resulting notions have been shown to exhibit a number of unintuitive and perhaps unsatisfactory features. In the "fairness up to one item" approach, EF1 has been generalized to *weighted EF1 (WEF1)* (Chakraborty et al. 2020) and PROP1 to *weighted PROP1 (WPROP1)* (Aziz, Moulin, and Sandomirskiy 2020). Yet, even though (weighted) envy-freeness implies (weighted) proportionality and EF1 implies PROP1, Chakraborty et al. (2020) have shown that, counterintuitively, WEF1 does not imply WPROP1. Moreover, while it is always possible to satisfy each of the two notions separately, these authors have demonstrated that it may be impossible to satisfy both simultaneously. In the "share-based" approach, MMS has been generalized to *weighted MMS (WMMS)* (Farhadi et al. 2019). A disadvantage of WMMS is that computing its value for an agent requires knowing not only the agent's own entitlement, but also the entitlements of all other agents. In addition, the definition of WMMS is rather difficult to understand and explain (especially when compared to MMS), thereby making it less likely to be adopted in practice.

In this paper, we expand and deepen our understanding of weighted fair division for indivisible items by generalizing existing weighted fairness notions, introducing new

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ones, and exploring the relationships between them. Along the way, we propose solutions to the aforementioned issues of existing notions, and uncover a wide range of features of the weighted setting that are not present in the unweighted case typically studied in the fair division literature.[1]

1.1 Our Contributions

As is commonly done in fair division, we assume throughout the paper that agents have additive utilities. In **Section 3**, we focus on fairness up to one item. The role of envy-freeness relaxations is to provide an upper bound on the amount of envy that is allowed between agents. In the unweighted setting, EF1 from agent i towards agent j requires that i's envy is at most i's highest utility for an item in j's bundle. Formally, denoting i's utility function by u_i and j's bundle by A_j, the envy should be at most $\max_{g \in A_j} u_i(g)$. In the weighted setting, however, envy is measured by comparing the scaled utilities $u_i(A_i)/w_i$ and $u_i(A_j)/w_j$. Therefore, one could reasonably argue that the amount of allowed envy should be similarly scaled to be either $u_i(g)/w_i$ or $u_i(g)/w_j$; the first scaling corresponds to (hypothetically) adding the value of g to A_i while the second scaling corresponds to removing the value of g from A_j. Clearly, the first scaling yields a smaller envy-allowance if and only if $w_i > w_j$, so it favors agents with larger weights, while the second scaling favors those with smaller weights. We generalize both extremes at once by defining the allowed envy to be a *weighted average* of the two quantities, i.e., $u_i(g) \cdot (x/w_j + y/w_i)$ for $x + y = 1$. We denote this envy-freeness relaxation by $\text{WEF}(x, y)$. Similarly, we define $\text{WPROP}(x, y)$ as a relaxation of proportionality.

For a fixed $x + y$, a higher x yields a stronger guarantee (e.g., low envy-allowance) for lower-entitlement agents, while a lower x yields a stronger guarantee for higher-entitlement agents. We show that $\text{WEF}(x, y)$ implies $\text{WPROP}(x, y)$ for all x, y. WEF1 corresponds to $\text{WEF}(1, 0)$ while WPROP1 corresponds to $\text{WPROP}(0, 1)$; this provides an explanation as well as a resolution of the counterintuitive fact that WEF1 is incompatible with WPROP1. With equal entitlements, $\text{WEF}(x, 1 - x)$ is equivalent to EF1 and $\text{WPROP}(x, 1 - x)$ is equivalent to PROP1 for all $x \in [0, 1]$. But with different entitlements, the conditions obtained for different values of x are different and incompatible. For each $x \in [0, 1]$, $\text{WEF}(x, 1 - x)$, and therefore $\text{WPROP}(x, 1 - x)$, can always be satisfied. On the other hand, for any distinct $x, x' \in [0, 1]$, an allocation that fulfills both $\text{WPROP}(x, 1 - x)$ and $\text{WPROP}(x', 1 - x')$ may not exist, and an analogous impossibility result holds for WEF relaxations.

In **Section 4**, we explore share-based fairness notions. We propose a new weighted generalization of MMS called *normalized MMS (NMMS)*, which is simply the MMS of an agent scaled by the agent's weight. Not only is NMMS intuitive, but its definition also depends only on the relative entitlement of the agent in question (that is, the enti-

tlement of the agent divided by the sum of entitlements), rather than on the entire vector of entitlements. We show that the best approximation factor that can always be attained for NMMS is $1/n$, matching the WMMS guarantee of Farhadi et al. (2019). As with WMMS, the $1/n$-NMMS guarantee can be achieved using the (unweighted) round-robin algorithm, with the agents taking turns in decreasing order of their weights. Interestingly, however, we show that WEF1 also implies $1/n$-NMMS—this means that a weighted round-robin algorithm as well as a generalization of Barman, Krisnamurthy, and Vaish (2018)'s market-based algorithm, which are known to ensure WEF1, also provide the worst-case optimal approximation of NMMS. In addition, we establish relations between several fairness notions, thereby extending the work of Amanatidis, Birmpas, and Markakis (2018) to the weighted setting.

