The Maximin Support Method: An Extension of the D'Hondt Method to Approval-Based Multiwinner Elections

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Abstract

We propose the maximin support method, a novel extension of the D'Hondt apportionment method to approval-based multiwinner elections. The maximin support method is a sequential procedure that aims to maximize the support of the least supported elected candidate. It can be computed efficiently and satisfies (adjusted versions of) the main properties of the original D'Hondt method: house monotonicity, population monotonicity, and proportional representation. We also establish a close relationship between the maximin support method and alternative D'Hondt extensions due to Phragmén.

1 Introduction

In a multiwinner election, the goal is to select a fixed number of candidates (a so-called committee) based on the preferences of a set of agents (Faliszewski et al. 2017). Multiwinner voting rules have a wide variety of applications including political elections (Brams, Kilgour, and Potthoff 2019), medical diagnostic decision-making (Gangl et al. 2019), and the selection of validators who participate in the consensus protocol of a blockchain (Cevallos and Stewart 2020).

Recent years have witnessed an increasing interest in settings where the agents express their preferences via approval ballots: For each candidate, an agent has the choice between approving or disapproving the candidate. A particular focus of the approval-based multiwinner voting literature (see Lackner and Skowron, 2020, for a recent survey) has been on the proportional representation of agents' preferences in the committee (Aziz et al. 2017; Sánchez-Fernández et al. 2017; Brill et al. 2017, 2020; Peters and Skowron 2020).

A simpler setting in which proportional representation has been extensively studied is that of apportionment (Balinski and Young 1982). Here, both candidates and agents have attributes and the goal is to select a committee such that the distribution over attributes in the committee resembles as closely as possible the distribution over attributes among the agents. In classical applications of apportionment, attributes refer to either geographical location or political party affiliation; proportionality then suggests, for instance, that the number of seats assigned to a state of a union in a representative body should be proportional to the population size of the state, or that the number of seats assigned to a political party in a parliament should be proportional to the number of votes the party received in an election. An important apportionment method is named after Victor D'Hondt.[1]

The apportionment problem has an illustrious history and has given rise to an elegant mathematical theory (Balinski and Young 1982; Pukelsheim 2014), but it is not without limitations. For example, requiring voters in a parliamentary election to choose among political parties is often described as restrictive, as it prevents them from expressive more fine-grained preferences (Renwick and Pilet 2016). Furthermore, apportionment methods are not applicable in scenarios where attributes (such as party affiliation) are not available. These limitations have led a number of scholars to explore more general settings (e.g., Hylland 1992; Kilgour, Brams, and Sanver 2006). One important generalization is the setting of approval-based multiwinner elections mentioned above. Apportionment problems constitute the special case where approval sets partition the candidate space. Thus, every approval-based multiwinner voting rule induces an apportionment method (Brill, Laslier, and Skowron 2018). Indeed, some of the most studied approval-based multiwinner voting rules, those of Phragmén (1894) and Thiele (1895), have been devised as extensions of the D'Hondt method of apportionment (Janson 2016).

In this paper, we introduce a novel approval-based multiwinner voting rule. Like Phragmén and Thiele, we take the D'Hondt apportionment method as our point of departure. In contrast to earlier proposals, we focus on a compelling "maximin" characterization of the method in terms of voter support: The D'Hondt method always selects committees maximizing the voter support for the least supported candidate in the committee. We generalize the notion of maximin support to approval-based multiwinner elections. When applying this concept iteratively (adding candidates to the committee one at a time), we obtain the *maximin support method (MMS)*. We establish that MMS is an efficiently computable extension of the D'Hondt method that satisfies committee monotonicity, (weak) support monotonicity, and proportional justified representation (PJR).

[1] In the US, the method is named after Thomas Jefferson, the third president of the United States. In fact, Jefferson introduced the method already in 1792, whereas D'Hondt described it in 1878.

We also establish a close relationship between the maximin support method and Phragmén's voting rules. In particular, we show that the (computationally intractable) non-sequential variant of Phragmén's rule always produces committees that optimize the maximin support objective globally. From this perspective, MMS can be considered an axiomatically desirable and polynomial-time computable approximation algorithm for the maximin support problem. Interestingly, recent independent[2] work by Cevallos and Stewart (2020) has shown that MMS strictly outperforms Phragmén's sequential rule regarding this perspective.

2 Preliminaries

Let C be a finite set of *candidates* and $N = \{1, \ldots, n\}$ be a set of n *voters*. Furthermore, $k \in \mathbb{N}$ denotes the number of winners to be selected. We assume $1 \leq k \leq |C|$ and $n \geq 1$.

For each $i \in N$, we let $A_i \subseteq C$ denote the *approval ballot* of voter i. That is, A_i is the subset of candidates that voter i approves of. An *approval profile* is a list $A = (A_1, \ldots, A_n)$ of approval ballots, one for each voter $i \in N$. Given an approval profile A and a candidate c, we let N_c denote the set of approvers of c and we call $|N_c|$ the *approval score* of c.

