Focused Inference and System P

Marco Wilhelm and Gabriele Kern-Isberner
Department of Computer Science, TU Dortmund, Dortmund, Germany
marco.wilhelm@tu-dortmund.de, gabriele.kern-isberner@cs.tu-dortmund.de

Abstract

We bring in the concept of focused inference into the field of qualitative nonmonotonic reasoning by applying focused inference to System P. The idea behind drawing focused inferences is to concentrate on knowledge which seems to be relevant for answering a query while completely disregarding the remaining knowledge even at the risk of missing some meaningful information. Focused inference is motivated by mimicking snap decisions of human reasoners and aims on rapidly drawing still reasonable inferences from large sets of knowledge. In this paper, we define a series of query-dependent, syntactically-driven focused inference relations, elaborate on their formal properties, and show that the series converges against System P. We take advantage of this result in form of an anytime algorithm for drawing inferences which is accompanied by a thorough complexity analysis.

Introduction

Knowledge-based systems (Rajendra and Sajja 2009) represent a subfield of artificial intelligence the essence of which is to infer novel information of high quality from usually uncertain or vague knowledge bases. System P (Adams 1975; Kraus, Lehmann, and Magidor 1990) constitutes a well-established standard of formal quality criteria which is used to evaluate such nonmonotonic inferences of knowledge-based systems. A drawback of high quality inference formalisms is that they are typically based on rich epistemic structures which are computationally expensive. Hence, it is of great interest from both an epistemic and a computational point of view to *focus on relevant knowledge* when drawing inferences. The most radical way of focusing on knowledge is to disregard the remaining knowledge completely. Surprisingly, there is only few work which investigates this idea of focusing on knowledge resp. of limiting beliefs (cf., e.g., (Kern-Isberner and Brewka 2017; Schwering 2017; Lakemeyer and Levesque 2020; Rosales and Jaakkola 2005; Chechetka and Guestrin 2010)). Certainly, one reason is that the high quality of inferences cannot be guaranteed in general when ignoring information. Nevertheless, in order to counter ever-growing knowledge bases and time limitation in real-time applications of knowledge-based systems, it is necessary to develop pragmatic inference strategies.

In this paper, we define a notion of knowledge in focus which depends on a syntactical distance to the query, investigate formal properties of inferences that are drawn from knowledge in focus, and provide an anytime algorithm which calculates focused inferences and converges against System P. A complexity analysis shows that our algorithm is able to compute System P inferences faster than direct computations if the relevant knowledge is syntactically close to the query.

Preliminaries

We consider a propositional language $\mathcal{L}$ which is defined over a finite set of *(propositional) variables* Σ. *Formulas* in $\mathcal{L}$ are variables or *negations* $(\neg A)$, *conjunctions* $(A \wedge B)$, or *disjunctions* $(A \vee B)$ of formulas $A, B \in \mathcal{L}$. We abbreviate $\neg A$ with $\overline{A}$, and $A \wedge B$ with AB. The semantics of formulas is given by *interpretations* $\mathcal{I} : \mathcal{L} \to \{0,1\}$ as usual in propositional logics. *Classical entailment* between formulas A, B is defined by $A \models B$ iff $\mathcal{I}(A) = 1$ implies $\mathcal{I}(B) = 1$ for all interpretations $\mathcal{I}$. If both A entails B and B entails A, then A and B are *logically equivalent*, written $A \equiv B$. Tautologies (formulas that are true in all interpretations) are denoted with $\top$ and contradictions (formulas that are false in all interpretations) with $\bot$.

Conditionals $(B|A)$ with $A, B \in \mathcal{L}$ are formal representations of default options of the form "If A holds, then usually B holds, too." and lead to a three-valued evaluation:

$$\mathcal{I}(B|A) = \begin{cases} 1 & \text{iff } \ \mathcal{I}(AB) = 1 & \text{(verification)} \\ 0 & \text{iff } \ \mathcal{I}(A\overline{B}) = 1 & \text{(falsification)} \\ u & \text{iff } \ \mathcal{I}(\overline{A}) = 1 & \text{(non-applicability)} \end{cases} .$$

A semantics of conditionals is provided by *ranking functions over possible worlds* (Spohn 2012). Here, *possible worlds* are simply propositional interpretations represented by complete conjunctions of *literals*, i.e. positive or negated variables. The set of all possible worlds is denoted with $\Omega(\Sigma)$. A *ranking function* $\kappa : \Omega(\Sigma) \to \mathbb{N}_0$ maps possible worlds to a degree of implausibility and is normalized by the requirement $\kappa^{-1}(0) \neq \emptyset$. Hence, $\kappa^{-1}(0)$ is the set of the most plausible worlds. Ranking functions are extended to formulas $A \not\equiv \bot$ by $\kappa(A) = \min\{\kappa(\omega) \mid \omega \in \Omega(\Sigma),\ \omega \models A\}$ and to formulas $A \equiv \bot$ by $\kappa(A) = \infty$. A ranking function κ is a *model* of a conditional, written $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$ or $A \equiv \bot$.

Input: Knowledge base Δ
Output: Z-partition of Δ if Δ is consistent; empty list otherwise

output $\leftarrow [\,]$
while $\Delta \neq \emptyset$
  $\Delta' \leftarrow \emptyset$
  for $r \in \Delta$
    if Δ tolerates r **then** $\Delta' \leftarrow \Delta' \cup \{r\}$
  if $\Delta' = \emptyset$ **then** output $\leftarrow [\,]$ and **break**
  output $\leftarrow$ output.append(Δ')
  $\Delta \leftarrow \Delta \setminus \Delta'$
return output

Algorithm 1: Consistency test for a knowledge base Δ which returns the Z-partition of Δ in case of consistency.

A finite set of conditionals is called *knowledge base*. With $\Sigma(\Delta)$ we denote the *signature* that is induced by the knowledge base Δ, i.e., $\Sigma(\Delta)$ is the set of variables which are mentioned in at least one conditional in Δ. Analogously, we write $\Sigma(X)$ for the signature of X, whether X is a formula, a conditional, or a query. A ranking function over $\Omega(\Sigma(\Delta))$ is a *model* of a knowledge base Δ iff it is a model of every conditional in Δ. A knowledge base is *consistent* iff it has at least one model. Otherwise, it is *inconsistent*.

A *query* $?A \mathrel{|\!\sim} B$ with $A, B \in \mathcal{L}$ asks whether B can be inferred from A. The answer depends on the underlying inference formalism which reflects the reasoner's inference behavior. Here, we rely on the nonmonotonic inference relation $\mathrel{|\!\sim}_{\Delta} \in \mathcal{L} \times \mathcal{L}$ which is defined wrt. a knowledge base Δ:

$$A \mathrel{|\!\sim}_{\Delta} B \quad \text{iff} \quad \kappa \models (B|A) \quad \text{for all models } \kappa \text{ of } \Delta.$$

For any fixed inference relation $\mathrel{|\!\sim}_x$, we denote the respective *instance* of the query $?A \mathrel{|\!\sim} B$ by $?A \mathrel{|\!\sim}_x B$. The instance $?A \mathrel{|\!\sim}_{\Delta} B$ is answered with `true` iff $A \mathrel{|\!\sim}_{\Delta} B$ holds and with `false` otherwise. The inference relation $\mathrel{|\!\sim}_{\Delta}$ is of high relevance in nonmonotonic reasoning as it is a semantical characterization of *System P* (Adams 1975; Kraus, Lehmann, and Magidor 1990) which provides an important standard for plausible nonmonotonic inference.

