

# Power in Liquid Democracy

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## Abstract

The paper develops a theory of power for delegable proxy voting systems. We define a power index able to measure the influence of both voters and delegators. Using this index, which we characterize axiomatically, we extend an earlier game-theoretic model by incorporating power-seeking behavior by agents. We analytically study the existence of pure strategy Nash equilibria in such a model. Finally, by means of simulations, we study the effect of relevant parameters on the emergence of power inequalities in the model.

## Introduction

Liquid democracy (Blum and Zuber 2016) is a form of proxy voting (Miller 1969; Tullock 1992; Alger 2006; Green-Armytage 2015; Cohensius et al. 2017) where each proxy is delegable, thereby giving rise to so-called transitive delegations. In such a system each voter may choose to cast her vote directly, or to delegate her vote to a proxy, who may in turn decide whether to vote or delegate, and so pass the votes she has accrued further to yet another proxy. The voters who decide to retain their votes—the so-called gurus—cast their ballots, which now carry the weight given by the number of delegations they accrued. Liquid democracy has been an influential proposal in recent public debates on democratic reform across the world, thanks also to platforms for democratic decision support such as, in particular, LiquidFeedback (Behrens et al. 2014)<sup>1</sup>. In the last couple of years it has enjoyed considerable attention from researchers in political science and eDemocracy, as well as artificial intelligence (see, for an overview, Paulin (2020)).

**Contribution** The starting point of our paper is a controversial feature of liquid democracy: transitive delegations may in principle lead to disproportionate accrual of power, thereby harming the democratic legitimacy of the resulting vote. To the best of our knowledge, this issue has received only limited attention. A notable exception is the work of Kling et al. (2015), which provided an empirical analysis of power and influence in liquid democracy based on data from the German Pirate Party. However, a formal theory of power in voting systems with delegable proxy is lacking. We aim

at providing such a theory here, and use it to gain insights into how power may happen to be distributed among agents involved in decision-making with liquid democracy.

*First*, we provide a generalization of the power index known as Banzhaf index to account for delegations in voting with quota rules. This novel index—called delegative Banzhaf index—measures not only the influence of voters, but also that of delegators. We characterize this index axiomatically (Theorem 1) and highlight how the index responds intuitively to the way in which delegations may be structured (Fact 1-3).

*Second*, we extend the strategic model of liquid democracy developed by Bloembergen, Grossi, and Lackner (2019) to account for power-seeking behavior by agents. In our model, agents want to vote truthfully in order to relay correct information to the mechanism, but they do so by also considering how much power they retain in the system. We carry out an equilibrium analysis (pure strategy Nash equilibrium) of the model. We show equilibria may not exist if delegations are constrained (Theorem 2), but they do when everybody is allowed to delegate to anybody (Theorem 3).

*Finally*, we simulate our game theoretic model and study how two key parameters of the model influence the distribution of power both in equilibrium and after one-shot interaction. Our experiments show that limiting the level of connectivity of the underlying network has a beneficial effect in limiting the emergence of inequalities in the distribution of power (measured by Gini coefficient). Perhaps less intuitively, the extent by which agents are motivated by the accumulation of power has a similar effect: groups where agents are more power-greedy appear to achieve more equal distributions of power.

Proofs are sketched or omitted. All details about our proofs and experiments can be found in the full version<sup>2</sup>.

**Related Work** The idea of voting with delegable proxy can be traced back to Dodgson (1884) and has been object of study in the political sciences (Green-Armytage 2015). In the last couple of years, several papers in the artificial intelligence community (and in particular the computational social choice one (Brandt et al. 2016)) have focused on liquid democracy. Two lines of research have broadly been

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<sup>1</sup><https://liquidfeedback.org/>

<sup>2</sup><https://arxiv.org/pdf/2010.07070.pdf>

pursued. On the one hand papers have pointed to potential weaknesses of voting by liquid democracy, e.g.: delegation cycles and the failure of individual rationality in multi-issue voting (Christoff and Grossi 2017; Brill and Talmon 2018); poor accuracy of group decisions as compared to those achievable via direct voting in non-strategic settings (Kahng, Mackenzie, and Procaccia 2018; Caragiannis and Micha 2019), as well as strategic ones (Bloembergen, Grossi, and Lackner 2019). On the other hand a number of papers have focused on the development of better behaved delegation schemes, e.g.: delegations with preferences over trustees (Brill and Talmon 2018) or over gurus (Escoffier, Gilbert, and Pass-Lanneau 2019, 2020); multiple delegations (Gölz et al. 2018); complex delegations like delegations to a majority of trustees (Colley, Grandi, and Novaro 2020); dampened delegations (Boldi et al. 2011); breadth-first delegations (Kotsialou and Riley 2020).

Our paper is a contribution to the first line of research. The possibility of large power imbalances is recognized as a potential problem for liquid democracy, although experimental work has argued the issue may be limited in practice (Kling et al. 2015). We aim at putting the discussion on power in liquid democracy on a precise footing and gain insights into how power imbalances may arise or be contained.

## Preliminaries: A Model of Liquid Democracy

Our model is based on the binary voting setting for truth-tracking (Condorcet 1785; Grofman, Owen, and Feld 1983; Elkind and Slinko 2016). The setting has already been applied to the study of liquid democracy by Kahng, Mackenzie, and Procaccia (2018); Bloembergen, Grossi, and Lackner (2019); Caragiannis and Micha (2019).

**Binary voting by truth-tracking agents** A finite set of agents  $N = \{1, 2, \dots, n\}$  has to vote on whether to accept or reject an issue. The vote is supposed to track the correct state of the world—that is whether it is ‘best’ to accept or reject the issue. The agents’ ability to make the right choice (i.e., the agents’ error model) is represented by the agent’s *accuracy*  $q_i \in (\frac{1}{2}, 1]$ , for  $i \in N$ .

We assume the result of such an election to be determined by a quota rule with quota  $\beta \in (\frac{n}{2}, n]$ . That is, the issue is accepted if and only if there are at least  $\beta$  agents supporting it. We will also be working with the more general setting in which each agent is endowed with a weight. Let  $\omega : N \rightarrow \mathbb{R}_+$  be a weight function assigning a positive weight to every agent.<sup>3</sup> Then the quota is  $\beta \in (\frac{\sum_{i \in N} \omega(i)}{2}, \sum_{i \in N} \omega(i)]$ . That is, an issue is accepted if and only if the weight it collects from individual votes matches or exceeds the quota (cf. Chalkiadakis, Elkind, and Wooldridge (2012)).