An *(approval-based multiwinner) election* E can be represented by a tuple $E = (N, C, A, k)$. Since N and C can be inferred from A, we often simply refer to an election by (A, k). An *(approval-based multiwinner voting) rule* R is a function that maps an election $E = (N, C, A, k)$ to a subset $R(E) \subseteq C$ of candidates of size $|R(E)| = k$, referred to as the *committee*. We often refer to committee members as *winners*. During the execution of a voting rule, ties between candidates can occur. We assume that ties are broken using a fixed priority ordering over the candidates. An example of a priority ordering is the lexicographic order, which we use in our examples.

An important subdomain of approval-based multiwinner elections is defined by *party-list elections*, where the set of candidates is partitioned into parties and voters can vote for exactly one party. Formally, a party-list election satisfies $C = P_1 \dot\cup P_2 \dot\cup \ldots \dot\cup P_p$ and every approval ballot A_i coincides with one party list P_j. The ballot profile for a party-list election can be summarized by a *vote vector* $V = (v_1, v_2, \ldots, v_p)$, where v_j is the number of votes for party P_j (i.e., $v_j = |\{i \in N : A_i = P_j\}|$).

An *apportionment method* takes as input a vote vector $V = (v_1, v_2, \ldots, v_p)$ and a natural number k and outputs a *seat distribution* $x = (x_1, \ldots, x_p) \in \mathbb{N}_0^p$ with $\sum_{j=1}^{p} x_i = k$. The interpretation is that party P_j is allocated x_j seats. Apportionment methods have been extensively studied in the literature (Balinski and Young 1982; Pukelsheim 2014). Since the party-list setting is a special case of the general approval-based multiwinner setting, every approval-based multiwinner rule induces an apportionment method (Brill, Laslier, and Skowron 2018). An approval-based multiwinner rule is called an *extension* of an apportionment method if it induces it. In this paper, we will introduce a novel extension of the apportionment method due to D'Hondt.

[2]The work of Cevallos and Stewart (2020) is based on a preprint of this paper, which has been publicly available for some time.

Parties		P_1	P_2	P_3
Votes (v_j)		5 100	3 150	1 750
Divisors	$d_1 = 1$	**5 100.0**	**3 150.0**	**1 750.0**
	$d_2 = 2$	**2 550.0**	1 575.0	875.0
	$d_3 = 3$	**1 700.0**	1 050.0	583.3
	$d_4 = 4$	1 275.0	787.5	437.5
	$d_5 = 5$	1 020.0	630.0	350.0
Seats (x_j)		3	1	1

Table 1: Example of the use of the D'Hondt method. The $k = 5$ highest quotients are marked in bold and correspond to the seat distribution $(3, 1, 1)$.

The *D'Hondt method* (aka *Jefferson method*) is a particular example from a family of apportionment methods known as *divisor methods*. These methods assign seats to parties based on a sequence of *divisors* $(d_1, d_2, d_3, \ldots)$, and different divisor methods differ in their choice of this sequence. Divisor methods can be illustrated by constructing a table in which columns correspond to parties and rows correspond to divisors. The entry in row i and column j is given by v_j/d_i, i.e., the number of votes of party P_j divided by the i-th divisor. The divisor method then assigns the k seats to the parties corresponding to the k highest quotients in this table.

The D'Hondt method is defined via the divisor sequence $(d_1, d_2, d_3, \ldots) = (1, 2, 3, \ldots)$. An example of the use of the D'Hondt method is shown in Table 1, where five seats need to be assigned to three parties. As shown in Table 1, the D'Hondt method assigns three seats to party P_1, one to party P_2, and one to party P_3.

An important proportionality axiom for apportionment methods is *lower quota*, which requires that each party P_j is allocated *at least* $\lfloor k \frac{v_j}{n} \rfloor$ seats. It is well known that the D'Hondt method is the only divisor method satisfying lower quota. Moreover, the D'Hondt method satisfies two prominent monotonicity properties: *house monotonicity*, which states that no party loses a seat when the house size k is increased, and *population monotonicity*, which states that if the ratio $\frac{v_i}{v_j}$ increases, then it should not be the case that x_i decreases and x_j increases (Balinski and Young 1982).

3 A Formal Model of Support

In this section, we formalize the notion of *support*, on which our extension of the D'Hondt method will be based.

Interpreting a vote for a party as support for the elected members of that party, and assuming that the v_j votes for party P_j are evenly distributed among the x_j seats assigned to that party, one can characterize the D'Hondt method as the unique apportionment method that maximizes the support of the least supported selected candidate. That is, the D'Hondt method chooses seat distributions x maximizing $\min_j v_j/x_j$. For instance, in the example illustrated in Table 1, each of the three elected candidates from party P_1 is supported by $5100/3 = 1700$ voters, the elected candidate from party P_2 is supported by 3150 voters, and the elected candidate from party P_3 is supported by 1750 voters. Therefore, $\min_j v_j/x_j = \min\{1700, 3150, 1750\} = 1700$, and all other seat distributions would lead to smaller values.

We now generalize this notion of support to approval profiles, by distributing votes in the form of approval ballots among subsets of candidates. In general, there will be many different ways of distributing the support of a voter among the candidates approved by the voter, leading to different support values for candidates. Later, we will focus on "optimal" ways of distributing the support.