Definition 1 (System P)**.** *Let* $\mathrel{|\!\sim} \subseteq \mathcal{L} \times \mathcal{L}$ *be an inference relation, and let* $A, B, C \in \mathcal{L}$*. Then,* System P *is the collection of the following inference rules:*

$$A \mathrel{|\!\sim} A, \qquad \text{(Reflexivity)}$$
$$AB \mathrel{|\!\sim} C,\ A \mathrel{|\!\sim} B \textit{ imply } A \mathrel{|\!\sim} C, \qquad \text{(Cut)}$$
$$A \mathrel{|\!\sim} B,\ A \mathrel{|\!\sim} C \textit{ imply } AB \mathrel{|\!\sim} C, \qquad \text{(Cautious Monotony)}$$
$$A \mathrel{|\!\sim} B,\ B \models C \textit{ imply } A \mathrel{|\!\sim} C, \qquad \text{(Right Weakening)}$$
$$A \mathrel{|\!\sim} C,\ B \mathrel{|\!\sim} C \textit{ imply } A \vee B \mathrel{|\!\sim} C, \qquad \text{(Or)}$$
$$A \equiv B,\ B \mathrel{|\!\sim} C \textit{ imply } A \mathrel{|\!\sim} C. \qquad \text{(Left Logical Equivalence)}$$

The inference relation $\mathrel{|\!\sim}$ satisfies *System P iff it satisfies all inference rules in System P.*

It is well known that $A \mathrel{|\!\sim}_{\Delta} B$ holds iff $\Delta \cup \{(\overline{B}|A)\}$ is inconsistent (Goldszmidt and Pearl 1991). (In-)consistency of a knowledge base can be decided with Algorithm 1 which makes use of the notions *tolerance* and *Z-partition*.[1] A knowledge base Δ *tolerates* a conditional $r = (B|A)$ iff there is a possible world in which r is verified and no conditional from Δ is falsified, i.e., iff $AB \wedge \bigwedge_{(B'|A') \in \Delta} (\overline{A'} \vee B')$ is satisfiable. An ordered partition $(\Delta_0, \Delta_1, \ldots, \Delta_m)$ of Δ is a *tolerance partition* of Δ iff, for $i = 0, \ldots, m$, every conditional in Δ_i is tolerated by $\bigcup_{j=i}^{m} \Delta_j$. We call m the *rank* of the tolerance partition. The *Z-partition* of Δ is the unique tolerance partition of Δ where the rank is minimal.

Proposition 1. *(Goldszmidt and Pearl 1991) A knowledge base Δ is consistent iff it has a tolerance partition.*

Consequently, $A \mathrel{|\!\sim}_{\Delta} B$ holds iff Algorithm 1 applied to $\Delta \cup \{(\overline{B}|A)\}$ returns no Z-partition, i.e., iff it returns an empty list. Algorithm 1 takes $\mathcal{O}(|\Delta|^2)$-many SAT tests.

Focused Inference

When drawing inferences in practice, human reasoners typically do not take their whole knowledge into account but focus on that part of their knowledge which they rate as relevant for the query. Ignoring knowledge which is suspected of being irrelevant reduces complexity and allows humans to make snap decisions in a short period of time. On the downside, if one narrows the focus too much, then one probably misses information which turns out to be relevant in the end.

The vast majority of formal approaches to nonmonotonic reasoning does not provide the feature of focusing on a certain part of knowledge. While most approaches certainly aim on working out the relevant knowledge, they usually do not disregard irrelevant knowledge completely but give lower weight to it. This way of handling relevance is meaningful when drawing very well-considered inferences but results in rather costly calculations. One possibility of disregarding irrelevant knowledge completely is provided by the concept of *syntax splitting* (Parikh 1999). The idea behind syntax splitting is to partition the knowledge base Δ into syntactically independent subsets and to exploit only those subsets that are syntactically linked to the query when drawing inferences. Formally, $(\Delta_1, \ldots, \Delta_m)$ is a *syntax splitting* of a knowledge base Δ iff $\{\Delta_1, \ldots, \Delta_m\}$ is a partition of Δ and $\{\Sigma(\Delta_1), \ldots, \Sigma(\Delta_m)\}$ is a partition of $\Sigma(\Delta)$.

Example 1. *Consider $\Delta = \{r_1, \ldots, r_6\}$ from Table 1. $\{\Delta_{\mathsf{penguin}}, \Delta_{\mathsf{amphibian}}\}$ is a syntax splitting of Δ because $\{\Sigma(\Delta_{\mathsf{penguin}}), \Sigma(\Delta_{\mathsf{amphibian}})\} = \{\{b, f, p, w\}, \{a, s\}\}$ is a partition of $\Sigma(\Delta)$. Due to $\{p, f\} \cap \Sigma(\Delta_{\mathsf{amphibian}}) = \emptyset$, the knowledge in $\Delta_{\mathsf{amphibian}}$ should be irrelevant when asking for $?f \mathrel{|\!\sim} \bar{p}$, i.e. if flying individuals are usually not penguins. We later show that this is indeed the case in System P and the query $?f \mathrel{|\!\sim} \bar{p}$ can be answered solely based on $\Delta_{\mathsf{penguin}}$.*

Syntax splitting is a powerful tool when organizing knowledge bases which address several rather independent topics. However, in practice, knowledge bases are usually fully connected and syntax splitting becomes futile as the finest syntax splitting of a fully connected knowledge base Δ is $\{\Delta\}$.

[1] Z-partitions are named after System Z (Pearl 1990) as they can be used to establish System Z.

Conditional	Meaning and appearance
$r_1 = (b\|\top)$	An individual is usually a bird.
$r_2 = (f\|b)$	Birds are usually able to fly.
$r_3 = (w\|b)$	Birds usually have wings.
	$\Delta_{\mathsf{bird}} = \{r_1, r_2, r_3\}$
$r_4 = (b\|p)$	Penguins are usually birds.
$r_5 = (\overline{f}\|p)$	Penguins are usually not able to fly.
	$\Delta_{\mathsf{penguin}} = \{r_1, \dots, r_5\}$
$r_6 = (a\|s)$	Salamanders are usually amphibians.
	$\Delta_{\mathsf{amphibian}} = \{r_6\}$
$r_7 = (e\|b)$	Birds usually lay eggs.
$r_8 = (e\|a)$	Amphibians usually lay eggs.
	$\Delta_{\mathsf{oviparous}} = \{r_1, \dots, r_8\}$

Table 1: Conditionals that are used in the examples.

Example 2. *We recall the knowledge bases $\Delta_{\mathsf{penguin}}$ and $\Delta_{\mathsf{amphibian}}$ from Example 1. Adding the knowledge that both birds and amphibians usually lay eggs results in the knowledge bases $\Delta_{\mathsf{bird}} \cup \{r_7\}$ and $\Delta_{\mathsf{amphibian}} \cup \{r_8\}$ (cf. Table 1) which are no longer syntactically separated as they share the variable e, and performing syntax splitting on the whole knowledge base $\Delta = \Delta_{\mathsf{bird}} \cup \{r_7\} \cup \Delta_{\mathsf{amphibian}} \cup \{r_8\}$ does not create added value. Nevertheless, the knowledge about amphibians should still not be relevant for the query $?f \mid\!\sim \bar{p}$.*

Example 2 suggests that syntax splitting is too coarse in some cases. For this reason, we work out more precise foci on knowledge. In compliance with syntax splitting, we define foci on a syntactical level, i.e., a focus is a set of variables. This makes foci easy to compute.

Definition 2 (Direct Focus; Knowledge in Focus)**.** *Let Δ be a knowledge base, and let $\mathcal{A} \subseteq \mathcal{L}$. The* direct focus *on $\mathcal{A}$ is*

$$\mathcal{F}_0(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \Sigma(A),$$

and the knowledge in the direct focus *on $\mathcal{A}$ is*

$$\Delta_0(\mathcal{A}) = \{r \in \Delta \mid \Sigma(r) \cap \mathcal{F}_0(\mathcal{A}) \neq \emptyset\}.$$

According to Definition 2, the focus $\mathcal{F}_0(\mathcal{A})$ is the signature of the formulas in $\mathcal{A}$ and $\Delta_0(\mathcal{A})$ consists of those conditionals from Δ which share at least one variable with $\mathcal{A}$.