In **Section 5**, we consider the case of identical items, also known as *apportionment* (Balinski and Young 2001). In this case, if the items were divisible, it would be clear that each agent should receive exactly the *quota* proportional to her weight. Hence, two important desiderata in apportionment are that each agent should obtain at least her *lower quota* (the quota rounded down), and at most her *upper quota* (the quota rounded up). We show that $\text{WEF}(x, 1 - x)$ guarantees lower quota if and only if $x = 0$ and upper quota if and only if $x = 1$, whereas WMMS, NMMS, and $\text{WPROP}(x, 1 - x)$ for any $x \in [0, 1]$ do not guarantee either quota. Similarly, the *maximum weighted Nash welfare* rule does not always respect either quota. Nevertheless, we introduce a new allocation rule, called *weighted egalitarian (WEG)*, which aims to maximize the leximin vector of the agents' utilities, with a carefully chosen normalization. We show that, in contrast to all other rules and notions studied herein, the WEG rule satisfies both quotas. This indicates that the egalitarian approach may be worthy of further study in settings with different entitlements.

2 Preliminaries

We consider a setting with a set $N = [n]$ of agents and a set $M = [m]$ of items, where $n \geq 2$ and $[k] := \{1, 2, \dots, k\}$ for each positive integer k. A subset of items is called a *bundle*. The *entitlement* or *weight* of each agent $i \in N$ is denoted by $w_i > 0$. For any subset of agents $N' \subseteq N$, we denote $w_{N'} := \sum_{i \in N'} w_i$. An *allocation* $A = (A_1, A_2, \dots, A_n)$ is a partition of M into m bundles such that bundle A_i is assigned to agent i. Each agent $i \in N$ has a *utility function* u_i, which we assume to be additive. This means that for any $M' \subseteq M$, $u_i(M') = \sum_{j \in M'} u_i(\{j\})$. For simplicity, we will sometimes write $u_i(j)$ instead of $u_i(\{j\})$ for $j \in M$.

3 Fairness up to One Item

In this section, we define the envy-freeness and proportionality relaxations $\text{WEF}(x, y)$ and $\text{WPROP}(x, y)$, and study them particularly for the case where $x + y = 1$.

3.1 Weighted Envy-Freeness Notions

We define a continuum of weighted envy-freeness relaxations parameterized by two nonnegative real values, x and

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[1] For an overview of previous work on weighted fair division of indivisible items, we refer to the related work sections of (Babaioff, Ezra, and Feige 2021) and (Chakraborty, Schmidt-Kraepelin, and Suksompong 2021).

y. We require that for any pair of agents i and j, for some item in j's bundle, after removing an x fraction of the value of this item and adding a y fraction of this value to i's bundle, i does not envy j. Formally:

Definition 3.1 (WEF(x, y))**.** For $x, y \in [0, 1]$, an allocation $(A_1, \ldots, A_n)$ is said to satisfy *WEF(x, y)* if for any $i, j \in N$, there exists $B \subseteq A_j$ with $|B| \leq 1$ such that[2]

$$\frac{u_i(A_i) + y \cdot u_i(B)}{w_i} \geq \frac{u_i(A_j) - x \cdot u_i(B)}{w_j}.$$

Equivalently, the condition can be stated in terms of the maximum allowed weighted envy:

$$\frac{u_i(A_j)}{w_j} - \frac{u_i(A_i)}{w_i} \leq \left(\frac{y}{w_i} + \frac{x}{w_j} \right) \cdot u_i(B).$$

WEF(x, y) generalizes and interpolates between several previously studied notions. In particular, WEF$(0, 0)$ is the same as weighted envy-freeness, WEF$(1, 0)$ is equivalent to the notion WEF1 proposed by Chakraborty et al. (2020), and WEF$(1, 1)$ corresponds to what these authors called "transfer weighted envy-freeness up to one item". The WWEF1 condition of Chakraborty et al. (2020) is strictly weaker than WEF$(x, 1-x)$ for every $x \in [0, 1]$; see the full version of our paper (Chakraborty, Segal-Halevi, and Suksompong 2021).

WEF(x, y) requires that the *weighted envy* of i towards j, defined as $\max \left\{ 0, \frac{u_i(A_j)}{w_j} - \frac{u_i(A_i)}{w_i} \right\}$, should be at most $\left(\frac{y}{w_i} + \frac{x}{w_j} \right) \cdot u_i(B)$. It is evident that, with equal entitlements, this condition depends only on the sum $x + y$; in particular, whenever $x + y = 1$, WEF(x, y) is equivalent to EF1. However, with different entitlements, every selection of x, y, even with $x + y = 1$, leads to a different condition: a higher x yields a stronger guarantee (i.e., lower allowed envy) for agent i when $w_i < w_j$, while a higher y yields a stronger guarantee for the agent when $w_i > w_j$. This raises two natural questions. First, for what values of x, y can WEF(x, y) be guaranteed? Second, is it possible to guarantee WEF(x, y) for multiple pairs (x, y) simultaneously?