For an approval profile A and a nonempty subset $D \subseteq C$ of candidates, we define the family $\mathcal{F}_{A,D}$ of *support distribution functions for* (A, D) as the set of all functions that distribute support *only* among the candidates in D. Formally, $\mathcal{F}_{A,D}$ consists of all functions $f : (N \times D) \to [0, 1]$ satisfying $f(i, c) = 0$ for all $i \in N$ and $c \in D \setminus A_i$, and

$$\sum_{c \in A_i \cap D} f(i, c) = 1 \quad \text{for all } i \in N \text{ with } A_i \cap D \neq \emptyset.$$

For each voter $i \in N$, $f(i, c)$ is the fraction of voter i's vote that is "assigned" to candidate c. Note that the definition requires that $f(i, c) = 0$ whenever $c \notin A_i$. Thus, the support of a voter is distributed only among those candidates that are approved by the voter. Given a support distribution function $f \in \mathcal{F}_{A,D}$ and a candidate $c \in D$, we let $\mathit{supp}_f(c)$ denote the total support received by c under f, i.e.,

$$\mathit{supp}_f(c) = \sum_{i \in N} f(i, c).$$

Example 1. Consider the following approval profile A over the candidate set $C = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$:

$10\,000 \times \{c_1, c_2\}$ $\quad 6\,000 \times \{c_1, c_3\}$ $\quad 4\,000 \times \{c_2\}$
$5\,500 \times \{c_3\}$ $\quad 9\,500 \times \{c_4\}$ $\quad 3\,000 \times \{c_5\}$
$5\,000 \times \{c_5, c_6, c_7\}$

Consider the candidate subset $D = \{c_1, c_3, c_5\}$ and let f be the (unique) function in $\mathcal{F}_{A,D}$ with $f(i, c_1) = 0.4$ for each voter i with $A_i = \{c_1, c_3\}$ (thus $f(i, c_3) = 0.6$ for those voters). Thus, f assigns 2400 out of the 6000 $\{c_1, c_3\}$-votes to c_1 and the remaining 3600 $\{c_1, c_3\}$-votes to c_3, resulting in the following support values:

$$\mathit{supp}_f(c_1) = 10\,000 + 2\,400 = 12\,400,$$
$$\mathit{supp}_f(c_3) = 3\,600 + 5\,500 = 9\,100,$$
$$\mathit{supp}_f(c_5) = 5\,000 + 3\,000 = 8\,000.$$

We will be interested in those support distribution functions in $\mathcal{F}_{A,D}$ that maximize the support for the least supported candidate in D. To this end, let $\mathit{maximin}(A, D)$ denote the maximal support for the least supported candidate in D, where the maximum is taken over all support distribution functions in $\mathcal{F}_{A,D}$. Formally,

$$\mathit{maximin}(A, D) = \max_{f \in \mathcal{F}_{A,D}} \min_{c \in D} \mathit{supp}_f(c).$$

Furthermore, we let $\mathcal{F}^{\text{opt}}_{A,D}$ denote the nonempty[3] set of *optimal* support distribution functions for (A, D), i.e.,

$$\mathcal{F}^{\text{opt}}_{A,D} = \{f \in \mathcal{F}_{A,D} \colon \mathit{supp}_f(c) \geq \mathit{maximin}(A, D) \forall c \in D\}.$$

The support distribution function specified in Example 1 is optimal, as $|N_{c_5}| = 8000$ and the approval score of a candidate in D is a natural upper bound for $\mathit{maximin}(A, D)$.

In the remainder of this paper, we will be interested in finding committees W with a large maximin support value $\mathit{maximin}(A, W)$ for a given approval profile A. An interesting rationale for maximizing the minimum support was given by Cevallos and Stewart (2020): The maximin support value of a committee puts a limit on the overrepresentation of voter groups. To illustrate this, consider a committee W together with an optimal support distribution function f for (A, W). Let $D \subseteq W$ be a subset of winning candidates, and consider the set $N_D = \bigcup_{c \in D} N_c$ of voters that approve at least one candidate in D. It follows from the definition of a support distribution function that the total support for candidates in D is upper-bounded by $|N_D|$, i.e., $\sum_{c \in D} \mathit{supp}_f(c) \leq |N_D|$. On the other hand, $\sum_{c \in D} \mathit{supp}_f(c) \geq |D| \cdot \mathit{maximin}(A, W)$. Combining these inequalities yields $|D| \leq |N_D|/\mathit{maximin}(A, W)$. In other words, the voter group N_D cannot have a number of representatives in the committee that is higher than the size of the voter group divided by the maximin support value.

4 The Maximin Support Method

We now propose an extension of the D'Hondt method to approval-based multiwinner elections. It is based on the same principle as the D'Hondt method, in that the support for the least supported elected candidate should be as large as possible. We therefore refer to this novel method as *maximin support method* (MMS). The maximin support method chooses candidates sequentially[4] until the desired number k of candidates has been selected. In every iteration, a candidate with the greatest support is chosen, under the condition that only support distribution functions maximizing the support for the least supported candidate are considered.

Given an approval-based multiwinner election $E = (N, C, A, k)$, the set $W = \mathit{MMS}(E)$ is determined by starting with $W = \emptyset$ and iteratively adding candidates until $|W| = k$. In each iteration, we add to W an unelected candidate receiving the greatest support, under the condition that only optimal support distribution functions are considered.[5] More precisely, for each candidate $c \in C \setminus W$, we compute an optimal support distribution function f_c for the set $W \cup \{c\}$ and determine the total support $\mathit{supp}_{f_c}(c)$ that c receives under f_c. The candidate maximizing this value is then added to W. See Algorithm 1 for a formal description.