When drawing inferences we are interested in the knowledge that is linked to a query $?A \mid\!\sim B$ and, hence, we focus on $\mathcal{A} = \{A, B\}$. The focus $\mathcal{F}_0(A, B)$[2] solely consists of those variables that are mentioned in the query $?A \mid\!\sim B$ and therefore is the smallest reasonable focus for answering this query.

Example 3. *Consider $\Delta_{\mathsf{oviparous}}$ (cf. Table 1). The direct focus on the query $?f \mid\!\sim \bar{p}$ is $\mathcal{F}_0(f, \bar{p}) = \{f, p\}$ and the knowledge in this focus is $\Delta_0(f, \bar{p}) = \{(f|b), (b|p), (\overline{f}|p)\}$.*

The idea of knowledge in focus can be generalized by iteration: In the focus $\mathcal{F}_1$ not only conditionals are taken into account which are directly linked to the query but also conditionals that share variables with other conditionals which are directly linked to the query, and so on.

Definition 3 (Iterated Focus; Iterated Knowledge in Focus)**.** *Let Δ be a knowledge base, and let $\mathcal{A} \subseteq \mathcal{L}$. For $i \in \mathbb{N}$, the* i-th focus *on $\mathcal{A}$ is*

$$\mathcal{F}_i(\Delta, \mathcal{A}) = \mathcal{F}_0(\mathcal{A}) \cup \bigcup_{r \in \Delta_{i-1}(\mathcal{A})} \Sigma(r),$$

which depends on the $(i-1)$-th knowledge in focus on $\mathcal{A}$, namely $\Delta_{i-1}(\mathcal{A})$. The i-th knowledge in focus *on $\mathcal{A}$ is*

$$\Delta_i(\mathcal{A}) = \{r \in \Delta \mid \Sigma(r) \cap \mathcal{F}_i(\Delta, \mathcal{A}) \neq \emptyset\}.$$

We omit the argument Δ in $\mathcal{F}_i(\Delta, \mathcal{A})$ and write $\mathcal{F}_i(\mathcal{A})$ instead if the respective knowledge base is clear from the context.

Example 4. *We abbreviate $\Delta = \Delta_{\mathsf{oviparous}}$ (cf. Table 1). The first iterated foci on the query $?f \mid\!\sim \bar{p}$ and the knowledge in these foci are*

$$\begin{aligned} &\mathcal{F}_1(f, \bar{p}) = \{b, f, p\}, && \Delta_1(f, \bar{p}) = \{r_1, \dots, r_5, r_7\}, \\ &\mathcal{F}_2(f, \bar{p}) = \{b, e, f, p, w\}, && \Delta_2(f, \bar{p}) = \Delta_1(f, \bar{p}) \cup \{r8\}, \\ &\mathcal{F}_3(f, \bar{p}) = \Sigma \setminus \{s\}, && \Delta_3(f, \bar{p}) = \Delta_{\mathsf{oviparous}}, \\ &\mathcal{F}_4(f, \bar{p}) = \Sigma. && \end{aligned}$$

One has $\mathcal{F}_i(\mathcal{A}) \subseteq \mathcal{F}_{i+1}(\mathcal{A})$ and $\Delta_i(\mathcal{A}) \subseteq \Delta_{i+1}(\mathcal{A})$ for $i \in \mathbb{N}_0$ and $\mathcal{A} \subseteq \mathcal{L}$. Because every focus and knowledge in focus is bounded above by a finite set, namely by Σ resp. Δ, the series $(\mathcal{F}_i(\mathcal{A}))_{i \in \mathbb{N}_0}$ and $(\Delta_i(\mathcal{A}))_{i \in \mathbb{N}_0}$ become stationary after at most $\min\{|\Sigma|, |\Delta|\}$-many iterations. The limits are

$$\mathcal{F}_\infty(\mathcal{A}) = \bigcup_{i \in \mathbb{N}_0} \mathcal{F}_i(\mathcal{A}),$$

$$\Delta_\infty(\mathcal{A}) = \bigcup_{i \in \mathbb{N}_0} \Delta_i(\mathcal{A}).$$

Example 5. *Consider $\Delta_{\mathsf{oviparous}}$ and the query $?f \mid\!\sim \bar{p}$ (cf. Example 4). One has $\mathcal{F}_\infty(f, \bar{p}) = \mathcal{F}_4(f, \bar{p}) = \Sigma$ and $\Delta_\infty(f, \bar{p}) = \Delta_3(f, \bar{p}) = \Delta_{\mathsf{oviparous}}$.*

Note that the focus in the limit $\mathcal{F}_\infty$ and the knowledge in the focus in the limit Δ_∞ are not necessarily Σ resp. Δ but are the focus and the knowledge in focus which are obtained by syntax splitting.

Definition 4 (Focused Inference)**.** *Let Δ be a knowledge base. For $i \in \mathbb{N}_0^\infty$, we define the* focused inference relation

$$\mid\!\sim^i_\Delta \; = \{(A, B) \in \mathcal{L} \times \mathcal{L} \mid A \mid\!\sim_{\Delta_i(A,B)} B\}.$$

If $A \mid\!\sim^i_\Delta B$ holds, we say that B follows from A in the i-th focus*.*

Every query instance $?A \mid\!\sim^i_\Delta B$ can be answered based on an inference relation which satisfies System P, namely $\mid\!\sim_{\Delta_i(A,B)}$. However, this does not mean that $\mid\!\sim^i_\Delta$ satisfies System P because the focus of $\Delta_i(A, B)$ is query-dependent and the inference relation which is selected to answer the query can differ from query to query.

Example 6. *From $\Delta = \Delta_{\mathsf{oviparous}}$ (cf. Table 1) the inference $f \mid\!\sim^0_\Delta \bar{p}$ can be drawn because $\Delta' = \Delta_0(f, \bar{p}) \cup \{(p|f)\} =$*

[2] Note that we omit set braces if the set occurs as the only argument of a mapping in order to shorten expressions.

$\{(f|b),(b|p),(\bar{f}|p),(p|f)\}$ is inconsistent (no conditional in Δ' is tolerated by Δ'). Some further inferences that can be drawn from Δ by direct applications of inference rules from System P are $a \vee b \mathrel{|\!\sim}^0_\Delta e$, $bf \mathrel{|\!\sim}^0_\Delta w$, and $p \mathrel{|\!\sim}^0_\Delta b \vee \bar{f}$. An inference that cannot be drawn in the direct focus is $w \mathrel{|\!\sim}^0_\Delta f$ because $\Delta_0(f,w) \cup \{(\bar{f}|w)\} = \{(f|b),(w|b),(\bar{f}|p),(\bar{f}|w)\}$ is consistent (consider any ranking function with $\kappa^{-1}(0) = \{\bar{b}\bar{f}pw\}$ and $\kappa^{-1}(1) = \{bf\bar{p}w\}$). However, $w \mathrel{|\!\sim}^1_\Delta f$ holds because $\Delta_1(f,w) \cup \{(\bar{f}|w)\} = \Delta_{\mathsf{penguin}} \cup \{r_7\} \cup \{(\bar{f}|w)\}$ is inconsistent: In the most plausible worlds b holds (due to r_1) and then f (due to r_2) and w (due to r_3) hold as well.

From an intuitive point of view, the inference behavior of $\mathrel{|\!\sim}^i_\Delta$ should become "better" with increasing i, i.e., the larger the focus is and, hence, the more knowledge is taken into account. In the next section, we investigate this conjecture based on formal criteria.