For the first question, whenever $x + y < 1$, a standard example of two agents with equal weights and one valuable item shows that WEF(x, y) cannot always be satisfied. We will show next that WEF(x, y) can always be satisfied when $x + y = 1$, thereby implying existence for $x + y > 1$ as well. To this end, we characterize picking sequences[3] whose output is guaranteed to satisfy WEF$(x, 1-x)$; this generalizes the WEF$(1, 0)$ characterization of Chakraborty, Schmidt-Kraepelin, and Suksompong (2021, Thm. 3.1). For brevity, we say that a picking sequence satisfies a fairness notion if its output always satisfies that notion. The proof of this theorem, along with all other missing proofs, can be

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[2]We use a set $B \subseteq A_j$ with $|B| \leq 1$ instead of an item $g \in A_j$ in order to handle the case where $A_j = \emptyset$.

[3]A *picking sequence* is a sequence $(p_1, \ldots, p_m)$ where $p_i \in N$ for each i. The *output* of a picking sequence is the allocation resulting from the process in which, in the i-th turn, agent p_i picks her favorite item from the remaining items, breaking ties in a consistent manner (for example, in favor of lower-numbered items).

found in the full version of our paper (Chakraborty, Segal-Halevi, and Suksompong 2021).

Theorem 3.2. *Let $x \in [0, 1]$. A picking sequence π satisfies WEF$(x, 1-x)$ if and only if for every prefix P of π and every pair of agents i, j, where agent i has t_i picks in P and agent j has t_j picks in P, we have $t_i + (1-x) \geq \frac{w_i}{w_j} \cdot (t_j - x)$.*

We can now prove that a WEF$(x, 1-x)$ allocation exists in every instance.

Theorem 3.3. *Let $x \in [0, 1]$. Consider a picking sequence π such that in each turn, the pick is assigned to an agent i with the smallest $\frac{t_i + (1-x)}{w_i}$, where t_i is the number of times agent i has picked so far. Then, π satisfies WEF$(x, 1-x)$.*

Proof. Consider any pair of agents i, j. It suffices to show that after every pick of agent j, the condition in Theorem 3.2 is satisfied. Suppose that after j's pick, the two agents have picked t_i and t_j times. Since j was assigned the pick, it must be that $\frac{(t_j + (1-x)) - 1}{w_j} \leq \frac{t_i + (1-x)}{w_i}$. In other words, we have $t_i + (1-x) \geq \frac{w_i}{w_j} \cdot (t_j - x)$, as desired. $\square$

In particular, Webster's apportionment method, which corresponds to the picking sequence in Theorem 3.3 with $x = 1/2$, satisfies WEF$(1/2, 1/2)$. This strengthens a result of Chakraborty, Schmidt-Kraepelin, and Suksompong (2021) that the method guarantees WWEF1.

While every WEF$(x, 1-x)$ notion can be satisfied on its own, it is unfortunately impossible to guarantee WEF$(x, 1-x)$ for two different values of x. We prove this strong incompatibility result even for the weaker notion of WPROP(x, y), which we define next.

3.2 Weighted Proportionality Notions

Similarly to WEF(x, y), we define a continuum of weighted proportionality relaxations.

Definition 3.4 (WPROP(x, y))**.** For $x, y \in [0, 1]$, an allocation $(A_1, \ldots, A_n)$ is said to satisfy *WPROP(x, y)* if for any $i \in N$, there exists $B \subseteq M \setminus A_i$ with $|B| \leq 1$ such that

$$\frac{u_i(A_i) + y \cdot u_i(B)}{w_i} \geq \frac{u_i(M) - n \cdot x \cdot u_i(B)}{w_N},$$

or equivalently,

$$u_i(A_i) \geq \frac{w_i}{w_N} \cdot u_i(M) - \left(\frac{w_i}{w_N} \cdot n \cdot x + y \right) \cdot u_i(B).$$

WPROP$(0, 0)$ is the same as weighted proportionality, while WPROP$(0, 1)$ is equivalent to the notion WPROP1 put forward by Aziz, Moulin, and Sandomirskiy (2020). As noted above, an equivalent condition is that, if the utility that agent i derives from her bundle is less than her *(weighted) proportional share* $\frac{w_i}{w_N} \cdot u_i(M)$, then the amount by which it falls short should not exceed $\left(\frac{w_i}{w_N} \cdot n \cdot x + y \right) \cdot u_i(B)$. The factor n can be thought of as a normalization factor, since B is removed from the entire set of items M, which is distributed among n agents, rather than from another agent's bundle as in the definition of WEF(x, y). This normalization

ensures that, similarly to the WEF(x, y) family, when all entitlements are equal, WPROP(x, y) reduces to PROP1 whenever $x + y = 1$. With different entitlements and $x + y = 1$, a higher x yields a stronger guarantee for agent i (i.e., the agent's utility cannot be far below her proportional share) when $w_i < w_N/n$, while a higher y yields a stronger guarantee for the agent when $w_i > w_N/n$ (note that the quantity w_N/n is the average weight of the n agents). This raises the same two questions that we posed for WEF(x, y): For which pairs (x, y) can WPROP(x, y) always be attained? And is it possible to attain it for different pairs (x, y) simultaneously?