Since the set $\mathcal{F}^{\text{opt}}_{A,W \cup \{c\}}$ of optimal support distribution functions may contain more than one function, the value of $\mathit{supp}_{f_c}(c)$ could potentially depend on the choice of f_c. The following result implies that this is not the case.

[3]Since $\mathcal{F}_{A,D}$ may be an infinite set, we need to make sure that the function $\min_{c \in D} \mathit{supp}_f(c)$ attains a maximum over this set. We will see in the proof of Theorem 3 that the corresponding optimization problem can be formulated as a feasible and bounded linear program. It follows that $\mathcal{F}^{\text{opt}}_{A,D} \neq \emptyset$ and that $\max_{f \in \mathcal{F}_{A,D}} \min_{c \in D} \mathit{supp}_f(c)$ indeed exists.

[4]One can also define a non-sequential (optimization) variant of the maximin support method; we discuss this variant in Section 6.1.

[5]Restricting attention to optimal support distribution functions ensures that support for previously elected candidates is not ignored when searching for new support distribution function.

Algorithm 1: Maximin Support Method (MMS)

Data: approval-based multiwinner election (A, k)
Result: subset $W \subseteq C$ of candidates with $|W| = k$

```
1 W = ∅
2 for j = 1 to k do
3   | foreach c ∈ C \ W do
4   |   | compute f_c ∈ F^opt_{A,W∪{c}}
5   |   | s_c = supp_{f_c}(c)
6   | w ∈ arg max_{c∈C\W} s_c
7   | W = W ∪ {w}
8 return W
```

Theorem 2. *Let (A, k) be an approval-based multiwinner election. The following holds for each $j \in \{0, \dots, k-1\}$.*

Let W^j denote the set of the first j candidates chosen by the maximin support method when applied to (A, k). Then, for each candidate $c \in C \setminus W^j$ and for each optimal support distribution function $f_c \in \mathcal{F}^{opt}_{A,(W^j \cup \{c\})}$,

$$supp_{f_c}(c) = maximin(A, W^j \cup \{c\}).$$

Theorem 2 states that in every iteration the candidate c added to W is among the least supported candidates under every optimal support distribution function. The support of this candidate thus equals $maximin(A, W \cup \{c\})$, which (by definition) is independent of the particular $f_c \in \mathcal{F}^{\text{opt}}_{A,W\cup\{c\}}$ that was chosen in line 4 of the algorithm. The proof of Theorem 2 employs linear programming duality theory and can be found in the full version of this paper (Sánchez-Fernández et al. 2021).

This result gives rise to an interesting alternative formulation of the maximin support method. In this equivalent formulation, there is no need to choose an optimal support distribution function for $(A, W \cup \{c\})$; rather, s_c is directly defined as $maximin(A, W \cup \{c\})$. A natural interpretation of this definition is that the value s_c measures the effect that the addition of a potential candidate would have on the maximal support for the least supported candidate.

The next theorem establishes that the maximin support method can be computed efficiently.

Theorem 3. *The maximin support method can be computed in polynomial time.*

Proof. It is sufficient to show that, for any subset $D \subseteq C$ of candidates, an optimal support distribution function $f \in \mathcal{F}^{\text{opt}}_{A,D}$ can be computed in polynomial time. For a given approval profile A and a subset $D \subseteq C$ of candidates, consider the following linear program, containing a variable $f(i, c)$ for each $i \in N$ and $c \in A_i \cap D$, and an additional variable s.

$$\begin{aligned}
&\text{maximize} && s \\
&\text{subject to} && \sum_{i \in N : c \in A_i \cap D} f(i, c) \geq s, && \text{for all } c \in D \\
& && \sum_{c \in A_i \cap D} f(i, c) = 1, && \text{for all } i \in N \text{ with } A_i \cap D \neq \emptyset \\
& && f(i, c) \geq 0, && \text{for all } i \in N \text{ and } c \in D
\end{aligned}$$

The first set of constraints require that the support for the least supported candidate in D is at least s, while the remaining constraints ensure that the variables $f(i, c)$ encode a valid support distribution function.[6] Therefore, optimal solutions of this linear program correspond to optimal support distribution functions. Since linear programming problems can be solved in polynomial time (Khachian 1979), this concludes the proof. □

We conclude this section by illustrating the maximin support method with an example.

Example 4. Consider the election $E = (A, k)$, where A is the approval profile from Example 1 and $k = 3$.

In the first iteration, the value $s_c = maximin(A, \{c\})$ equals the approval score of candidate c, i.e., $s_c = |N_c|$ for all c. Therefore, the approval winner c_1 (with $s_{c_1} = 16000$) is chosen. The corresponding support distribution function f satisfies $f(i, c_1) = 1$ for all $i \in N_{c_1}$.