**Liquid democracy elections** When agent  $i \in N$  delegates to agent  $j \in N$  we write  $d_i = j$ . We admit the possibility for an agent to abstain by delegating to a nul agent 0. This feature will be of technical use for the characterization of the power index we are going to introduce. Then  $\mathbf{d} = (d_1, d_2, \dots, d_n)$  is called a *delegation profile* (or simply

*profile*) and is a vector describing each agent’s delegation. Equivalently, delegation profiles can be usefully thought of as maps  $\mathbf{d} : N \rightarrow N \cup \{0\}$ , where  $\mathbf{d}(i) = d_i$ . When  $d_i = i$ , agent  $i$  votes on her own behalf. We call such an agent a *guru*. On the other hand, any agent who is not a guru, is called a *delegator*. For profile  $\mathbf{d}$ , and  $C \subseteq N$ , let  $C^{\mathbf{d}}$  denote all gurus in  $C$  in profile  $\mathbf{d}$ , i.e.,  $C^{\mathbf{d}} = \{i \in C \mid d_i = i\}$ . A delegation profile in which all agents are gurus (i.e., for all  $i \in N$   $d_i = i$ ) is said to be *trivial*.

We call a *liquid democracy election* (LDE) the tuple  $V = \langle N, \omega, \mathbf{d}, \beta \rangle$ , where  $N$  is the set of agents with weights according to  $\omega$ ,  $\mathbf{d}$  is a delegation profile, and  $\beta$  is the quota. Let then  $\mathbb{V}$  denote the set of all LDEs. Clearly, LDEs with trivial profiles are instances of standard weighted voting.

**Gurus, chains and cycles** Any profile  $\mathbf{d}$  can also be represented by a directed graph. An edge from agent  $i$  to  $j$  ( $i \rightarrow j$ ) exists whenever  $d_i = j$ . Consider then a profile  $\mathbf{d}$  where a path exists from  $i$  to  $j$ , i.e.,  $i \rightarrow i_1 \rightarrow \dots \rightarrow i_k \rightarrow j$ . We call such paths *delegation chains*. When such a chain from  $i$  to guru  $j$  exists, every agent on this delegation chain (indirectly) delegates to  $j$ , and we denote  $i$ ’s guru by  $d_i^* = \mathbf{d}^*(i) = j$ . Additionally, the set of agents between any pair of agents on the delegation chain are called the *intermediaries* between the two agents. For example, suppose the above delegation chain occurs in profile  $\mathbf{d}$ . Then the set of intermediaries between  $i$  and  $j$  is  $\{i_1, \dots, i_k\}$ , and it is denoted by  $\delta_{\mathbf{d}}(i, j)$ . The sum of the weights of the intermediaries between two agents  $i$  and  $j$  and the weight of  $j$ , is called the *delegation distance* from  $i$  to  $j$  and is denoted by  $\Delta_{\mathbf{d}}(i, j) = \sum_{a \in (\delta_{\mathbf{d}}(i, j) \cup \{j\})} \omega(a)$ . A *delegation cycle* is a chain where the first and last agents coincide. In such a case, no agent in the chain is linked to a guru. Therefore no agent linked via a delegation chain to an agent in a delegation cycle has a guru. For  $C \subseteq N$ , we write  $\mathbf{D}(C) = \{j \in N \mid \exists k \in C, d_j^* = k\}$  to denote the set of agents that directly or indirectly delegate to a guru in  $C$ . If  $C = \{i\}$  we write  $\mathbf{D}(i)$  for the set of agents whose guru is  $i$ .

One last piece of notation: we will need to consider what happens to delegation chains when we restrict to certain subsets of agents. For instance, given the chain  $i \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \rightarrow j$ , if  $\{i, i_2, \dots, i_k, j\} \subseteq C \subseteq N$  but  $i_1 \notin C$ , then  $i$  is not able to delegate to  $j$  within  $C$  as she has no access to intermediary  $i_1$  in such subset. For  $C \subseteq N$  we write  $\mathbf{d}_C^*(i_2) = j$  to denote that  $j$  is the guru of  $i_2$  and the chain from  $i_2$  to  $j$  contains only elements of  $C$ . Then we write  $\hat{\mathbf{D}}(C) = \{j \in N \mid \exists k \in C, \mathbf{d}_C^*(j) = k\}$  for the set of agents that directly or indirectly delegate to some agent in  $C$  through intermediaries contained in  $C$ . Intuitively, this captures the support accrued by gurus in  $C$  via agents in  $C$ .

## A Power Index for Liquid Democracy

Once delegations are settled, liquid democracy results in weighted voting where only gurus vote with the sum of weights they accrued from direct or indirect delegations. From a voting perspective, gurus are therefore the only agents who retain voting power after the delegation phase.

<sup>3</sup>In the ‘one-voter-one-vote’ setting,  $\omega(i) = 1$  for all  $i \in N$ .

However, this neglects the power that delegators actually have within liquid democracy by being able to control large number of votes. By means of a simple example: a guru  $i$  obtaining  $m$  direct delegations is intuitively more ‘powerful’ than a guru obtaining  $m$  delegations via an intermediary  $j$ , who is in turn recipient of  $m - 1$  direct delegations. Most of  $i$ ’s power depends then on  $j$  (see also Example 1 below).

So in this section we generalize the Banzhaf index (Penrose 1946; Banzhaf 1965) to the delegable proxy voting setting.<sup>4</sup> The Banzhaf index has already been used to study the power of gurus in liquid democracy by Kling et al. (2015).

### Delegative Banzhaf Index: Definition

We briefly recall the definition of the Banzhaf index. A simple game is a tuple  $\mathcal{G} = \langle N, \nu \rangle$ , where  $N$  is the set of agents ( $|N| = n$ ) and  $\nu$  is the characteristic function  $\nu : 2^N \rightarrow \{0, 1\}$ . For any  $C \subseteq N$ , if  $\nu(C) = 1$  then  $C$  is said to be *winning*, otherwise it is said to be *losing*. An agent  $i$  is called a *swing agent* for coalition  $C$  if  $\nu(C) - \nu(C \setminus \{i\}) = 1$ . Then in a simple game  $\mathcal{G}$ , the Banzhaf index  $B_i(\mathcal{G})$  of agent  $i \in N$  is:  $B_i(\mathcal{G}) = \frac{1}{2^{n-1}} \sum_{C \subseteq N \setminus \{i\}} (\nu(C \cup \{i\}) - \nu(C))$ , i.e.,  $i$ ’s probability of being swing for a random coalition.