Formal Properties of Focused Inference

We investigate the quality of the focused inference relations $\mathrel{|\!\sim}^i_\Delta$ for $i \in \mathbb{N}_0^\infty$ with regard to System P. We begin with the inference relation $\mathrel{|\!\sim}^\infty_\Delta$. Before we show that $\mathrel{|\!\sim}^\infty_\Delta$ equals $\mathrel{|\!\sim}_\Delta$ and, hence, satisfies System P, we prove some technical lemmas.

Lemma 1. *Let Δ_1 and Δ_2 be knowledge bases, and let $A, B \in \mathcal{L}$. Then, $A \mathrel{|\!\sim}_{\Delta_1} B$ and $\Delta_1 \subseteq \Delta_2$ imply $A \mathrel{|\!\sim}_{\Delta_2} B$. This property is known as the* semi-monotony *of $\mathrel{|\!\sim}_\Delta$.*

Proof. From $A \mathrel{|\!\sim}_{\Delta_1} B$ it follows that $\Delta_1 \cup \{(\overline{B}|A)\}$ is inconsistent. As every superset of an inconsistent knowledge base is inconsistent and $\Delta_1 \cup \{(\overline{B}|A)\} \subseteq \Delta_2 \cup \{(\overline{B}|A)\}$ holds, $\Delta_2 \cup \{(\overline{B}|A)\}$ is inconsistent, too. Consequently, it follows that $A \mathrel{|\!\sim}_{\Delta_2} B$ is true as well. □

We now show that the models of syntactically independent knowledge bases Δ_1 and Δ_2 define a model of $\Delta_1 \cup \Delta_2$.

Lemma 2. *Let Δ be a consistent knowledge base which syntactically splits into $\{\Delta_1, \Delta_2\}$, and let $\kappa_i : \Omega(\Sigma(\Delta_i)) \to \mathbb{N}_0$ be a model of Δ_i for $i = 1, 2$. Then, the sum of both, i.e. $(\kappa_1 + \kappa_2)(\omega_1\omega_2) = \kappa_1(\omega_1) + \kappa_2(\omega_2)$, is a model of Δ.*

Proof. Let A be a formula which is defined over Σ_1. Then,

$$\begin{aligned}(\kappa_1+\kappa_2)(A) &= \min_{\substack{\omega_1\omega_2 \models A \\ \omega_1 \in \Omega(\Sigma(\Delta_1)),\, \omega_2 \in \Omega(\Sigma(\Delta_2))}} (\kappa_1+\kappa_2)(\omega_1\omega_2) \\ &= \min_{\substack{\omega_1 \models A \\ \omega_1 \in \Omega(\Sigma(\Delta_1)),\, \omega_2 \in \Omega(\Sigma(\Delta_2))}} \Big(\kappa_1(\omega_1) + \kappa_2(\omega_2)\Big) \\ &= \min_{\omega_1 \models A,\, \omega_1 \in \Omega(\Sigma(\Delta_1))} \kappa_1(\omega_1) = \kappa_1(A).\end{aligned}$$

The third equality holds because κ_2 is normalized and the minimum is built over all possible worlds in $\Omega(\Sigma(\Delta_2))$. As a consequence, one has $(\kappa_1+\kappa_2)(AB) = \kappa_1(AB) < \kappa_1(A\overline{B}) = (\kappa_1+\kappa_2)(A\overline{B})$ for all conditionals $(B|A) \in \Delta_1$. The proof for conditionals in Δ_2 is analogous. □

We eventually prove $\mathrel{|\!\sim}^\infty_\Delta = \mathrel{|\!\sim}_\Delta$.

Proposition 2. *Let Δ be a consistent knowledge base. Then, $\mathrel{|\!\sim}^\infty_\Delta = \mathrel{|\!\sim}_\Delta$. In particular, $\mathrel{|\!\sim}^\infty_\Delta$ satisfies System P.*

Proof. From $A \mathrel{|\!\sim}^\infty_\Delta B$ it follows that $A \mathrel{|\!\sim}_{\Delta_\infty(A,B)} B$ holds. With $\Delta_\infty(A,B) \subseteq \Delta$ and Lemma 1, $A \mathrel{|\!\sim}_\Delta B$ follows. Now, let $A \mathrel{|\!\sim}_\Delta B$. We define $\Delta' = \Delta \setminus \Delta_\infty(A,B)$ such that Δ syntactically splits into $\{\Delta', \Delta_\infty(A,B)\}$ (otherwise $\Delta_\infty(A,B)$ would not be the limit of $(\Delta_i(A,B))_{i\in\mathbb{N}_0}$). Let κ be a model of $\Delta_\infty(A,B)$ and let κ' be a model of Δ'. Lemma 2 states that $\kappa + \kappa'$ is a model of Δ. Hence, $A \mathrel{|\!\sim}_\Delta B$ implies $(\kappa+\kappa')(AB) < (\kappa+\kappa')(A\overline{B})$. Because of $\Sigma(A \mathrel{|\!\sim}_\Delta B) \subseteq \Sigma(\Delta_\infty(A,B))$, one has $\kappa(AB) = (\kappa+\kappa')(AB) < (\kappa+\kappa')(A\overline{B}) = \kappa(A\overline{B})$ (cf. the proof of Lemma 2), i.e., $\kappa \models (B|A)$. As this result holds for arbitrary models κ of $\Delta_\infty(A,B)$, both $A \mathrel{|\!\sim}_{\Delta_\infty(A,B)} B$ and $A \mathrel{|\!\sim}^\infty_\Delta B$ follow. □

Now, we investigate the inference relations $\mathrel{|\!\sim}^i_\Delta$ with $i < \infty$. Example 6 gives reason to expect that these inference relations do not satisfy System P. Actually, for all knowledge bases Δ, there is a minimal $i \in \mathbb{N}_0$ such that $\mathrel{|\!\sim}^j_\Delta$ satisfies System P for all $j \geq i$, because there is $k \in \mathbb{N}_0$ with $\mathrel{|\!\sim}^k_\Delta = \mathrel{|\!\sim}^\infty_\Delta$. However, this index i depends on the knowledge base Δ. Nevertheless, some inference rules of System P hold in all foci.

Proposition 3. *Let Δ be a consistent knowledge base and let $i \in \mathbb{N}_0$. The inference relation $\mathrel{|\!\sim}^i_\Delta$ satisfies Reflexivity, Cautious Monotony, and Or.*

Proof. Reflexivity: If $A \equiv \bot$, then all ranking functions κ satisfy $\kappa \models (A|A)$ by definition and $A \mathrel{|\!\sim}^i_\Delta A$ holds. If $A \not\equiv \bot$, then $\kappa(AA) = \kappa(A) < \infty = \kappa(\bot) = \kappa(A\overline{A})$ for all ranking functions κ and, again, $A \mathrel{|\!\sim}^i_\Delta A$ holds.

Cautious Monotony (CM): $A \mathrel{|\!\sim}^i_\Delta B$ (resp. $A \mathrel{|\!\sim}^i_\Delta C$) is equivalent to $A \mathrel{|\!\sim}_{\Delta_i(A,B)} B$ (resp. $A \mathrel{|\!\sim}_{\Delta_i(A,C)} C$). With Lemma 1, $A \mathrel{|\!\sim}_{\Delta_i(A,B,C)} B$ (resp. $A \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$) follows. Because $\mathrel{|\!\sim}_{\Delta_i(A,B,C)}$ satisfies System P, in particular CM, $AB \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$ and, hence, $AB \mathrel{|\!\sim}^i_\Delta C$ hold.