To answer these questions, we first define a stronger version of WPROP(x, y) which we call WPROP$^*(x, y)$.

Definition 3.5 (WPROP$^*(x, y)$). For $x, y \in [0, 1]$, an allocation $(A_1, \ldots, A_n)$ is said to satisfy WPROP$^*(x, y)$ if for any $i \in N$, the following holds: There exists $B \subseteq M \setminus A_i$ with $|B| \leq 1$ and, for every $j \in N \setminus \{i\}$, there exists $B_j \subseteq A_j$ with $|B_j| \leq 1$ such that

$$\frac{u_i(A_i) + y \cdot u_i(B)}{w_i} \geq \frac{u_i(M) - x \cdot \sum_{j \in N \setminus \{i\}} u_i(B_j)}{w_N}.$$

Since $\sum_{j \in N \setminus \{i\}} u_i(B_j) \leq (n - 1) \cdot u_i(B) \leq n \cdot u_i(B)$, where we take B to be the singleton set containing i's most valuable item in $M \setminus A_i$ if $A_i \neq M$ and the empty set otherwise, WPROP$^*(x, y)$ is a strengthening of WPROP(x, y). Nevertheless, it is less intuitive than WPROP(x, y). As such, we do not propose WPROP$^*(x, y)$ as a major fairness desideratum in its own right, but it will prove to be a useful concept in establishing some of our results.

We first prove that for all x and y, WEF(x, y) implies WPROP$^*(x, y)$, which in turn implies WPROP(x, y); this generalizes the fact that weighted envy-freeness implies weighted proportionality, which corresponds to taking $x = y = 0$. Note that while we generally think of x and y as being constants not depending on n or $w_1, \ldots, w_n$, in the following three implications we will derive more refined bounds in which x and y may depend on these parameters.

Lemma 3.6. *For any $x, y \in [0, 1]$ and $y' \geq (1 - \frac{\min_{i \in N} w_i}{w_N}) y$, WEF$(x, y)$ implies WPROP$^*(x, y')$.*

In particular, WEF(x, y) implies WPROP$^*(x, y)$.

Lemma 3.7. *For any $x, y \in [0, 1]$ and $x' \geq \left(1 - \frac{1}{n}\right) x$, WPROP$^*(x, y)$ implies WPROP$^*(x', y)$.*

In particular, WPROP$^*(x, y)$ implies WPROP(x, y).

Corollary 3.8. *For any $x, y \in [0, 1]$, WEF(x, y) implies WPROP(x, y).*

Combining Corollary 3.8 and Theorem 3.3, we find that WPROP$(x, 1 - x)$ can be guaranteed for any fixed x. Can it be guaranteed for two different x simultaneously? Can it be guaranteed for x, y such that $x + y < 1$? The following theorem (along with the discussion after it) shows that the answer to both questions is no.

Theorem 3.9. *For any $x, x', y, y' \in [0, 1]$, if $x' + y < 1$ or $x + y' < 1$, there is an instance with identical items in which no allocation is both WPROP(x, y) and WPROP(x', y').*

Theorem 3.9 has several corollaries.

First, taking $y = 1 - x$ and $y' = 1 - x'$ yields that, if $x \neq x'$, there is an instance in which no allocation is both WPROP$(x, 1 - x)$ and WPROP$(x', 1 - x')$. Combining this with Corollary 3.8 gives the same incompatibility for WEF$(x, 1 - x)$ and WEF$(x', 1 - x')$.

Second, taking $x' = x$ and $y' = y$ yields that, when $x + y < 1$, a WPROP(x, y) allocation is not guaranteed to exist, and therefore a WEF(x, y) allocation may not exist.

Third, we can generalize Chakraborty et al. (2020)'s result that WEF$(1, 0)$ does not imply WPROP$(0, 1)$:

Corollary 3.10. *For any distinct $x, x' \in [0, 1]$, WEF$(x, 1 - x)$ does not imply WPROP$(x', 1 - x')$.*

Proof. Assume for contradiction that WEF$(x, 1 - x)$ implies WPROP$(x', 1 - x')$ for some $x \neq x'$. By Theorem 3.2, a WEF$(x, 1 - x)$ allocation always exists, and therefore such an allocation is WPROP$(x', 1 - x')$. By Corollary 3.8, this allocation is also WPROP$(x, 1 - x)$. Hence, an allocation that is both WPROP$(x, 1 - x)$ and WPROP$(x', 1 - x')$ always exists, contradicting the first corollary of Theorem 3.9 mentioned above. $\square$

3.3 Weighted Nash Welfare

The WEF and WPROP criteria consider individual agents or pairs of agents. A different approach, known in economics as "welfarism" (Moulin 2003), takes a global view and tries to find an allocation that maximizes a certain aggregate function of the utilities. A common aggregate function is the product of utilities, also called the *Nash welfare*. This notion extends to the weighted setting as follows.