There is one obvious way in which an LDE  $V$  induces a simple game: it is the simple game capturing the weighted voting occurring among gurus once delegations have been fixed, i.e.,  $\mathcal{G}_V = \langle N, \nu_V \rangle$  where, for any  $C \subseteq N$ :

$$\nu_V(C) = 1 \text{ iff } \sum_{i \in \mathbf{D}(C)} \omega(i) \geq \beta. \quad (1)$$

That is, a coalition wins whenever all gurus in it together accrue enough weight to meet the quota. In such a game only gurus may have positive power:  $i \in N^d$  if  $B_i(\mathcal{G}_V) > 0$ , as  $\mathcal{G}_V$  is silent about the influence that delegators have in determining the winning coalitions.

The influence of delegators can be captured by a different simple game  $\mathcal{G}'_V = \langle N, \nu'_V \rangle$  where, for any  $C \subseteq N$ :

$$\nu'_V(C) = 1 \text{ iff } \sum_{i \in \mathbf{D}(C)} \omega(i) \geq \beta. \quad (2)$$

That is, a coalition  $C$  is winning whenever the sum of weights accrued by the gurus in  $C$  from agents in  $C$ , meets the quota. According to this way of constructing the simple game, an agent’s weight is accrued in a coalition  $C$  if the agent, her guru, and all intermediaries between them are contained in  $C$ . We refer to  $\mathcal{G}'_V$  as the *delegative simple game* of LDE  $V$ . Clearly, if  $\mathbf{d}$  is trivial, all agents are gurus and therefore  $\mathcal{G}_V = \mathcal{G}'_V$ .

So, given an LDE  $V = \langle N, \omega, \mathbf{d}, \beta \rangle$ , we define the delegative Banzhaf index of an agent  $i$  in LDE  $V$  simply as the Banzhaf index of  $i$  in the delegative simple game of  $V$ :

$$DB_i(V) = B_i(\mathcal{G}'_V). \quad (3)$$

Observe that in LDEs  $V$  where the delegation profile is trivial, and therefore games  $\mathcal{G}(V)$  and  $\mathcal{G}'(V)$  coincide, the two indices coincide.

<sup>4</sup>Our approach can similarly be used to develop a generalization of the Shapley-Shubik power index (Shapley and Shubik 1954).

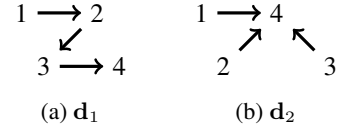


Figure 1: Profiles in Example 1

**Example 1.** Consider two LDEs,  $V_1 = \langle N, \omega, \mathbf{d}_1, \beta \rangle$  and  $V_2 = \langle N, \omega, \mathbf{d}_2, \beta \rangle$ , where  $N = \{1, 2, 3, 4\}$ ,  $\omega(i) = 1$  for all  $i \in N$ ,  $\beta = 3$  and  $\mathbf{d}_1$  and  $\mathbf{d}_2$  are represented in Fig. 1a and Fig. 1b, respectively.

We focus on the indices of agents 1 and 4. First consider  $V_1$ . Since no coalition  $C \subseteq N$  exists with  $\nu'_{V_1}(C) = 1$  and  $\nu'_{V_1}(C \setminus \{1\}) = 0$ ,  $DB_1(V_1) = 0$ . Then we compute  $DB_4(V_1)$ .  $\nu'_{V_1}(C) = 1$  and  $\nu'_{V_1}(C \setminus \{4\}) = 0$  iff  $C \in \{\{2, 3, 4\}, \{1, 2, 3, 4\}\}$ . Thus  $DB_4(V_1) = \sum_{C \subseteq N} (\nu'_{V_1}(C) - \nu'_{V_1}(C \setminus \{4\})) / 2^3 = 1/4$ . Then consider  $V_2$ . First for  $DB_1(V_2)$ ,  $\nu'_{V_2}(C) = 1$  and  $\nu'_{V_2}(C \setminus \{1\}) = 0$  iff  $C \in \{\{1, 2, 4\}, \{1, 3, 4\}\}$ . Therefore,  $DB_1(V_2) = \sum_{C \subseteq N} (\nu'_{V_2}(C) - \nu'_{V_2}(C \setminus \{1\})) / 2^3 = 1/4$ . For  $DB_4(V_2)$ , we have that if  $C \subseteq \{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$ ,  $\nu'_{V_2}(C) = 1$  and  $\nu'_{V_2}(C \setminus \{4\}) = 0$ . Then  $DB_4(V_2) = \sum_{C \subseteq N} (\nu'_{V_2}(C) - \nu'_{V_2}(C \setminus \{4\})) / 2^3 = 1/2$ .

DB depends on the structure of the delegation profile. For instance, in both  $V_1$  and  $V_2$ , agent 1 is a delegator with no incoming delegations, but  $DB_1(V_1) = 0$  while  $DB_1(V_2) = 1/4$ . In  $V_1$ , agent 1 is “far” from the guru and her vote does not matter for meeting the quota. In  $V_2$ , agent 1 delegates directly to the guru. Similarly, in both LDEs agent 4 collects 4 votes. However,  $DB_4(V_1) = 1/4$  but  $DB_4(V_2) = 1/2$  since in  $V_1$ , the delegation chain pointing to 4 is long, so that agent 4 depends on 3 for almost all her weight.

### Characterization of DB

To underpin (3) we present a characterization of the delegative Banzhaf index. We want to axiomatically identify DB among all functions  $f : \mathbb{V} \rightarrow (N \rightarrow \mathbb{R})$  for LDEs on  $N$ . To do so we borrow ideas and techniques from existing axiomatizations of the Banzhaf index for weighted voting games (Dubey and Shapley 1979; Nowak 1997; Lehrer 1988).