Or: $A \mathrel{|\!\sim}^i_\Delta C$ (resp. $B \mathrel{|\!\sim}^i_\Delta C$) is equivalent to $A \mathrel{|\!\sim}_{\Delta_i(A,C)} C$ (resp. $B \mathrel{|\!\sim}_{\Delta_i(B,C)} C$). With Lemma 1, $A \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$ (resp. $B \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$) follows. Because $\mathrel{|\!\sim}_{\Delta_i(A,B,C)}$ satisfies System P, in particular Or, $A \vee B \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$ and, hence, $A \vee B \mathrel{|\!\sim}^i_\Delta C$ hold. □

Proposition 4. *There is no $i \in \mathbb{N}_0$ such that for all consistent knowledge bases Δ, the inference relation $\mathrel{|\!\sim}^i_\Delta$ satisfies Cut, Right Weakening, or Left Logical Equivalence.*

Proof. We give counterexamples for the case $i = 0$ first and generalize them to $i > 0$ afterwards.

Cut: Consider $\Delta = \{(c|ab),(a|\top),(d|a),(b|d)\}$. Then, $ab \mathrel{|\!\sim}^0_\Delta c$ and $a \mathrel{|\!\sim}^0_\Delta b$ hold, but $a \mathrel{|\!\sim}^0_\Delta c$ does not hold. The inference $a \mathrel{|\!\sim}^0_\Delta b$ holds because in all models of Δ the most plausible worlds satisfy a (due to $(a|\top) \in \Delta$) and consequently d (due to $(d|a) \in \Delta$) as well as b (due to $(b|d) \in \Delta$). The inference $a \mathrel{|\!\sim}^0_\Delta c$ does not hold because $(b|d) \notin \Delta_0(a,c)$ implies that b cannot be inferred from a. For this reason, $\Delta_0(a,c) \cup \{(\bar{c}|a)\}$ is consistent (consider any ranking function κ with $\kappa^{-1}(0) = \{a\bar{b}\bar{c}d\}$ and $\kappa^{-1}(1) = \{abcd\}$). This counterexample generalizes to arbitrary $i \in \mathbb{N}$ by separating the conclusion b from the premise a through a transitive chain of conditionals: For

$$\Delta^i = \{(c|ab), (a|\top), (d_1|a), (d_2|d_1), \ldots, (d_{i+1}|d_i), (b|d_{i+1})\},$$

$(b|d_{i+1}) \notin \Delta_i^i(a,c)$ rules out the inference of b from a and, hence, of c from a.

Right Weakening: Let $\Delta = \{(a|\top), (b|a), (d|b), (bc|d)\}$. Then, $a \mathrel{|\!\sim}_\Delta^0 bc$ and $bc \models c$ hold, but $a \mathrel{|\!\sim}_\Delta^0 c$ does not hold. In the most plausible worlds, a holds (due to $(a|\top) \in \Delta$) as well as b (due to $(b|a) \in \Delta$) and d (due to $(d|b) \in \Delta$). Consequently, c follows from $bc \models c$ and $(bc|d) \in \Delta$. The inference $a \mathrel{|\!\sim}_\Delta^0 c$ does not hold because $(d|b) \notin \Delta_0(a,c)$. Hence, c cannot be inferred from a. This counterexample generalizes to arbitrary $i \in \mathbb{N}$:

$$\Delta^i = \{(a|\top), (b|a), (d_1|b), (d_2|d_1), \ldots, (d_{2i+1}|d_{2i}), (bc|d_{2i+1}\}$$

proves that $\mathrel{|\!\sim}_\Delta^i$ does not satisfy Right Weakening because $(d_{i+1}|d_i) \notin \Delta_i^i(a,c)$.

Left Logical Equivalence: Let $\Delta = \{(c|d), (d|a \vee \bar{a})\}$. Then, $a \vee \bar{a} \mathrel{|\!\sim}_\Delta^0 c$ and $a \vee \bar{a} \equiv \top$ hold, but $\top \mathrel{|\!\sim}_\Delta^0 c$ does not hold which is because of $(d|a \vee \bar{a}) \notin \Delta_0(c) = \{(c|d)\}$. This counterexample generalizes to arbitrary $i \in \mathbb{N}$:

$$\Delta^i = \{(c|d_1), (d_2|d_1), \ldots, (d_{i+1}|d_i), (d_{i+1}|a \vee \bar{a})\}$$

proves that $\mathrel{|\!\sim}_\Delta^i$ does not satisfy Left Logical Equivalence because $(d_{i+1}|a \vee \bar{a}) \notin \Delta_i^i(c)$. □

The inference rules Cut, Right Weakening, and Left Logical Equivalence aim on inferring a proposition C from a proposition A by making use of a "justification" B. If this justification is out of the focus $\mathcal{F}_i(A,C)$, then the inference $A \mathrel{|\!\sim}_\Delta^i C$ can (possibly) not be drawn. This is in compliance with human reasoning: The less present the justification for an inference is, the more likely it is to dismiss the inference. If the justification B is within the focus $\mathcal{F}_i(A,C)$, however, the inference $A \mathrel{|\!\sim}_\Delta^i C$ should be drawn. In compliance with this consideration, the following weaker versions of Cut, Right Weakening, and Left Logical Equivalence apply.

Proposition 5. *Let Δ be a consistent knowledge base, let $i \in \mathbb{N}_0$, and let $A, B, C \in \mathcal{L}$. If $\Sigma(B) \subseteq \mathcal{F}_i(A,C)$, then*

$$AB \mathrel{|\!\sim}_\Delta^i C,\ A \mathrel{|\!\sim}_\Delta^i B \quad \textit{imply} \quad A \mathrel{|\!\sim}_\Delta^i C, \qquad \text{(Focused Cut)}$$

$$A \mathrel{|\!\sim}_\Delta^i B,\ B \models C \quad \textit{imply} \quad A \mathrel{|\!\sim}_\Delta^i C, \qquad \text{(Focused RW)}$$

$$A \equiv B,\ B \mathrel{|\!\sim}_\Delta^i C \quad \textit{imply} \quad A \mathrel{|\!\sim}_\Delta^i C. \qquad \text{(Focused LLE)}$$

Proof. Focused Cut: From $AB \mathrel{|\!\sim}_\Delta^i C$ and $A \mathrel{|\!\sim}_\Delta^i B$ it follows that $AB \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$ and $A \mathrel{|\!\sim}_{\Delta_i(A,B,C)} B$ hold. Since $\mathrel{|\!\sim}_{\Delta_i(A,B,C)}$ satisfies System P, $A \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$ follows. Because of $\Sigma(B) \subseteq \mathcal{F}_i(A,C)$ one has $\Delta_i(A,C) = \Delta_i(A,B,C)$. Therefore, $A \mathrel{|\!\sim}_{f_i(\Delta)} C$ holds.

The proofs of Focused RW and Focused LLE are analogous to the proof of Focused Cut: From $A \mathrel{|\!\sim}_\Delta^i B$ and $B \models C$ (resp. $A \equiv B$ and $B \mathrel{|\!\sim}_\Delta^i C$), $A \mathrel{|\!\sim}_{\Delta_i(A,B,C)} B$ (resp. $B \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$) follows. In both cases $A \mathrel{|\!\sim}_{\Delta_i(A,B,C)} C$ follows with the same argument as in the proof of Focused Cut. Consequently, $A \mathrel{|\!\sim}_{f_i(\Delta)} C$ holds. □

Concerning Right Weakening we would like to point out that this inference rule is not entirely unquestioned (cf., e.g., (Casini, Meyer, and Varzinczak 2019)). Like Cut, it constitutes a weakened form of transitivity which is a typical property of monotonic reasoning. Focused RW and Focused LLE restrict transitivity more than Right Weakening and Left Logical Equivalence as they crop transitive chainings when they get out of focus (cf. the proof of Proposition 4).