Definition 3.11 (MWNW). *A maximum weighted Nash welfare* allocation is an allocation that maximizes the weighted product $\prod_{i \in N} u_i(A_i)^{w_i}$. [4]

With equal entitlements, MWNW implies EF1 (Caragiannis et al. 2019). However, with different entitlements, MWNW is incompatible with WEF$(1, 0)$ (Chakraborty et al. 2020) and with WPROP$(0, 1)$ (Chakraborty, Schmidt-Kraepelin, and Suksompong 2021). We generalize both of these incompatibility results at once.

Theorem 3.12. *For each $x \in [0, 1]$, there exists an instance with identical items in which every MWNW allocation is not WPROP$(x, 1 - x)$, and hence not WEF$(x, 1 - x)$.*

Proof. We prove the case $x < 1$ here and leave the case $x = 1$ to the full version. Consider an instance with n identical items and n agents, with $n > w_1 > \frac{2 - x}{1 - x}$ (the range is nonempty when n is sufficiently large), and $w_2 = \cdots = w_n = (n - w_1)/(n - 1)$, so $w_N = n$. Any MWNW allocation must give a single item to each agent. But this violates the WPROP$(x, 1 - x)$ condition for agent 1, since $\frac{1 + (1 - x)}{w_1} < \frac{2 - x}{\frac{2 - x}{1 - x}} = 1 - x = \frac{n - nx}{w_N}$. $\square$

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[4] If the maximum weighted product is zero (that is, when all allocations give a utility of zero to one or more agents), then the MWNW rule first maximizes the number of agents who get a positive utility, and subject to that, maximizes the weighted product for these agents.

4 Share-Based Notions

In this section, we turn our attention to share-based notions, which assign a threshold to each agent representing the agent's "fair share". Denote by $\Pi(M, n)$ the collection of all ordered partitions of M into n subsets. In the equal-entitlement setting, the *(1-out-of-n) maximin share* of an agent i is defined as follows:

$$\mathrm{MMS}_i^{1\text{-out-of-}n}(M) := \max_{(Z_1,\ldots,Z_n) \in \Pi(M,n)} \min_{j \in [n]} u_i(Z_j).$$

Sometimes we drop M and the superscript '1-out-of-n' and simply write MMS_i. An allocation is called *MMS-fair* or simply *MMS* if the utility that each agent receives is at least as high as the agent's MMS. Similarly, for a parameter α, an allocation is called *α-MMS-fair* or *α-MMS* if every agent receives utility at least α times her MMS. We will use analogous terminology for other share-based notions.

There are several ways to extend this notion to the unequal-entitlement setting. The first definition is due to Farhadi et al. (2019). Denote by $\mathbf{w} = (w_1, \ldots, w_n)$ the vector of weights.

Definition 4.1 (WMMS). The *weighted maximin share* of an agent $i \in N$ is defined as:

$$\mathrm{WMMS}_i^{\mathbf{w}}(M) := \max_{(Z_1,\ldots,Z_n) \in \Pi(M,n)} \min_{j \in [n]} \frac{w_i}{w_j} \cdot u_i(Z_j)$$

$$= w_i \cdot \max_{(Z_1,\ldots,Z_n) \in \Pi(M,n)} \min_{j \in [n]} \frac{u_i(Z_j)}{w_j}.$$

Sometimes we will drop $\mathbf{w}$ and M from the notation when these are clear from the context; the same convention applies to other notions.

Intuitively, WMMS tries to find the most proportional allocation with respect to all agents' weights and agent i's utility function. Note that the WMMS of agent i depends not only on i's entitlement, but also on the entitlements of all other agents. In particular, even if i's entitlement remains fixed, her WMMS might vary due to changes in the other weights. This can be seen as a disadvantage of the WMMS notion. Farhadi et al. (2019) showed that a $1/n$-WMMS allocation always exists, and this guarantee is tight for every n.

The second definition was implicitly considered by Babaioff, Nisan, and Talgam-Cohen (2021), who did not give it a name. Following Segal-Halevi (2019), we call it the *ordinal maximin share*. Babaioff, Ezra, and Feige (2021) called it the *pessimistic share*. To define this share, we first extend the notion of 1-out-of-n MMS as follows. For any positive integers $\ell \leq d$,

$$\mathrm{MMS}_i^{\ell\text{-out-of-}d}(M) := \max_{P \in \Pi(M,d)} \min_{Z \in \mathrm{Unions}(P,\ell)} u_i(Z),$$

where the minimum is taken over all unions of ℓ bundles from a given d-partition P. Based on this generalized MMS notion, we define the ordinal MMS:

Definition 4.2 (OMMS). The *ordinal maximin share* of an agent $i \in N$ is defined as:

$$\mathrm{OMMS}_i^{\mathbf{w}}(M) := \max_{\ell,d: \frac{\ell}{d} \leq \frac{w_i}{w_N}} \mathrm{MMS}_i^{\ell\text{-out-of-}d}(M).$$

With equal entitlements, the OMMS is equal to the MMS. Therefore, an OMMS allocation always exists for agents with identical valuations, but may not exist for $n \geq 3$ agents with different valuations (Kurokawa, Procaccia, and Wang 2018). With different entitlements, it is an open question whether an OMMS allocation always exists for agents with identical valuations.