The strategy we follow consists in generalizing a known characterization of the Banzhaf index for standard weighted voting due to Barua, Chakravarty, and Roy (2005). We use the same axioms of that characterization (Axioms 2-5 below), with the addition of one axiom for so-called dummy agents (Axiom 1). Crucially, however, we show how to adapt the key definitions upon which the axioms are based from the standard weighted voting setting to LDEs. This concerns in particular the definitions of composition and bloc formation (Definitions 6 and 7) which play an important role in the proof. As a result one can retrieve the known characterization of the standard Banzhaf index from ours, by simply restricting to the class of LDEs where profiles are trivial, and therefore delegations do not matter.

**Preliminary Definitions** We start by introducing standard definitions from the theory of simple games. Assume an LDE  $V = \langle N, \omega, \mathbf{d}, \beta \rangle$  be given.

**Definition 1** (Dummy Agent). *An agent  $i \in N$  is dummy if for any  $C \subseteq N$  ( $i \in C$ ),  $\nu'_V(C) = \nu'_V(C \setminus \{i\})$ , where  $\nu'$  is the characteristic function of the delegative simple game of  $V$ . Let  $N^{dum}$  denote all dummy agents.*

That is, an agent is dummy whenever she cannot influence  $\nu'_V(C)$  by quitting or joining any coalition  $C \subseteq N$ . It is worth observing that, in LDEs there are three ways in which an agent can be dummy: if the agent abstains (i.e., delegates to 0); if the agent is linked by a chain to a delegation cycle; if the agent—call it  $i$ —is such that  $\Delta_{\mathbf{d}}(i, d_i^*) \geq \beta$ , that is, the delegation distance between  $i$  and her guru in  $\mathbf{d}$  is larger than  $\beta$ . We call such an agent *distant* (in  $\mathbf{d}$ ).

**Definition 2** (Dictator). *An agent  $i \in N$  is a dictator if  $\nu'_V(C) = 1$  if and only if  $i \in C$ , for any  $C \subseteq N$ .*

That is, an agent  $i$  is a dictator of  $V$  whenever it belongs to all and only the winning coalitions of the delegative simple game of  $V$ . In an LDE this occurs if the dictator  $i$  is a guru and  $\beta \leq \omega(i)$ , that is,  $i$  meets the quota on her own.

**Definition 3** (Symmetric Agents). *Any two agents  $i, j \in N$  are symmetric if for all  $C \subseteq N \setminus \{i, j\}$ ,  $\nu'_V(C \cup \{i\}) = \nu'_V(C \cup \{j\})$ .*

Symmetric agents are swing for exactly the same coalitions in the delegative simple game of  $V$ . Note that a pair of symmetric agents do not necessarily have the same weight.

**Example 2** (Example 1 continued). *Consider  $V_1$  in Figure 1a. Since  $\beta = 3$  and the delegation distance  $\Delta_{\mathbf{d}}(1, 4) = 3$ , agent 1 is a distant (and therefore dummy) agent. Next consider agents 1 and 2 (or any pair of  $\{1, 2, 3\}$ ) in  $V_2$ , each of whom directly delegates to agent 4. For any coalition  $C \subseteq N \setminus \{1, 2\}$ ,  $\nu'_{V_2}(C \cup \{1\}) = \nu'_{V_2}(C \cup \{2\})$ , thus 1 and 2 are symmetric. There is no dictator in Example 1.*

The following definitions generalize the standard theory of simple games to account for delegations.

**Definition 4** (Minimally Winning Coalition). *A coalition  $C \subseteq N$  is a minimally winning coalition if for any  $i \in \hat{\mathbf{D}}(C)$ ,  $\nu'_V(C) = 1$  and  $\nu'_V(C \setminus \{i\}) = 0$ .*

That is, a coalition  $C$  is minimally winning if it is winning (in the delegative simple game of  $V$ ), but becomes losing if any agent who is linked to a guru in  $C$  via agents in  $C$  is removed. So a minimally winning coalition is a coalition that contains just enough gurus with just enough support through intermediaries in the same coalition to meet the quota. It follows that no distant agent may be included in a minimally winning coalition. Notice, however, that such a coalition may contain agents that are not linked to gurus in  $C$  by intermediaries in  $C$  (i.e., that do not belong to  $\hat{\mathbf{D}}(C)$ ) and therefore it may not be minimal w.r.t. set inclusion.

**Definition 5** (Unanimity LDE (ULDE)).  *$V$  is a unanimity LDE if the quota  $\beta = \sum_{i \in \mathbf{D}(N)} \omega(i)$ . We call such a quota unanimity quota and denote it by  $\beta^U$ .*

That is, in a ULDE the quota equals the sum of weights of all agents who directly or indirectly delegate to gurus.

The last two definitions concern operations on LDEs: how to combine two LDEs into a new one; and how to build an LDE from another one by merging two agents into a ‘bloc’.

**Definition 6** (Composition). *Let two LDEs  $V_1 = \langle N_1, \omega_1, \mathbf{d}_1, \beta_1 \rangle$  and  $V_2 = \langle N_2, \omega_2, \mathbf{d}_2, \beta_2 \rangle$  be given, such that for any  $i \in N_1 \cap N_2$ , if  $\mathbf{d}_1(i) = j$  and  $j \in N_2$  (resp.  $\mathbf{d}_2(i) = j$  and  $j \in N_1$ ),  $\mathbf{d}_2(i) = j$  (resp.  $\mathbf{d}_1(i) = j$ ), otherwise  $\mathbf{d}_2(i) = 0$  (resp.  $\mathbf{d}_1(i) = 0$ ), and  $\omega_1(i) = \omega_2(i)$ . We define two new LDEs  $V_1 \wedge V_2 = \langle N_1 \cup N_2, \omega_{1 \wedge 2}, \mathbf{d}_{1 \wedge 2}, \beta_1 \wedge \beta_2 \rangle$  and  $V_1 \vee V_2 = \langle N_1 \cup N_2, \omega_{1 \vee 2}, \mathbf{d}_{1 \vee 2}, \beta_1 \vee \beta_2 \rangle$ , where:*