Left Logical Equivalence can also be guaranteed by restricting the language that is used to describe knowledge. The reason why Left Logical Equivalence is not satisfied by focused inference in general is that foci are defined on a syntactical level and it is possible that a formula A is within a focus $\mathcal{F}$ while an equivalent formula B is not. This, however, only happens when at least one of the formulas uses redundant variables. For instance, consider $A = a$ and $B = a(b \vee \bar{b})$. Then, $A \equiv B$ holds and B is considered in all foci $\mathcal{F}$ with $a \in \mathcal{F}$ or $b \in \mathcal{F}$ but A is considered only in foci $\mathcal{F}$ with $a \in \mathcal{F}$. We avoid this problem by introducing a language $\mathcal{L}'$ which is free of such 'syntactical sugar'.

Definition 5 (Propositional Language $\mathcal{L}'$)**.** *The propositional language $\mathcal{L}' \subseteq \mathcal{L}$ is defined by*

$$A \in \mathcal{L}' \quad \textit{iff} \quad A \in \mathcal{L} \textit{ and } \forall B \in \mathcal{L}: B \equiv A \Rightarrow \Sigma(A) \subseteq \Sigma(B).$$

Formulas in $\mathcal{L}'$ mention only those variables that are relevant for the evaluation of the formula. For instance, with respect to the abovementioned example, $A = a$ is in $\mathcal{L}'$ but $B = a(b \vee \bar{b})$ is not.

Proposition 6. *Let Δ be a consistent knowledge base, and let $i \in \mathbb{N}_0$. If $B \in \mathcal{L}'$, then*

$$A \equiv B,\ B \mathrel{|\!\sim}_\Delta^i C \quad \textit{imply} \quad A \mathrel{|\!\sim}_\Delta^i C. \qquad \text{(Reduced LLE)}$$

Proof. $B \in \mathcal{L}'$ and $A \equiv B$ imply $\Sigma(B) \subseteq \Sigma(A)$ and, hence, $\Sigma(B) \cup \Sigma(C) \subseteq \Sigma(A) \cup \Sigma(C)$ holds. Consequently, $\Delta_i(B,C) \subseteq \Delta_i(A,C)$ holds, too. Because of $A \equiv B$ and $B \mathrel{|\!\sim}_\Delta^i C$ one has $B \mathrel{|\!\sim}_{\Delta_i(B,C)} C$ and, as $\mathrel{|\!\sim}_{\Delta_i(B,C)}$ satisfies System P and in particular Left Logical Equivalence, $A \mathrel{|\!\sim}_{\Delta_i(B,C)} C$. Finally, since $\mathrel{|\!\sim}_\Delta$ is semi-monotonous, $A \mathrel{|\!\sim}_{\Delta_i(A,C)} C$ and $A \mathrel{|\!\sim}_\Delta^i C$ follow. □

So far we have learned that for all knowledge bases Δ there is a minimal index $i \in \mathbb{N}_0$ with $i < \min\{|\Sigma|, |\Delta|\}$ such that $\mathrel{|\!\sim}_\Delta^i$ satisfies System P. Even if one does not want to miss any relevant knowledge for answering the query $?A \mathrel{|\!\sim}_\Delta B$, it is sufficient to focus on $\Delta_i(A,B)$ instead of considering Δ because $\mathrel{|\!\sim}_\Delta = \mathrel{|\!\sim}_\Delta^i$ holds for this index i. The next proposition states that even if the focus $\mathcal{F}_j$ is too small to draw inferences compliant with System P, i.e., $\mathcal{F}_j \subset \mathcal{F}_i$, the focused inference relation $\mathrel{|\!\sim}_\Delta^j$ does not provide "false positives".

Proposition 7. *Let Δ be a knowledge base and let $i, j \in \mathbb{N}_0$. Then, $j \leq i$ implies $\mathrel{|\!\sim}_\Delta^j \subseteq \mathrel{|\!\sim}_\Delta^i$.*

Input: Z-partition $(\Delta'_1, \ldots, \Delta'_m)$ of a consistent knowledge base Δ'; knowledge base Δ''
Output: Z-partition of $\Delta = \Delta' \cup \Delta''$ if Δ is consistent; empty list otherwise

1 output $\leftarrow [\,]$
2 **for** $i = 0, \ldots, m$
3   **while** $\Delta'_i \neq \emptyset$
4         $\Delta'_{\_} \leftarrow \emptyset$; $\Delta''_{\_} \leftarrow \emptyset$; boolean $\leftarrow$ `false`
5   **for** $r \in \Delta''$
6   **if** $\Delta'' \cup \bigcup_{j=i}^{m} \Delta'_j$ tolerates r
7                 $\Delta'_{\_} \leftarrow \Delta'_{\_} \cup \{r\}$; boolean $\leftarrow$ `true`
8   **for** $r \in \Delta'_i$
9   **if** $\Delta'' \cup \bigcup_{j=i}^{m} \Delta'_j$ tolerates r
10                 $\Delta''_{\_} \leftarrow \Delta''_{\_} \cup \{r\}$; boolean $\leftarrow$ `true`
11   **if** boolean $=$ `false`
12   output $\leftarrow [\,]$
13   **break**
14   output $\leftarrow$ output.append($\Delta'_{\_} \cup \Delta''_{\_}$)
15         $\Delta'_i \leftarrow \Delta'_i \setminus \Delta'_{\_}$; $\Delta'' \leftarrow \Delta' \setminus \Delta''_{\_}$
16   **if** $\Delta'' = \emptyset$
17   output $\leftarrow$ output.extend($[\Delta'_{i+1}, \ldots, \Delta'_m]$)
18   **break**
19 **if** $\Delta'' \neq \emptyset$
20   output.extend(Algorithm_1(Δ''))
21 **return** output

Algorithm 2: Consistency test for a knowledge base Δ which exploits the Z-partition of a consistent subset $\Delta' \subseteq \Delta$ and returns the Z-partition of Δ.

Proof. This proposition directly follows from $\Delta_j(A, B) \subseteq \Delta_i(A, B)$ for $j \leq i$ and the semi-monotony of $\mathrel{|\!\!\sim}_\Delta$. $\square$

As a consequence of Proposition 7, the sequence $(\mathrel{|\!\!\sim}^i_\Delta)_{i \in \mathbb{N}_0}$ forms an ascending chain ($\mathrel{|\!\!\sim}^0_\Delta \subseteq \mathrel{|\!\!\sim}^1_\Delta \subseteq \ldots$) and converges to $\mathrel{|\!\!\sim}^\infty_\Delta$. In the next section, we make use of this result and present an algorithm for drawing inferences in System P based on focused inference. For this, we test whether a query holds wrt. $\mathrel{|\!\!\sim}^0_\Delta$. If the answer is `false`, we iteratively enlarge the focus until we reach $\mathrel{|\!\!\sim}^\infty_\Delta$. If the inference can still not be drawn wrt. $\mathrel{|\!\!\sim}^\infty_\Delta$, then the query is `false` wrt. System P. In all other cases the query is `true`.