In the second iteration, we have $W = \{c_1\}$ and we need to compute the value $s_x = maximin(A, \{c_1, x\})$ for all $x \in C \setminus \{c_1\}$. For example, for candidate c_2 we get $s_{c_2} = maximin(A, \{c_1, c_2\}) = 10000$; the corresponding support distribution function f assigns 4000 out of the 10000 $\{c_1, c_2\}$-votes to c_1 and the remaining 6000 to c_2. A better value is achieved by candidate c_3. The support distribution realizing $s_{c_3} = maximin(A, \{c_1, c_3\}) = 10750$ assigns all 10000 $\{c_1, c_2\}$-votes to c_1, all 5500 $\{c_3\}$-votes to c_3, and divides the 6000 $\{c_1, c_3\}$-votes between c_1 and c_3 such that both candidates have a total support of 10750 each. Computing the other values, we get $s_{c_4} = 9500$, $s_{c_5} = 8000$, and $s_{c_6} = s_{c_7} = 5000$. Therefore, c_3 is selected.

In the third iteration, we have $W = \{c_1, c_3\}$ and we need to compute the value $s_x = maximin(A, \{c_1, c_3, x\})$ for all $x \in C \setminus \{c_1, c_3\}$. It can be checked that $s_{c_2} = 8500$, $s_{c_4} = 9500$, $s_{c_5} = 8000$, and $s_{c_6} = s_{c_7} = 5000$. Thus, candidate c_4 is chosen. There are several optimal support distribution functions f realizing $maximin(A, \{c_1, c_3, c_4\}) = 9500$; each of them assigns all 9500 $\{c_4\}$-votes to c_4 and distributes the 6000 votes that approve of c_1 and c_3 in such a way that c_1 and c_3 have a total support of at least 9500 each.

In summary, we have $MMS(E) = \{c_1, c_3, c_4\}$.

5 Axiomatic Properties of MMS

In this section, we show that the maximin support method is indeed an extension of the D'Hondt method, and that it satisfies (adjusted versions of) several important properties that the latter satisfies. In particular, we show that the maximin support method satisfies *committee monotonicity*, *weak support monotonicity* (a variant of population monotonicity), and *proportional justified representation*.

Committee Monotonicity Committee monotonicity requires that all selected candidates are still selected when the committee size k is increased. Since the maximin support method selects winners iteratively, committee monotonicity is trivially satisfied.

[6]Note that constraints of the form $f(i, c) \leq 1$ are not necessary because each variable $f(i, c)$ is non-negative and appears in a constraint of the form $\sum_{c \in A_i \cap D} f(i, c) = 1$.

Observation 5. *The maximin support method satisfies committee monotonicity.*

Support Monotonicity Support monotonicity axioms for approval-based multiwinner elections have been proposed by Sánchez-Fernández and Fisteus (2019). Informally, these axioms require that when a subset of the winners in an election see their support increased (either because a voter adds this subset of candidates to her approval set, or because a new voter enters the election and approves of precisely this subset of candidates), then (some of) those candidates must remain in the committee. We establish that MMS satisfies an axiom known as *weak support monotonicity*.[7]

Theorem 6. *The maximin support satisfies weak support monotonicity.*

The formal definition of this axiom can be found in the full version of this paper (Sánchez-Fernández et al. 2021), together with a discussion of other support monotonicity axioms and the proof of Theorem 6.

Proportional Representation Axioms capturing the proportional representation of voter preferences in approval-based multiwinner elections have been studied extensively in recent years (Aziz et al. 2017; Brill et al. 2017; Peters and Skowron 2020). In this paper, we focus on an axiom known as *proportional justified representation (PJR)* (Sánchez-Fernández et al. 2017). PJR is a generalizations of the lower quota axiom to the general approval-based multiwinner setting: If a voting rule satisfies PJR, then its induced apportionment method satisfies lower quota (Brill, Laslier, and Skowron 2018).

In order to define PJR, we need some terminology. Consider an election (N, C, A, k). Given a positive integer $\ell \in \{1, \ldots, k\}$, we say that a subset $N^* \subseteq N$ of voters is *ℓ-cohesive* if $|N^*| \geq \ell \frac{n}{k}$ and $|\bigcap_{i \in N^*} A_i| \geq \ell$. A subset $D \subseteq C$ of candidates provides *proportional justified representation (PJR)* if for all $\ell \in \{1, \ldots, k\}$ and all ℓ-cohesive subsets $N^* \subseteq N$, it holds that $|D \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$.

Definition 7. An approval-based multiwinner voting rule R satisfies *proportional justified representation (PJR)* if $R(E)$ provides PJR for every election $E = (N, C, A, k)$.

In order to prove that the maximin support method satisfies PJR, we make use of the recently introduced notion of *priceability* (Peters and Skowron 2020). The definition of priceability is based on the notion of a *price system*. A price system is a pair $(p, \{p_i\}_{i \in N})$, where $p > 0$ and for each voter $i \in N$, the function $p_i : C \to [0, 1]$ satisfies $p_i(c) = 0$ if $i \in N \setminus N_c$ and $\sum_{c \in C} p_i(c) \leq 1$.

Definition 8. Given an approval profile A, a set $W \subseteq C$ of candidates is *priceable* if there exists a price system $(p, \{p_i\}_{i \in N})$ such that the following conditions hold:

1. $\sum_{i \in N} p_i(c) = p$ for all c in W;
2. $\sum_{i \in N} p_i(c) = 0$ for all c in $C \setminus W$; and
3. $\sum_{i \in N_c} [1 - \sum_{c' \in W} p_i(c')] \leq p$ for all c in $C \setminus W$.