- for any  $i \in N_1$  (resp.  $i \in N_2$ ),  $\omega_{1 \vee 2}(i) = \omega_{1 \wedge 2}(i) = \omega_1(i)$  (resp.  $\omega_{1 \vee 2}(i) = \omega_{1 \wedge 2}(i) = \omega_2(i)$ );
- for any  $i \in N_1 \setminus N_2$  (resp.  $N_2 \setminus N_1$ ),  $\mathbf{d}_{1 \vee 2}(i) = \mathbf{d}_{1 \wedge 2}(i) = \mathbf{d}_1(i)$  (resp.  $\mathbf{d}_{1 \vee 2}(i) = \mathbf{d}_{1 \wedge 2}(i) = \mathbf{d}_2(i)$ ), and for any  $i \in N_1 \cap N_2$ , if  $\mathbf{d}_1(i) = \mathbf{d}_2(i)$  then  $\mathbf{d}_{1 \vee 2}(i) = \mathbf{d}_{1 \wedge 2}(i) = \mathbf{d}_1(i) = \mathbf{d}_2(i)$ , otherwise  $\mathbf{d}_{1 \vee 2}(i) = \mathbf{d}_{1 \wedge 2}(i) = \mathbf{d}_k(i)$  where  $k \in \{1, 2\}$  and  $\mathbf{d}_k(i) \neq 0$ ;
- $\beta_1 \wedge \beta_2$  (resp.  $\beta_1 \vee \beta_2$ ) is met by  $C \subseteq N_1 \cup N_2$  iff  $\sum_{i \in \hat{\mathbf{D}}_{C \cap N_1}(C \cap N_1)} \omega_1(i) \geq \beta_1$  and (resp. or)  $\sum_{i \in \hat{\mathbf{D}}_{C \cap N_2}(C \cap N_2)} \omega_2(i) \geq \beta_2$ .

Two LDEs can be composed provided the delegation graphs at their intersection coincide or, if they do not, provided that this is because an agent in  $N_1 \cap N_2$  delegates outside the intersection under one profile while she abstains (i.e., delegates to 0) under the other profile. The condition is required to guarantee the coherency of delegations in the composition. Then quotas in the composition are so defined as to guarantee that coalitions in the delegative simple game of the composition are winning iff they are winning in both, or at least one of, the delegative simple games of the LDEs (cf. proof of Lemma 1).

**Definition 7** (Bloc formation). *Given  $V = \langle N, \omega, \mathbf{d}, \beta \rangle$  and for any  $i, j \in N$  such that  $d_i = j$  or  $i, j \in N^{\mathbf{d}}$ ,  $V' = \langle N', \omega', \mathbf{d}', \beta \rangle$  is called the bloc LDE joining  $i$  and  $j$  into a bloc  $ij$ , where*

- $N' = N \setminus \{i, j\} \cup \{ij\}$ ;
- For  $\mathbf{d}'$ , if  $d_i = j$ ,  $d'_{ij} = d_j$ , and for any  $a \in N$ , such that  $d_a = i$  or  $d_a = j$ ,  $d'_a = ij$ , but if  $i, j \in N^{\mathbf{d}}$ , for any  $a \in N$  such that  $d_a = i$  or  $d_a = j$ ,  $d'_a = ij$ ;
- $\omega'(ij) = \omega(i) + \omega(j)$ .

A bloc LDE treats two agents  $i$  and  $j$ , who are either adjacent in the delegation graph or both gurus, as one new agent  $ij$ . By applying the operation in Definition 7 repeatedly, it is possible to coalesce all agents who share the same guru into one bloc. Furthermore, any pair of delegation chains can also be joined into one bloc by joining their gurus into one bloc. Such operations play an important role in the proof of our characterization result (cf. proof of Lemma 2).

**Axioms** We can now introduce the axioms of our characterization. Assume again that an LDE  $V$  is given.

We assign minimum power to dummy agents, maximum to dictators, and identical power to symmetric agents:

**Axiom 1** (No Power (NP)). *If  $i \in N^{dum}$ ,  $f_i(V) = 0$ .*

**Axiom 2** (Maximum Power (MP)). *The power index of a dictator is 1.*

**Axiom 3 (Equal Treatment (ET)).** For any pair of symmetric agents  $i, j \in N$ ,  $f_i(V) = f_j(V)$ .

The last two axioms concern how the index should behave with respect to composition and bloc formation.

**Axiom 4 (Bloc Principle (BP)).** For any two agents  $i, j \in N$  such that  $d_i = j$ , or  $i, j \in N^d$ , let  $V'$  be the bloc LDE by joining  $i$  and  $j$  into bloc  $ij$ . Then  $f_{ij}(V') = f_i(V) + f_j(V)$ .

**Axiom 5 (Sum Principle (SP)).** For any pair of LDEs  $V_1, V_2 \in \mathbb{V}$ , such that any  $i \in N_1 \cap N_2$  satisfies the condition in Definition 6,  $f_i(V_1 \wedge V_2) + f_i(V_1 \vee V_2) = f_i(V_1) + f_i(V_2)$  for any  $i \in N_1 \cup N_2$ .

Intuitively, Axiom 4 requires the power of the bloc, which is formed by two gurus or two adjacent agents on a delegation chain, to be the sum of their individual powers. Axiom 5 requires that the sum of any agent's power in  $V_1 \wedge V_2$  and  $V_1 \vee V_2$  to be the sum of her power in  $V_1$  and  $V_2$ .

**Characterization** The result is based on two lemmas.

**Lemma 1.** DB satisfies NP, MP, ET, BP, and SP.

*Proof sketch.* We prove the claim for SP as it provides a nice illustration of the workings of our definitions. The other cases are provided in the appendix. To show that DB satisfies the SP, one first has to show that by the way in which weights  $\beta_1 \wedge \beta_2$  and  $\beta_1 \vee \beta_2$  are set in Definition 6, we have that for any coalition  $C \subseteq N_1 \cup N_2$ ,  $\nu_{V_1 \wedge V_2}^C(C) = 1$  iff  $\nu_{V_1}^C(C \cap N_1) = 1$  and  $\nu_{V_2}^C(C \cap N_2) = 1$ , and  $\nu_{V_1 \vee V_2}^C(C) = 1$  iff  $\nu_{V_1}^C(C \cap N_1) = 1$  or  $\nu_{V_2}^C(C \cap N_2) = 1$ . The proof can then proceed with a standard argument.