An Algorithmic View on Focused Inference

The fact that the focused inference relations $\mathrel{|\!\!\sim}^i_\Delta$ for $i \in \mathbb{N}_0$ form an ascending chain which converges against the System-P-conform inference relation $\mathrel{|\!\!\sim}_\Delta$ makes it possible to iteratively approximate System P with increasing accuracy: In order to answer a query wrt. $\mathrel{|\!\!\sim}_\Delta$ one tests if this query holds wrt. $\mathrel{|\!\!\sim}^0_\Delta$. If this is the case, the query also holds wrt. $\mathrel{|\!\!\sim}_\Delta$ due to the semi-monotony of $\mathrel{|\!\!\sim}_\Delta$. Otherwise, one tests the query wrt. $\mathrel{|\!\!\sim}^1_\Delta$ and so on. For this procedure to be efficient, it is necessary to reuse the calculations for answering the query instance $?A \mathrel{|\!\!\sim}^i_\Delta B$ when $?A \mathrel{|\!\!\sim}^{i+1}_\Delta B$ has to be tested. In other words, one has to take advantage of the consistency of $\Delta_i(A, B) \cup \{(\overline{B}|A)\}$ when deciding consistency of $\Delta_{i+1}(A, B) \cup \{(\overline{B}|A)\}$. More general, an efficient algorithm is needed which decides consistency of a knowledge base Δ provided that a subset Δ' of Δ is already known to be consistent. We develop such an algorithm in this section step by step. For that purpose the Z-partition of Δ' is a useful auxiliary structure: The idea is to sort the conditionals from $\Delta \setminus \Delta'$ into the Z-partition $(\Delta'_1, \ldots, \Delta'_m)$ of Δ'. During this process it can happen that (a) conditionals from $\Delta \setminus \Delta'$ can simply be incorporated into a partition set Δ'_i, (b) Δ'_i has to be fanned out into several smaller subsets before applying (a), or (c) conditionals from $\Delta \setminus \Delta'$ constitute a completely novel partition set. What can *not* happen, however, is that two conditionals r_1 and r_2 from Δ' switch their order in the Z-partition, i.e., r_1 occurs in a set with a lower index than r_2 within the partition of Δ' but in a set with a higher index within the partition of Δ (cf. the notion of tolerance). The whole procedure is implemented in Algorithm 2.

Proposition 8. *Algorithm 2 terminates and is correct. Its worst-case time-complexity is in*

$$\mathcal{O}\left(\left(\frac{|\Delta'|^2}{m} + |\Delta'| \cdot |\Delta''| + |\Delta''|^2\right) \cdot \mathsf{SAT}(|\Sigma|)\right)$$

where $\mathsf{SAT}(|\Sigma|)$ *is the worst-case time-complexity of testing satisfiability of a formula with* $|\Sigma|$*-many variables.*

Proof (Sketch). Termination: The for-loops in Algorithm 2 are obviously finite. In every pass of the while-loop, one out of these three events happens: (a) An element is removed from Δ'', (b) an element is removed from Δ'_i, or (c) the loop aborts because the variable *boolean* is `false`. Since Δ'' and Δ'_i are finite sets, the termination condition $\Delta'_i = \emptyset$ eventuates after finitely many passes or the loop aborts due to case (c). Hence, the while-loop is finite, too.

Correctness: Algorithm 2 sorts the conditionals from Δ'' in the Z-partition $(\Delta'_1, \ldots, \Delta'_m)$ starting from Δ'_1 up to Δ'_m while fanning out the partition sets Δ'_i if necessary, such that each conditional in a new partition set Δ_i is tolerated by Δ_i, the remaining conditionals from Δ'_i as well as the conditionals from Δ'_j, $j > i$ (the tolerance tests are performed in the lines 6 and 9). When all conditionals from Δ'' are sorted in before the for-loop in line 2 ends, the partition sets from $(\Delta'_1, \ldots, \Delta'_m)$ which remained untouched are appended (line 17). When the for-loop ends before all conditionals from Δ'' are sorted in instead, the Z-partition of the remaining conditionals from Δ'' is appended (line 20). The outcome is a tolerance partition of Δ. It remains to show that this tolerance partition has minimal rank. This holds because every conditional from Δ'' is sorted in as early as possible and no conditional from Δ'_j can be tolerated before a conditional from Δ'_i when $j > i$ as $(\Delta'_1, \ldots, \Delta'_m)$ forms a tolerance partition and, consequently, all conditionals from Δ'_j have to appear in a higher partition of the Z-partition of Δ than conditionals from Δ'_i (if a conditional is not tolerated by a knowledge base Δ, then it is also not tolerated by any superset of Δ).

Input: Knowledge base Δ; query $?A \mathrel{|\!\sim}_{\Delta} B$
Output: `true` if $A \mathrel{|\!\sim}_{\Delta} B$ holds; `false` otherwise

output $\leftarrow$ `false`
$i \leftarrow 0$
$T \leftarrow$ Algorithm 1$(\Delta_i(A,B) \cup \{(\overline{B}|A)\})$
if $T = [\,]$ **then** output $\leftarrow$ `true`
while $\Delta_{i+1}(A,B) \neq \Delta_i(A,B) \wedge$ output $=$ `false`
    $T \leftarrow$ Algorithm_2$(T, \Delta_{i+1}(A,B) \setminus \Delta_i(A,B))$
    if $T = [\,]$ **then** output $\leftarrow$ `true`
    $i \leftarrow i+1$
return output

Algorithm 3: Iterative query answering in System P.

Complexity: In the worst case, the algorithm takes

$$\begin{aligned}&\mathcal{O}(\textstyle\sum_{i=0}^{m}(\sum_{k=1}^{|\Delta_i'|}(k+|\Delta''|))+|\Delta''|^2)\\ =\;&\mathcal{O}(\textstyle\sum_{i=0}^{m}(|\Delta_i'|^2+|\Delta_i'|\cdot|\Delta''|)+|\Delta''|^2)\\ =\;&\mathcal{O}(\textstyle\sum_{i=0}^{m}(\frac{|\Delta'|^2}{m^2}+\frac{|\Delta'|}{m}\cdot|\Delta''|)+|\Delta''|^2)\\ =\;&\mathcal{O}(\textstyle\frac{|\Delta'|^2}{m}+|\Delta'|\cdot|\Delta''|+|\Delta''|^2)\end{aligned}$$

many SAT tests. For obtaining the last equality, the method of Lagrangian multipliers (Boyd and Vandenberghe 2014) with the side condition $\sum_{i=0}^{m}|\Delta_i'| = |\Delta'|$ is applied. $\square$

The complexity of Algorithm 2 is the better the higher the rank m of the Z-partition of Δ' is. In the worst case, the Z-partition of Δ' is (Δ') and does not provide any helpful information. In this case, the complexity of Algorithm 2 is the same as the complexity of a direct computation of the Z-partition of $\Delta = \Delta' \cup \Delta''$ with Algorithm 1.

Eventually, Algorithm 3 recursively applies Algorithm 2 on Δ_i and the Z-partition of Δ_{i-1} for increasing index i in order to answer the query instance $?A \mathrel{|\!\sim}_{\Delta} B$.

Proposition 9. *Algorithm 3 terminates and is correct. Its worst-case time-complexity is in*

$$\mathcal{O}\left(\left(1+\frac{k}{m_k}\right)\cdot|\Delta_k(A,B)|^2\cdot\mathsf{SAT}(|\Sigma(\Delta_k(A,B))|)\right),$$

where k is such that $\mathcal{F}_k(A,B)$ is the smallest focus in which $?A \mathrel{|\!\sim}_{\Delta} B$ can be decided and m_k is the rank of the Z-partition of $\Delta_k(A,B)$.

Proof (Sketch). Termination: Algorithm 3 terminates because its only while-loop is finite; $(\Delta_i)_{i\in\mathbb{N}_0}$ converges after at most $\min\{|\Sigma|,|\Delta|\}$-many steps.

Correctness: Algorithm 3 recursively tests whether $A \mathrel{|\!\sim}^{i}_{\Delta} B$ holds for increasing $i \in \mathbb{N}_0$. If $A \mathrel{|\!\sim}^{i}_{\Delta} B$ holds, then $A \mathrel{|\!\sim}_{\Delta} B$ holds, too, and the algorithm returns `true`. If $A \mathrel{|\!\sim}^{i}_{\Delta} B$ does not hold, the index i is increased and the algorithm repeats. When $(\mathrel{|\!\sim}^{i}_{\Delta})_{i\in\mathbb{N}_0}$ becomes stationary at index j and $A \mathrel{|\!\sim}^{j}_{\Delta} B$ does not hold, then $A \mathrel{|\!\sim}_{\Delta} B$ does not hold either and the algorithm correctly returns `false`.