The third notion, the *AnyPrice Share*, is due to Babaioff, Ezra, and Feige (2021). Instead of partitioning the items into n disjoint bundles, an agent is allowed to choose any collection of (possibly overlapping) bundles. However, the agent must then assign a weight to each chosen bundle so that the sum of the bundles' weights is w_N, and each item belongs to bundles whose total weight is at most the agent's entitlement. The AnyPrice Share is the agent's utility for the least valuable chosen bundle. Formally:

Definition 4.3 (APS). The *AnyPrice share* of an agent $i \in N$ is defined as:

$$\mathrm{APS}_i^{\mathbf{w}}(M) := \max_{P \in \mathrm{AllowedBundleCollections}(M,w_i)} \min_{Z \in P} u_i(Z).$$

where the maximum is taken over all collections P of bundles such that for some assignment of weights to the bundles in P, the total weight of all bundles in P is w_N, and for each item, the total weight of the bundles to which the item belongs is at most w_i.

Observe that when all entitlements are equal (to w_N/n), the agent can choose any 1-out-of-n MMS partition and assign a weight of w_N/n to each part; this shows that the APS is at least the MMS in the equal-entitlement setting. Babaioff, Ezra, and Feige (2021) proved that a $3/5$-APS allocation exists for agents with arbitrary entitlements, and that the APS is always at least as large as the OMMS. Hence, their result implies the existence of a $3/5$-OMMS allocation. Babaioff et al. also gave an equal-entitlement example in which the APS is strictly larger than the MMS. Their example shows a disadvantage of the APS: an allocation that gives every agent her APS may not exist even when all agents have identical valuations and equal entitlements.

The fourth notion, which is new to this paper, is the *normalized maximin share*. The idea is that we take an agent's 1-out-of-n MMS and scale it according to the agent's entitlement.

Definition 4.4 (NMMS). The *normalized maximin share* of an agent $i \in N$ is defined as:

$$\mathrm{NMMS}_i^{\mathbf{w}}(M) := \frac{w_i}{w_N} \cdot n \cdot \mathrm{MMS}_i^{1\text{-out-of-}n}(M).$$

Compared to the previous three notions, the definition of NMMS is rather simple. Moreover, unlike WMMS, the NMMS of an agent depends only on the agent's relative entitlement (i.e., w_i/w_N).

APS-fairness and OMMS-fairness are both "ordinal" notions in the sense that, even though the APS and OMMS values are numerical, whether an allocation is APS- or OMMS-fair can be determined by only inspecting each agent's ordinal ranking over bundles. On the other hand, WMMS-fairness and NMMS-fairness are both "cardinal" notions,

since they depend crucially on the numerical utilities. We prove that there are no implication relations between ordinal and cardinal notions: each type of notions does not imply any nontrivial approximation of the other type.

Theorem 4.5. *(a) An APS-fair or OMMS-fair allocation does not necessarily yield any positive approximation of WMMS-fairness or NMMS-fairness.*

(b) A WMMS-fair or NMMS-fair allocation does not necessarily yield any positive approximation of APS-fairness or OMMS-fairness.

Given that NMMS is a new notion, an important question is whether any useful approximation of it can be ensured. Farhadi et al. (2019) proved that the best possible WMMS guarantee is $1/n$-WMMS. We prove that the same holds for NMMS, starting with the upper bound.

Theorem 4.6. *For each n, there is no NMMS guarantee over all instances with n agents better than $1/n$-NMMS.*

We establish a matching lower bound by proving that WPROP*$(1, 0)$, and hence WEF$(1, 0)$, implies $1/n$-NMMS.

Lemma 4.7. *WPROP*$(1, 0)$ implies $1/n$-NMMS.*

Theorem 4.8. *WEF$(1, 0)$ implies $1/n$-NMMS. In particular, every instance admits a $1/n$-NMMS allocation. The factor $1/n$ in both statements cannot be improved.*

Theorem 4.8 is an immediate consequence of Lemmas 3.6 and 4.7 and Theorem 4.6. It generalizes the result that EF1 implies $1/n$-MMS in the unweighted setting (Amanatidis, Birmpas, and Markakis 2018, Prop. 3.6), and stands in contrast to a result of Chakraborty et al. (2020, Prop. 6.2) that WEF$(1, 0)$ does not imply any positive approximation of WMMS. Since a weighted round-robin algorithm as well as a generalization of Barman, Krisnamurthy, and Vaish (2018)'s market-based algorithm ensure WEF$(1, 0)$ (Chakraborty et al. 2020), these algorithms guarantee $1/n$-NMMS as well. Moreover, we prove that if each agent values each item at most her NMMS, then the approximation factor can be improved to $1/2$, thereby providing a direct analog to a WMMS result by Farhadi et al. (2019, Thm. 3.2).