First consider any  $i \in N_1 - N_2$ , i.e., agent  $i$  is contained in  $N_1$  but not in  $N_2$ . Let  $m_i^V$  denote the number of times that agent  $i$  is swing in  $\mathcal{G}_V^i$  i.e.,  $m_i^V = |\{C \subseteq N \setminus \{i\} \mid \nu_V^C(C) = 0, \nu_V^C(C \cup \{i\}) = 1\}|$ . Then, if  $i$  is swing in  $C \subseteq N_1$  in LDE  $V_1$ , she is also swing in  $(C \cup C') \cap N_1$ , for any  $C' \subseteq N_2 - N_1$ . Therefore, in LDE  $V_1 \vee V_2$ ,  $m_i^{V_1 \vee V_2} = m_i^{V_1} 2^{|N_2 - N_1|}$ , where  $m_i^{V_1 \vee V_2}$  is the number of times that  $i$  is swing in LDE  $V_1 \vee V_2$ . Additionally, since  $i \in N_1 - N_2$ ,  $m_i^{V_2} = 0$ , that is,  $i$  cannot be swing in LDE  $V_2$ , which implies  $m_i^{V_1 \wedge V_2} = 0$ . Hence we have for  $i \in N_1 - N_2$ ,  $m_i^{V_1 \vee V_2} = m_i^{V_1} 2^{|N_2 - N_1|} + m_i^{V_2} 2^{|N_1 - N_2|} - m_i^{V_1 \wedge V_2}$ . Identical equations can be developed for agent  $i \in N_2 - N_1$  or  $i \in N_1 \cap N_2$ . Divide each side of the equation by  $2^{|N_1 \cup N_2| - 1}$  and obtain that, for any  $i \in N_1 \cup N_2$ ,  $\frac{m_i^{V_1 \vee V_2}}{2^{|N_1 \cup N_2| - 1}} = \frac{m_i^{V_1}}{2^{|N_1| - 1}} + \frac{m_i^{V_2}}{2^{|N_2| - 1}} - \frac{m_i^{V_1 \wedge V_2}}{2^{|N_1 \cup N_2| - 1}}$ , which implies that  $\text{DB}_i(V_1 \wedge V_2) + \text{DB}_i(V_1 \vee V_2) = \text{DB}_i(V_1) + \text{DB}_i(V_2)$ .  $\square$

**Lemma 2.** A power index  $f$  for LDEs satisfies NP, MP, ET, BP, and SP, only if it is DB.

*Proof sketch.* The proof consists of two claims: **claim 1** if  $f$  is DB for any ULDE, it is DB for any LDE; **claim 2**  $f$  is DB for any ULDE.

To prove **claim 1**, observe that any LDE  $V$  can be represented as the disjunction of  $m$  ULDEs, i.e.,  $V = V_1 \vee \dots \vee V_m$ , given  $V$  has  $m$  minimally winning coalitions

$\{C_1, \dots, C_m\}$ , where  $V_i = \langle C_i, \omega, \mathbf{d}_{C_i}, \beta^U \rangle$ . By induction over the number of disjunction ULDEs we show then that if  $f$  is DB for the disjunction of any  $k$  ( $1 \leq k < m$ ) ULDEs, it is also DB for the disjunction of any  $k + 1$  ULDEs. The claim holds by SP, since  $f_i(V_1 \vee \dots \vee V_{k+1}) = f_i(V_1 \vee \dots \vee V_k) + f_i(V_{k+1}) - f_i((V_1 \vee \dots \vee V_k) \wedge V_{k+1})$ , where  $f$  is DB for  $V_1 \vee \dots \vee V_k$  and  $V_{k+1}$  by assumption, as well as for  $(V_1 \vee \dots \vee V_k) \wedge V_{k+1}$  because it is also disjunction of  $k$  ULDEs:  $(V_1 \vee \dots \vee V_k) \wedge V_{k+1} = (V_1 \wedge V_{k+1}) \vee \dots \vee (V_k \wedge V_{k+1})$ .

To prove **claim 2** consider an ULDE  $V = \langle N, \omega, \mathbf{d}, \beta^U \rangle$ . We need to show that  $f_i(V) = 1/2^{n'-1}$  for any non-dummy agent, and  $f_i(V) = 0$  for any dummy agent, where  $n'$  is the number of non-dummy agents in the ULDE, i.e.,  $n' = |N \setminus N^{dum}|$ . The proof is conducted by induction on  $|N|$ . As the basis,  $f_i(V) = 1$  if  $|N| = 1$  by MP since  $i$  is a dictator. Assume then that the claim holds when  $|N| = k$ , we show it also holds if  $|N| = k + 1$ . For any non-dummy agent, if only one non-dummy agent exists, the claim is obvious by MP. If  $|N \setminus N^{dum}| \geq 2$ , we exploit BP to join two gurus, or a delegator with her trustee, into a bloc, then obtain an LDE with  $k$  agents where the hypothesis holds. Then the claim follows by ET. For dummy agents the claim is proven by exploiting NP.  $\square$

**Theorem 1.** A power index for liquid democracy satisfies MP, NP, SP, ET, and BP, if and only if it is DB.

*Proof.* It follows from Lemma 1 and Lemma 2.  $\square$

**Further Properties of DB** Besides the above axioms, it is worth mentioning a few other properties of the index that highlight its dependence on the delegation graph. Here  $(\mathbf{d}_{-i}, d'_i)$  is the profile obtained by changing  $\mathbf{d}(i)$  to  $d'_i$  in  $\mathbf{d}$ .

**Fact 1 (Delegation & Power Loss).** For any pair of LDEs  $V = \langle N, \omega, \mathbf{d}, \beta \rangle$  and  $V' = \langle N, \omega, \mathbf{d}', \beta \rangle$ , such that  $\mathbf{d}' = (\mathbf{d}_{-i}, d'_i)$ ,  $\mathbf{d}(i) = i$  and  $d'_i = j$  ( $i \neq j$ ), we have that  $\text{DB}_i(V) \geq \text{DB}_i(V')$ .

That is, delegations never lead to an increase in power for the delegator. In fact one can show that the inequality  $\text{DB}_i(V) \geq \text{DB}_i(V')$  can be strict.

The last two facts show that agents closer to the guru have more power and that, power-wise, delegating directly to a guru is better than doing that indirectly.

**Fact 2 (Power Monotonicity).** For any pair of agents  $i, j \in N$ , such that  $d_i = j$ ,  $\text{DB}_i(V) \leq \text{DB}_j(V)$ .

**Fact 3 (Direct vs. Indirect Delegation).** Let  $V$  be a LDE, in which there exists three agents  $i, j, k \in N$ , such that  $\mathbf{d}(i) = j$  and  $\mathbf{d}(k) = i$ . Let then  $V' = \langle N, \omega, \mathbf{d}', \beta \rangle$ , such that  $\mathbf{d}' = (\mathbf{d}_{-k}, d'_k)$  where  $d'_k = j$ . Then,  $\text{DB}_k(V') \geq \text{DB}_k(V)$ .