[7]Despite its name, weak support monotonicity is slightly stronger than the basic version of population monotonicity that only considers increased support for single candidates.

Peters and Skowron (2020) proved that if a committee of size k is priceable for A, then it provides PJR for (A, k). Therefore, we can prove that the maximin support satisfies PJR by showing that it always returns priceable committees.

Theorem 9. *Let E be an approval-based multiwinner election. Then, $MMS(E)$ is priceable.*

Proof. Let $E = (N, C, A, k)$ be an approval-based multiwinner election and $W = MMS(E)$.

We establish the priceability of W by defining a price system $(p, \{p_i\}_{i \in N})$ as follows. The price p is defined by $p = maximin(A, W)$. In order to define the functions p_i, let f be any optimal support distribution function for (A, W) and set $p_i(c) = f(i, c)$ for all $i \in N$ and all $c \in W$. For each $c \in W$ such that $supp_f(c) > p$, we arbitrarily reduce the value of some $p_i(c)$ such that $\sum_{i \in N} p_i(c) = p$.

For the sake of contradiction, suppose that there exists a candidate c in $C \setminus W$ with $\sum_{i \in N_c} [1 - \sum_{c' \in W} p_i(c')] > p$. Consider a support distribution function f' for $(A, W \cup \{c\})$ given by $f'(i, c) = 1 - \sum_{c' \in W} p_i(c')$ for each voter $i \in N_c$, $f'(i, c) = 0$ for each voter $i \in N \setminus N_c$, and $f'(i, c') = p_i(c')$ for each voter $i \in N$ and each candidate $c' \in W$. If there is a voter i with $A_i \cap W \neq \emptyset$ and $\sum_{c' \in W \cup \{c\}} f(i, c') < 1$, we arbitrarily increase the support that voter i gives to some candidates in $A_i \cap W$ so that $\sum_{c' \in W \cup \{c\}} f(i, c') = 1$.

Observe that $supp_{f'}(c) > p$ and $supp_{f'}(c') \geq p$ for each $c' \in W$. Let c^* denote the candidate chosen by the maximin support method in the last iteration. We now have the following two possibilities:

Case 1: f' is not an optimal support distribution function for $(A, W \cup \{c\})$. Then, since $supp_{f'}(c') \geq p$ for each c' in $W \cup \{c\}$, we have $maximin(A, W \cup \{c\}) > p$. But then $maximin(A, W \setminus \{c^*\} \cup \{c\}) \geq maximin(A, W \cup \{c\}) > p$, contradicting the assumption that candidate c^* was selected in the last iteration.

Case 2: f' is an optimal support distribution function for $(A, W \cup \{c\})$. Then, Theorem 2 implies that $maximin(A, W \cup \{c\}) = supp_{f'}(c) > p$. By the same argument as in the first case, we reach a contradiction. ☐

Corollary 10. *The maximin support method satisfies proportional justified representation.*

In the same spirit as lower quota, PJR ensures that each cohesive group of voters is represented in the committee by *at least* the number of candidates that is proportional to the group size. In light of the discussion at the end of Section 3, it can thus be argued that MMS strikes an attractive compromise between two competing representation goals.

D'Hondt Extension Theorem 9 can also be used to show that the maximin support indeed extends the apportionment method of D'Hondt: Peters and Skowron (2020) proved that in party-list elections, the only priceable committees are those selected by the D'Hondt method. As a consequence, the maximin support method coincides with the D'Hondt method for such elections.

Corollary 11. *The maximin support method is an extension of the D'Hondt apportionment method.*

6 Relationship to Phragmén's Rules

The maximin support method is by no means the only way to extend the D'Hondt method to approval-based multiwinner elections. As mentioned in the introduction, the rules of Phragmén and Thiele also generalize the D'Hondt method. In this section we focus on Phragmén's rules.[8]

Phragmén's methods can be described as *load distribution* methods (Brill et al. 2017). Every selected candidate induces one unit of *load*, and this load needs to be distributed among the approvers of that candidate. For example, if there are 6 voters approving candidate c and we decide to select c for the committee, then one possible way of distributing the load would be to give a load of $\frac{1}{6}$ to each of those voters. However, it is not required that the load is distributed evenly among the approvers: different approvers of c could be assigned different (non-negative) loads, as long as the loads associated with each selected candidate sum up to 1. The goal is to choose a committee such that the load distribution is as *balanced* as possible. Different interpretations of balancedness lead to different optimization goals; the most relevant variant *minimizes the maximal load* of a voter.

In particular, *minimax-Phragmén* is the rule that returns committees corresponding to load distributions minimizing the maximal voter load. And *seq-Phragmén* is a sequential version of minimax-Phragmén; it selects candidates iteratively, in each round adding a candidate to the committee such that the new maximal voter load is as small as possible.