Theorem 4.9. *Given an instance, if $u_i(g) \leq \text{NMMS}_i$ for all $i \in N$ and $g \in M$, then there exists a $1/2$-NMMS allocation.*

Unlike WEF$(1, 0)$, we prove that for each $x \in [0, 1)$, WEF$(x, 1 - x)$ does not imply any positive approximation of NMMS, so the same holds for WPROP$(x, 1 - x)$. In addition, the same holds even for WPROP$(1, 0)$, which explains why using WPROP*$(1, 0)$ is necessary in Lemma 4.7. Further, MWNW does not imply any positive approximation of NMMS, thereby providing evidence that $1/n$-NMMS is not trivial to achieve. This also contrasts with the unweighted setting, where maximum Nash welfare implies $\Theta(1/\sqrt{n})$-MMS (Caragiannis et al. 2019). The proofs for all of these results can be found in the full version of our paper (Chakraborty, Segal-Halevi, and Suksompong 2021).

Farhadi et al. (2019, Thm. 2.2) established the $1/n$-WMMS guarantee through a well-known algorithm, the *(unweighted) round-robin algorithm*, which simply lets agents take turns picking their favorite item from the remaining

items. A crucial specification required for their guarantee to work is that the agents must take turns in non-increasing order of their weights. This motivates the following definition:

Definition 4.10 (OEF1). An allocation is *ordered-EF1* if

(i) it is EF1 when we disregard weights, and
(ii) the agents can be renumbered so that $w_1 \geq w_2 \geq \cdots \geq w_n$ and no agent $i \in N$ has (unweighted) envy towards any later agent $j \in \{i + 1, i + 2, \ldots, n\}$.

The standard proof that the unweighted round-robin algorithm outputs an EF1 allocation[5] implies that the algorithm with the aforementioned ordering specification outputs an OEF1 allocation. Moreover, it follows from Farhadi et al. (2019)'s proof that OEF1 implies $1/n$-WMMS. We establish next that, interestingly, OEF1 also implies $1/n$-NMMS, which means that the same algorithm guarantees the optimal approximation of both WMMS and NMMS.

Theorem 4.11. *Any OEF1 allocation is also $1/n$-NMMS.*

WMMS and NMMS are both cardinal extensions of MMS to the weighted setting. However, we prove in the next two theorems that the relationship between them is rather weak.

Theorem 4.12. *WMMS implies $1/n$-NMMS, and the factor $1/n$ is tight.[6]*

Theorem 4.13. *For $n \geq 3$, NMMS does not imply any positive approximation of WMMS.*

Chakraborty et al. (2020) showed that WEF$(1, 0)$ does not guarantee any positive approximation of WMMS. In the full version, we prove a similar negative result for WEF$(x, 1 - x)$ and WPROP$(x, 1 - x)$ for all $x \in [0, 1]$.

5 Identical Items: Lower and Upper Quota

We have seen many fairness notions and proved that several of them are incompatible with one another. In this section, we add another dimension to the comparison between fairness notions by focusing on the case where all items are identical. The motivation for studying this case is twofold. First, due to its simplicity, it is easier to agree on a fairness criterion. If the items were divisible, agent i should clearly receive her *quota* of $q_i := \frac{w_i}{w_N} \cdot m$ items. With indivisible items, one may therefore expect agent i to receive either her *lower quota* $\lfloor q_i \rfloor$ or her *upper quota* $\lceil q_i \rceil$. Second, the case of identical items is practically relevant when allocating parliament seats among states or parties, a setting commonly known as *apportionment* (Balinski and Young 1975, 2001). Both quotas are frequently considered in apportionment.

We determine whether each fairness notion implies lower quota (resp., upper quota) in the identical-item setting, i.e., whether it ensures that each agent $i \in N$ receives at least $\lfloor q_i \rfloor$ items (resp., at most $\lceil q_i \rceil$ items). Our results are summarized in Table 1; all proofs are in the full version of our paper (Chakraborty, Segal-Halevi, and Suksompong 2021).

To illustrate the ideas behind some of these proofs, consider an instance with $n = m = 3$ and entitlements 4, 1, 1.

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[5]See, e.g., (Caragiannis et al. 2019, p. 7).

[6]Note that the tightness does not follow from Theorem 4.6, since a WMMS allocation does not always exist.

Notion	Lower quota	Upper quota
WEF$(x, 1-x)$	Yes if $x = 0$ No if $x \in (0,1]$	Yes if $x = 1$ No if $x \in [0,1)$
WPROP$(x, 1-x)$	No	No
MWNW	No	No
WMMS	No	No
NMMS	No	No
OMMS	Yes	No
APS	Yes	No

Table 1: Summary of whether each fairness notion implies lower or upper quota in the identical-item setting.

The quotas are 2, 0.5, 0.5. We denote an allocation by $(a_1, \ldots, a_n)$, where $a_i = |A_i| =$ the number of items that agent i receives.