## A Game-theoretic Model

We will now use the DB index to extend the game-theoretic model of liquid democracy by Bloembergen, Grossi, and Lackner (2019), referred to as delegation game. Like in that model, we will be assuming that delegations are constrained by an underlying social network represented by a directed graph  $\langle N, E \rangle$ , where each agent is a node in the network

and for any  $i, j \in N$  ( $i \neq j$ ), if there is an edge from  $i$  to  $j$ , i.e.  $(i, j) \in E$ ,  $j$  is called a *neighbor* of  $i$ . Let  $E(i)$  denote all neighbors of agent  $i$ , i.e.,  $E(i) = \{j \in N \mid (i, j) \in E\}$ .

**Delegation Games** In the delegation game by Bloembergen, Grossi, and Lackner (2019) agents' payoffs in an LDE depend solely on the accuracy of their gurus, and the effort agents incur should they vote directly. Here we abstract from the effort element of the model and focus instead on incorporating a power-seeking element in agents' utilities. The key intuition behind our extension is to model agents that are not only interested in voting accurately, but also in their own influence during the vote. So our agents choose their delegations by aiming at maximizing the trade-off between pursuing high accuracy and seeking more power.

**Definition 8.** A delegation game is a tuple  $\mathcal{D} = \langle N, G, \{q_i\}_{i \in N}, \Sigma, \beta, u \rangle$ , where:  $N$  is a finite set of agents;  $G = \langle N, E \rangle$  is a directed graph;  $q_i$  is  $i$ 's accuracy;  $\Sigma_i = E(i) \cup \{i\}$  is  $i$ 's delegation strategy space;  $\beta \in (\frac{n}{2}, n]$  is a quota;  $u$  is the utility function, defined as follows:

$$u_i(\mathbf{d}) = \text{DB}_i(\mathbf{d}) \cdot q_{d_i^*} \quad (4)$$

Observe that the strategy profiles of this game are delegation profiles. Each such profile  $\mathbf{d}$  induces an LDE  $\langle N, \omega, \mathbf{d}, \beta \rangle$  where we assume  $\omega$  to be the standard weight function assigning weight 1 to each agent. The utility of profile  $\mathbf{d}$  for  $i$  is the accuracy that  $i$  acquires in  $\mathbf{d}$ , multiplied by  $i$ 's power in  $\mathbf{d}$ , measured by  $\text{DB}_i(\mathbf{d})$ .<sup>5</sup> Notice that, therefore, the utility of a dummy agent is 0 and that the utility of a dictator equals her accuracy.

For our experiments we will be using the more general form of (4) given by  $\text{DB}_i(\mathbf{d})^\alpha \cdot q_{d_i^*}$ , with  $\alpha \in [0, 1]$ . Intuitively, parameter  $\alpha$  will be used to control how much agents are influenced by power in the range going from no influence to influence equal to that of accuracy.

**Equilibrium Analysis** In this section, we ask the natural question of whether the games of Definition 8 have a Nash equilibrium (NE) in pure strategies. In general, the answer to this question is negative:

**Theorem 2.** *There are delegation games that have no (pure strategy) NE.*

*Sketch of proof.* The proof consists in providing a delegation game and then showing that it is possible to construct a profitable deviation for some agent, for every possible delegation profile. The game is  $N = \{1, 2, 3, 4, 5, 6\}$ ,  $q_1 = 0.51$ ,  $q_2 = 0.7$ ,  $q_3 = 0.9$ ,  $q_4 = 0.6$ ,  $q_5 = 0.7$ ,  $q_6 = 0.9$ ,  $\beta = 4$ , and the underlying directed graph is  $G = \langle N, E = \{(1, 3), (2, 3), (4, 6), (5, 6)\} \rangle$ .  $\square$

However NE can be guaranteed to exist when the underlying network is complete.

**Theorem 3.** *In any delegation game  $\mathcal{D} = \langle N, \{q_i\}_{i \in N}, G, \Sigma, \beta, u \rangle$ , where  $G$  is a complete network, there exists at least one (pure strategy) NE.*

<sup>5</sup>Notice that we slightly abuse notation here by using  $\mathbf{d}$  directly as input for the index, instead of the corresponding LDE.

*Sketch of proof.* First of all observe that on a complete network, if an equilibrium exists, it must be such that all delegation chains are of length 1 by Fact 3. The theorem is then proven by construction. We construct a profile by letting each agent choose, in turn according to a given sequence, whether to delegate to agent  $i^* \in N$ , who has the highest accuracy, or to be a guru. If an agent decides to delegate to  $i^*$ , this delegation is assumed to be fixed. So the set of delegators (to  $i^*$ ) monotonically increases. The process continues until no guru wants to delegate. We show that the profile constructed in such a way is a NE by showing that: [1]  $i^*$  has no profitable deviation because were she deviating to a delegator, she would form a cycle (thereby obtaining utility 0), and were she delegating to another guru she would inherit a lower accuracy (by assumption) and have lower power (by Fact 1). [2] No delegator has a profitable deviation, neither by delegating to another delegator (by Fact 3), nor by delegating to another guru (as she would inherit lower accuracy and obtain lower power), nor by becoming herself a guru. The latter claim requires some work and can be shown by proving that as more agents delegate to  $i^*$ , the power of delegators does not decrease while that of gurus (except for  $i^*$ ) does. [3] Finally, no guru has a profitable deviation, neither by delegating to  $i^*$  (by construction), nor to another guru (as she would then be better off delegating to  $i^*$ ), nor by delegating to a delegator (again, by Fact 3).  $\square$

## Experiments

We use the above model to study, by means of experiments, how the distribution of power in profiles is affected by specific parameters. In particular, we are interested in gaining insights into: whether higher connectivity of the underlying network increases power imbalances; whether the more agents are driven by power the less they tend to delegate.

**Setup** Starting from the trivial profile, we generate profiles in two ways: by one-shot interaction (OSI), in which each agent selects her neighbor (including herself) which maximises her utility; and by iterated better response dynamics (IBRD), in which each agent iteratively selects one neighbor at random and delegates to her only if this increases her utility, until a stable state (equilibrium) is reached.