Load Distributions Given an approval profile A and a subset $D \subseteq C$ of candidates, a *load distribution for D given A* is a two-dimensional array $\ell = (\ell_{i,c})_{i\in N, c\in D}$ satisfying[9]

$$\begin{aligned} 0 \le \ell_{i,c} \le 1 \qquad & \text{for all } i \in N \text{ and } c \in D, \\ \ell_{i,c} = 0 \qquad & \text{for all } i \in N \text{ and } c \in D \setminus A_i, \quad \text{and} \\ \sum_{i\in N} \ell_{i,c} = 1 \qquad & \text{for all } c \in D. \end{aligned}$$

We let $\mathcal{L}_{A,D}$ denote the set of all load distributions for (A, D). For a load distribution $\ell \in \mathcal{L}$, the *total load of voter i under ℓ*, denoted ℓ_i, is given by $\ell_i = \sum_{c\in D} \ell_{i,c}$. Note that $\sum_{i\in N} \ell_i = |D|$ for all $\ell \in \mathcal{L}_{A,D}$. Finally, a load distribution is called *optimal* for (A, D) if the maximal total voter load $\max_{i\in N} \ell_i$ is as small as possible. $\mathcal{L}^{\text{opt}}_{A,D}$ denotes the set of all optimal load distribution functions for (A, D).

We are now going to establishing a close connection between load distributions and support distribution functions.

Lemma 12. *Let A be an approval profile and $D \subseteq C$ a subset of candidates. Then, the following statements hold.*

1. *For every support distribution function $f \in \mathcal{F}_{A,D}$, there is a load distribution $\ell^f \in \mathcal{L}_{A,D}$ such that*
$$\max_{i\in N} \ell^f_i \le \frac{1}{\min_{c\in D} \mathit{supp}_f(c)}.$$
2. *For every load distribution $\ell \in \mathcal{L}_{A,D}$, there is a support distribution function $f^\ell \in \mathcal{F}_{A,D}$ such that*
$$\min_{c\in D} \mathit{supp}_{f^\ell}(c) \ge \frac{1}{\max_{i\in N} \ell_i}.$$

Proof. For a given support distribution function $f \in \mathcal{F}_{A,D}$, define the load distribution $\ell^f \in \mathcal{L}_{A,D}$ by setting $\ell^f_{i,c} = \frac{f(i,c)}{\mathit{supp}_f(c)}$ for each $i \in N$ and $c \in D$.[10] It follows that the total load of a voter is upper bounded by $\frac{1}{\mathit{supp}_f(c^*)}$, where c^* is a candidate with minimal support (recall that $\sum_c f(i, c) = 1$ for each voter i such that $A_i \cap D \neq \emptyset$).

For a given load distribution $\ell \in \mathcal{L}_{A,D}$, define a support distribution function $f^\ell \in \mathcal{F}_{A,D}$ by setting $f^\ell(i, c) = \frac{\ell_{i,c}}{\ell_i}$ for each voter $i \in N$ such that $\ell_i > 0$. That is, the support for a candidate is proportional to the load received from that candidate, scaled such that the total support by the voter is 1. It follows that the minimal support of a candidate is lower bounded by $\frac{1}{\ell_{i^*}}$, where i^* is a voter with maximal load. To see this, let i^* denote a voter with maximal load and let $N_\ell = \{i \in N : \ell_i > 0\}$. For $c \in D$, we get

$$\mathit{supp}_{f^\ell}(c) = \sum_{i\in N} f^\ell(i, c) \ge \sum_{i\in N_\ell} \frac{\ell_{i,c}}{\ell_i} \ge \frac{1}{\ell_{i^*}} \sum_{i\in N} \ell_{i,c} = \frac{1}{\ell_{i^*}}.$$

□

6.1 Phragmén's Optimal Rule

Lemma 12 has particularly interesting implications for load distributions and support distribution functions that are optimal: The construction used in the proof of Lemma 12 establishes a one-to-one correspondence between elements of $\mathcal{L}^{\text{opt}}_{A,D}$ and elements of $\mathcal{F}^{\text{opt}}_{A,D}$. Therefore, the objective of minimizing the maximal voter load is equivalent to the objective of maximizing the minimal support. As a consequence, minimax-Phragmén (the method that globally minimizes the maximal voter load) is identical to the rule that globally maximizes the minimal support.

Theorem 13. *Let $E = (N, C, A, k)$ be an approval-based multiwinner election. Then, minimax-Phragmén(E) = $\arg\max_{W\subseteq C, |W|=k} \mathit{maximin}(A, W)$.*

Since it is NP-hard to compute winners under minimax-Phragmén (Brill et al. 2017), the same is true for finding a set of candidates maximizing the minimum support. Brill et al. (2017) proved that minimax-Phragmén satisfies PJR (when combined with an appropriate tie-breaking rule) but not EJR. With respect to monotonicity axioms, Mora and Oliver (2015) proved that minimax-Phragmén

[8] Thiele's non-sequential rule, known as *Proportional Approval Voting (PAV)*, does not satisfy committee monotonicity, whereas Thiele's sequential rule (*sequential PAV*) does not satisfy PJR. On the other hand, PAV satisfies a stronger proportionality axiom known as *extended justified representation (EJR)*. Phragmén's rules and MMS fail EJR.

[9] Throughout this section we assume that $N_c \neq \emptyset$ for all $c \in D$; otherwise, load distributions cannot be defined.

[10] If $\mathit{supp}_f(c) = 0$ for some candidate c, the first part of the lemma trivially holds.

fails committee monotononicity and Sánchez-Fernández and Fisteus (2019) have extended previous results by Phragmén (1896) showing that minimax-Phragmén satisfies weak support monotonicity.