Complexity: For $i \in \mathbb{N}_0$, we denote $|\Delta_i(A,B)|$ with d_i and the rank of the Z-partition of $\Delta_i(A,B)$ with m_i. Then, Algorithm 3 takes

$$\begin{aligned}&\mathcal{O}(d_0^2+\textstyle\sum_{i=1}^{k}(\frac{d_{i-1}^2}{m_{i-1}}+(d_{i-1}+d_i-d_{i-1})\cdot(d_i-d_{i-1})))\\ =\;&\mathcal{O}(d_0^2+\textstyle\sum_{i=1}^{k}(\frac{d_{i-1}^2}{m_{i-1}}+d_i\cdot(d_i-d_{i-1})))\\ \leq\;&\mathcal{O}(d_0^2+\textstyle\frac{k}{m_{k-1}}\cdot d_{k-1}^2+d_k\cdot\sum_{i=1}^{k}(d_i-d_{i-1}))\\ =\;&\mathcal{O}(d_0^2+\textstyle\frac{k}{m_{k-1}}\cdot d_{k-1}^2+d_k\cdot(d_k-d_0))\\ =\;&\mathcal{O}((1+\textstyle\frac{k}{m_k})\cdot d_k^2)\end{aligned}$$

many SAT tests. $\square$

In the worst case, $?A \mathrel{|\!\sim}_{\Delta} B$ can be answered with Algorithm 3 not before the first focus $\mathcal{F}_k$ with $\Delta_k(A,B) = \Delta$ is reached. Additionally, $k = |\Delta|$ and $m_k = 1$ could hold. In this case, the complexity of Algorithm 3 is in $\Theta(|\Delta(A,B)|^3\cdot\mathsf{SAT}(|\Sigma|))$ and worse than the complexity of a direct computation with Algorithm 1. In practical applications, however, $k \ll |\Delta|$ and $m_k > 1$ can be expected. If $A \mathrel{|\!\sim}_{\Delta} B$ does not hold, then Algorithm 3 proceeds until the first focus $\mathcal{F}_k$ with $\Delta_k(A,B) = \Delta_\infty(A,B)$ is reached because focused inference is *semi-decidable*: If $A \mathrel{|\!\sim}^{i}_{\Delta} B$ holds, then $A \mathrel{|\!\sim}_{\Delta} B$ holds as well, but if $i < k$ and $A \mathrel{|\!\sim}^{i}_{\Delta} B$ does not hold, nothing can be said about $A \mathrel{|\!\sim}_{\Delta} B$. This weakness is attenuated by testing $?A \mathrel{|\!\sim}^{i}_{\Delta} B$ and $?A \mathrel{|\!\sim}^{i}_{\Delta} \overline{B}$ in parallel and by aborting the procedure as soon as one of the query instances turns out to be true.

In the best case, the inference $A \mathrel{|\!\sim}_{\Delta} B$ holds and can be drawn in the direct focus. Then, the complexity of Algorithm 3 is in $\mathcal{O}(|\Delta_0(A,B)|^2\cdot\mathsf{SAT}(|\Sigma(\Delta_0(A,B)|)$. This result is promising not only because $|\Delta_0(A,B)| \ll |\Delta|$ holds in many cases and improves complexity directly but also because $|\Sigma(\Delta_0(A,B)| \ll |\Sigma|$ speeds up the SAT tests in Algorithm 3 in comparison to those in Algorithm 1.

Conclusion and Future Work

We analyzed the effect of focusing on knowledge when drawing inferences in System P. For this, we hierarchically organized the knowledge according to its syntactical distance to the query and defined focused inference relations which solely use the knowledge that is within reach of the query (within "focus") to draw the inference. All focused inferences are in compliance with System P but not all System P inferences are among the focused ones if the focus is not broad enough. Our focused inference approach benefits from the semi-monotony of System P but is not limited to System P.

In future work, we want to extend our approach to more expressive background logics, formalize a semantical variant of focused inference, and we want to apply focused inference to other inference formalisms which satisfy semi-monotony (e.g., Reiter's default logic for normal defaults (Reiter 1980)) and, more challenging, to formalisms which are not semi-monotonous (e.g., System Z (Pearl 1990) and c-representations (Kern-Isberner 2004)). We also see connections to other research topics such as forgetting (focused inference as a tunnel view) and paraconsistent logics (focus on a consistent part of the knowledge).

References

Adams, E. W. 1975. *The Logic of Conditionals*. Springer.

Boyd, S. P.; and Vandenberghe, L. 2014. *Convex Optimization*. Cambridge University Press.

Casini, G.; Meyer, T.; and Varzinczak, I. 2019. Simple Conditionals with Constrained Right Weakening. In Kraus, S., ed., *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2019, Macao, China, August 10-16, 2019*, 1632–1638. ijcai.org.

Chechetka, A.; and Guestrin, C. 2010. Focused Belief Propagation for Query-Specific Inference. In Teh, Y. W.; and Titterington, D. M., eds., *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics, AISTATS 2010, Chia Laguna Resort, Sardinia, Italy, May 13-15, 2010*, volume 9 of *JMLR Proceedings*, 89–96. JMLR.org.

Goldszmidt, M.; and Pearl, J. 1991. On the Consistency of Defeasible Databases. *Artif. Intell.* 52(2): 121–149.

Kern-Isberner, G. 2004. A Thorough Axiomatization of a Principle of Conditional Preservation in Belief Revision. *Ann. Math. Artif. Intell.* 40(1-2): 127–164.

Kern-Isberner, G.; and Brewka, G. 2017. Strong Syntax Splitting for Iterated Belief Revision. In Sierra, C., ed., *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017*, 1131–1137. ijcai.org.

Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic Reasoning, Preferential Models and Cumulative Logics. *Artif. Intell.* 44(1-2): 167–207.

Lakemeyer, G.; and Levesque, H. J. 2020. A First-Order Logic of Limited Belief Based on Possible Worlds. In Calvanese, D.; Erdem, E.; and Thielscher, M., eds., *Proceedings of the 17th International Conference on Principles of Knowledge Representation and Reasoning, KR 2020, Rhodes, Greece, September 12-18, 2020*, 624–635.

Parikh, R. 1999. *Beliefs, Belief Revision, and Splitting Languages*, 266–278. Center for the Study of Language and Information.

Pearl, J. 1990. System Z: A Natural Ordering of Defaults with Tractable Applications to Nonmonotonic Reasoning. In *Proceedings of the 3rd Conference on Theoretical Aspects of Reasoning about Knowledge*, TARK '90, 121–135. Morgan Kaufmann Publishers Inc.

Rajendra, A.; and Sajja, P. 2009. *Knowledge-Based Systems*. Jones and Bartlett Learning.

Reiter, R. 1980. A Logic for Default Reasoning. *Artif. Intell.* 13(1): 81–132.

Rosales, R.; and Jaakkola, T. S. 2005. Focused Inference. In Cowell, R. G.; and Ghahramani, Z., eds., *Proceedings of the Tenth International Workshop on Artificial Intelligence and Statistics, AISTATS 2005, Bridgetown, Barbados, January 6-8, 2005*. Society for Artificial Intelligence and Statistics.

Schwering, C. 2017. A Reasoning System for a First-Order Logic of Limited Belief. In Sierra, C., ed., *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017*, 1247–1253. ijcai.org.

Spohn, W. 2012. *The Laws of Belief: Ranking Theory and Its Philosophical Applications*. Oxford University Press.