As one might expect, the bottleneck in our experiments consists in the computation of DB in order to establish agents' utilities by Eq. (4). It is well-known that computing the Banzhaf index in weighted voting games is intractable (Matsui and Matsui 2001). We therefore implement the approximation method described by Bachrach et al. (2008). Each time we need to estimate the DB of an agent, 15000 coalitions are randomly sampled (by uniform distribution), and the ratio of the coalitions for which the agent is swing is used as the estimator of the DB. By the analytical bounds proven by Bachrach et al. (2008), with the above method we know that the correct DB is in the confidence interval  $[\widehat{\text{DB}} - 0.011, \widehat{\text{DB}} + 0.011]$  with probability of 0.95, where  $\widehat{\text{DB}}$  is the estimator. So it should be clear that the statistics presented in this section report on values that depend on the estimator  $\widehat{\text{DB}}$ , and that with high probability are close to the

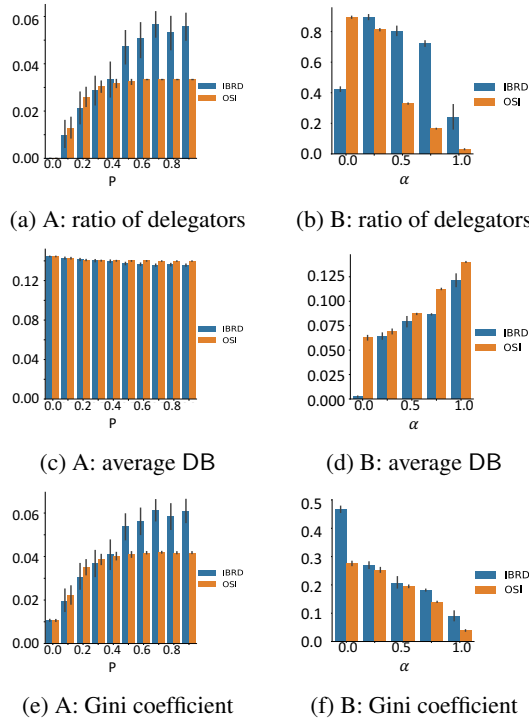


Figure 2: Selected plots from experiments A and B

exact intended values.

We will be working with two parameters. To test the effect of connectivity on power we assume that interaction happens on a random network and vary the probability  $p$  (see range in e.g., Fig. 2a) of any two agents being linked. To test the effect of different attitudes towards the importance of power for agents we work with the generalization  $DB_i(\mathbf{d})^\alpha \cdot q_{a_i}^*$  of (4) with  $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$  and assuming an underlying random network with  $p = 0.75$ .

We set  $|N| = 30$ , the quota  $\beta = 16$ , and for each parameter setting, we use an accuracy vector  $Q \in \mathbb{R}^{30}$ , where each element in  $Q$  is drawn from a Gaussian distribution  $\mathcal{N}(0.75, 0.125)$ . All statistics are the mean value over 50 instances for each parameter setting. Further details on the setup of our experiments, including pseudo-code for the algorithms of OSI and IRBD are provided in the appendix.

**Connectivity: experiment A** Fig. 2a shows that the higher the connectivity (the larger  $p$ ), the more agents tend to delegate both in equilibrium (IBRD) and in one-shot interaction (OSI). This is in line with expectations as agents have more chance to interact with high-accuracy agents. It is worth observing, however, that the ratio of delegators is very low (less than 0.06 on average). That is, very few agents delegate on average. This is in contrast with the behavior of the delegation game where utility is solely based on accuracy (cf. Bloembergen, Grossi, and Lackner (2019)). We will see with experiment B that the influence of power on agents' utility seems to be an important factor in limiting vs. facilitating delegations. Despite the small number of delegations we can still observe that increasing  $p$  lowers the mean value

of DB (Fig. 2c) and increases inequality in the distribution of DB, measured by the Gini coefficient (Fig. 2e), although it should be stressed the Gini coefficient remains very low due to the small fraction of delegators. Intuitively, more delegations enhance some agents' power, but reduce the power of other agents, be they gurus or delegators.

**Power: experiment B** Fig. 2b shows that larger values of  $\alpha$  correspond to significantly fewer delegators for OSI. As agents put more weight on power they are more reluctant to delegate in the initial profile (recall Fact 1). For IRBD this effect is observable only for  $\alpha$  in the upper half of the range of available values. We argue this may depend on the fact that IRBD, at the initial profile, allows for delegations to take place that only suboptimally improve utility, triggering then further delegations at later iterations. As  $\alpha$  grows, the average power increases (Fig. 2d) and inequality in the distribution of power decreases (Fig. 2f). The decreasing ratio of delegators (Fig. 2b) is another side of this trend: as  $\alpha$  decreases the group turns from consisting mostly of delegators ( $\alpha \in \{0, 0.25\}$ ) to low numbers of delegators comparable to those observed in experiment A ( $\alpha = 1$ ).

Interestingly, the average length of chains also significantly decreases from around 5.5 for  $\alpha = 0$  to 1.2 for  $\alpha = 0.25$ , and further mildly decreases to around 0.5 for  $\alpha = 1$ . This is in line with Fact 3: as power becomes more important, agents prefer shorter chains to their gurus.

We also ran experiments to identify the effects of varying quota  $\beta$  in the range  $\{0.6|N|, 0.8|N|, |N|\}$ . We identified a similar trend showing that higher quota tend to limit the amount of delegations, but our results were less robust than those reported in experiments A and B. The refinement of this experiment is left to future work.

## Conclusions

The paper developed a power index for voting with delegable proxy. We showed that the index generalizes the Banzhaf index for standard weighted voting and can be axiomatized in a similar fashion. We used the index to model a variant of delegation games for liquid democracy where agents seek to find a tradeoff between increasing their accuracy and acquiring power in the system. We showed equilibria for this sort of interaction exist under a full connectivity condition, but do not exist in general. Finally, two parameters of the model were shown, through simulations, to play an important role in containing the emergence of large inequalities in the distribution of power: the level of connectivity of an underlying (random) network, and the extent to which agents are motivated by the acquisition of power.

The paper opens several directions for future research of both theoretical and experimental kind. Here we mention one: it would be interesting to understand how much agents' attitude towards power could help in readdressing the deterioration of decision-making quality highlighted by Kahng, Mackenzie, and Procaccia (2018); Caragiannis and Michalek (2019), through its equalizing effect on power distribution.



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