6.2 Phragmén's Sequential Rule

The rule seq-Phragmén can be viewed as a greedy algorithm for minimax-Phragmén (for details, we refer to the paper by Brill et al. 2017). There is a close relationship between the maximin support method and Phragmén's sequential rule. Both MMS and seq-Phragmén construct the set of winners by iteratively adding candidates: MMS chooses candidates such that the minimal support of the new set is maximized; seq-Phragmén chooses candidates such that the maximal voter load incurred by the new set is minimized. However, there is a subtle difference between the two methods concerning the *redistribution* of support/load. Under MMS, support distributed to candidates in earlier rounds can be freely redistributed when looking for maximin support distributions for the new set of candidates. This is not the case for the loads under seq-Phragmén, however: once a voter is assigned some load from some candidate, this load is "frozen" and will always stay with the voter. As a consequence, the two methods might give different results, as the following example illustrates.

Example 14. Consider election $E = (N, C, A, k)$ with $k = 4$ and $C = \{a_1, a_2, a_3, c_1, c_2, c_3, c_4\}$. There are 16 voters casting the following ballots:

$$\begin{array}{ll} 5 \times \{a_1, c_1, c_2, c_3, c_4\} & 2 \times \{a_1\} \\ 4 \times \{a_2, c_1, c_2, c_3, c_4\} & 1 \times \{a_2\} \\ 3 \times \{a_3, c_1, c_2, c_3, c_4\} & 1 \times \{a_3\} \end{array}$$

The committee according to the maximin support method is given by $MMS(E) = \{c_1, a_1, a_2, a_3\}$ (selected in this order), while seq-Phragmén selects (in this order) $\{c_1, c_2, c_3, a_1\}$.

It is straightforward to check that seq-Phragmén can be computed in polynomial time (Brill et al. 2017). With respect to the axiomatic properties considered in this paper, seq-Phragmén is indistinguishable from the maximin support method: seq-Phragmén satisfies committee monotonicity by definition; it satisfies PJR but fails EJR (Brill et al. 2017); and results by Phragmén (1896), Mora and Oliver (2015), and Janson (2016) imply that seq-Phragmén satisfies weak support monotonicity.

An interesting distinction between seq-Phragmén and the maximin support method concerns their ability to approximate the optimal solution of the maximin support problem. As recently shown by Cevallos and Stewart (2020), MMS provides a 2-approximation for this problem, whereas seq-Phragmén does not offer a constant-factor approximation.[11] To state these results formally, we let $OPT(A, k)$ denote the optimal maximin support value $\max_{W \subseteq C, |W|=k} maximin(A, W)$ for election (A, k), and H_k the k-th harmonic number $H_k = \sum_{i=1}^{k} 1/i$.

[11]Due to Lemma 12, the same bounds hold for the problem of minimizing the maximal voter load.

Proposition 15 (Cevallos and Stewart 2020)**.**

1. *$maximin(A, MMS(E)) \geq \frac{1}{2} OPT(A, k)$ for each election $E = (A, k)$.*
2. *For each committee size $k \in \mathbb{N}$ and each $\epsilon > 0$, there is an election $(A^{(k)}, k)$ such that $OPT(A^{(k)}, k) \geq (H_k - \epsilon) \cdot maximin(A^{(k)}, W_k)$, where W_k is the committee selected by seq-Phragmén in election $(A^{(k)}, k)$.*

Proposition 15 implies that seq-Phragmén can behave arbitrarily worse than minimax-Phragmén (and also than the maximin support method) in terms of maximizing the minimum support. We note that the maximin support value of a committee can be seen as a measure of its representativeness: the optimal value of $|N|/k$ can only be achieved when all voters are represented in the committee (in the sense that each voter approves at least one winning candidate) and, furthermore, the support can be evenly distributed among the committee members. Committees with smaller maximin support values can thus be interpreted as providing a lesser degree of representation. From this perspective, Proposition 15 shows an important advantage of the maximin support method compared to seq-Phragmén. Note, however, that this advantage comes at the price of increased computational complexity (see also Cevallos and Stewart 2020): as we have seen in Section 4, in each iteration of MMS, we need to solve one linear program for every remaining candidate.

7 Conclusion

We have proposed the maximin support method (MMS) as a novel extension of the D'Hondt method to approval-based multiwinner elections. Like the method of D'Hondt, MMS aims to maximize the support of the least supported winning candidate. We have shown that MMS can be computed efficiently and satisfies an attractive combination of axiomatic properties. In particular, we have argued that MMS strikes a balance between sufficiently representing the interests of cohesive voter groups, while at the same time trying not to overrepresent groups. We have also established a close relationship between MMS and Phragmén's rules. This novel connection allows us to formulate Phragmén's rules as support maximization (rather than load minimization) problems, and to view MMS as a tractable approximation of Phragmén's (intractable) optimal rule.

There are several intriguing questions for future work, including the following: Is the approximation factor of MMS for the optimal maximin support problem tight? Do there exist polynomial-time computable (and axiomatically desirable) voting rules providing a better approximation factor? Do there exist committee-monotonic rules satisfying stronger proportionality guarantees such as EJR?

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(Lund, June 2016). In this earlier version, the maximin support method was referred to as the *Open D'Hondt (ODH)* method. We would like to thank the anonymous reviewers at AAAI-21 and the participants of the COMSOC Video Seminar (December 2020) for valuable feedback and discussions.

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