The unique allocation satisfying WEF$(1,0)$ is $(1,1,1)$: any agent that gets no item would have weighted envy towards any agent that gets two or more items, even after removing a single item. Note that $(1,1,1)$ is also the unique MWNW allocation as well as the unique allocation satisfying WMMS- and NMMS-fairness, since the WMMS and NMMS are positive for all agents. However, this allocation violates the lower quota of agent 1, which is 2 items.

In contrast, the unique allocation satisfying WEF$(0,1)$ is $(3,0,0)$: if agent 1 gets two or fewer items, she feels weighted envy—even after getting an additional item—towards another agent who gets (at least) one item. The allocation $(3,0,0)$ also satisfies OMMS- and APS-fairness, as the OMMS and APS of agents 2 and 3 are both 0. However, it violates the upper quota of agent 1.

No notion that we have seen so far guarantees both lower and upper quotas. In light of this, we introduce a new rule based on the well-known *leximin* principle (Moulin 2003). With equal entitlements, leximin aims to maximize the smallest utility, then the second smallest utility, and so on. To extend it to the setting with different entitlements, we have to carefully consider how the utilities should be normalized. A first idea is to normalize the agents' utilities by their proportional share, that is:

$$\arg\max_{(A_1,\ldots,A_n)} \operatorname*{leximin}_{i \in N} \frac{w_N \cdot u_i(A_i)}{w_i \cdot u_i(M)}.$$

where "leximin" means that we maximize the smallest value, then the second smallest, and so on. However, this rule yields a blatantly unfair outcome when there is a single item and two agents: since the smallest utility is always 0, the item is allocated to the agent with a *smaller* entitlement, as this makes the ratio w_N / w_i larger. Therefore, we instead maximize the minimum *difference* between the agents' normalized utilities and their proportional shares.

Definition 5.1 (WEG). A *weighted egalitarian* allocation is an allocation in

$$\arg\max_{(A_1,\ldots,A_n)} \operatorname*{leximin}_{i \in N} \left(\frac{u_i(A_i)}{u_i(M)} - \frac{w_i}{w_N} \right).$$

This rule is similar to the "Leximin rule" for apportionment (Biró, Kóczy, and Sziklai 2015); the latter rule minimizes the leximin vector of the *departures*, defined as the absolute difference $\left| \frac{w_i \cdot u_i(M)}{w_N \cdot u_i(A_i)} - 1 \right|$.

Note that in the instance with one item and two agents, a WEG allocation gives the item to the agent with a larger entitlement, as one would reasonably expect. Moreover, in the aforementioned example with $n = m = 3$, the WEG allocations are $(2,1,0)$ and $(2,0,1)$, which satisfy both quotas. The next theorem shows that this holds in general.

Theorem 5.2. *With identical items, every WEG allocation satisfies both lower and upper quota.*[7]

In contrast to share-based notions, a WEG allocation always exists by definition. Moreover, it is relatively easy to explain such an allocation. For example, if a WEG allocation in a particular instance yields a minimum difference $\frac{u_i(A_i)}{u_i(M)} - \frac{w_i}{w_N} = -0.05$, one can explain to the agents that "each of you receives only 5% less than your proportional share, and there is no allocation with a smaller deviation". This makes the egalitarian approach attractive for further study in the context of unequal entitlements.

6 Conclusion and Future Work

In this paper, we have revisited known fairness notions for the setting where agents can have different entitlements to the resource, and introduced several new notions for this setting. Our work further reveals the richness of weighted fair division that has been uncovered by several recent papers. Indeed, when all agents have the same weight, WEF$(x, 1-x)$ reduces to EF1 and WPROP$(x, 1-x)$ reduces to PROP1 for all $x \in [0,1]$, while WMMS, NMMS, and OMMS all reduce to MMS. We believe that the concepts we introduced add meaningful value beyond those proposed in prior work. In particular, the notions WEF$(x, 1-x)$ and WPROP$(x, 1-x)$ allow us to choose the degree to which we want to prioritize agents with larger weights in comparison to those with smaller weights. A natural middle ground is $x = 1/2$—the notion WEF$(1/2, 1/2)$ is particularly appealing because Webster's apportionment method, which satisfies it (Theorem 3.3), is known to be the unique unbiased method under various definitions of bias in the apportionment context (Balinski and Young 2001, Sec. A.5). Furthermore, our NMMS notion provides an intuitive generalization of the well-studied MMS criterion for which a nontrivial approximation can always be attained.

An interesting direction for future work is to study our new fairness notions in conjunction with other properties, for example the economic efficiency property of *Pareto optimality (PO)*. Chakraborty et al. (2020) have shown, by means of generalizing Barman, Krisnamurthy, and Vaish (2018)'s market-based algorithm, that WEF$(1,0)$ is compatible with PO, but it remains unclear whether their argument can be further generalized to work for WEF$(x, 1-x)$ when $x < 1$. Extending some of our results to non-additive utilities is a challenging but important direction as well.

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[7]In the full version we show that, in the more general case of *binary* valuations, a WEG allocation satisfies APS and OMMS.

Acknowledgments

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