

Dividing a Graphical Cake

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Abstract

We consider the classical cake-cutting problem where we wish to fairly divide a heterogeneous resource, often modeled as a cake, among interested agents. Work on the subject typically assumes that the cake is represented by an interval. In this paper, we introduce a generalized setting where the cake can be in the form of the set of edges of an undirected graph, allowing us to model the division of road networks. Unlike in the canonical setting, common fairness criteria such as proportionality cannot always be satisfied in our setting if each agent must receive a connected subgraph. We determine the optimal approximation of proportionality that can be obtained for any number of agents with arbitrary valuations, and exhibit a tight guarantee for each graph in the case of two agents. In addition, when more than one connected piece per agent is allowed, we establish the best egalitarian welfare guarantee for each total number of connected pieces. We also study a number of variants and extensions, including when approximate equitability is considered, or when the item to be divided is undesirable (also known as chore division).

1 Introduction

Cake cutting refers to the problem of fairly allocating a divisible resource, often modeled as a cake, among agents with varying preferences. The problem dates back to shortly after the end of World War II (Steinhaus 1948) and, not surprisingly given its wide range of applications, still enjoys significant attention in mathematics, computer science, economics, and political science to this day (Brams and Taylor 1996; Robertson and Webb 1998; Procaccia 2016).

What does it mean for an allocation to be fair? Steinhaus (1948) proposed the following definition of fairness: if the cake is divided between n agents, each agent should receive a part that she values at least $1/n$ of the entire cake. This definition became known as *proportionality*, and is one of the most fundamental notions in the literature of fair division. In his seminal article, Steinhaus showed that a proportional allocation can be found for any number of agents with arbitrary preferences over the cake—Steinhaus’ method, which he attributed to Knaster and Banach, was later formulated as a moving-knife procedure by Dubins and Spanier (1961). The procedure works by having a referee move a knife over

the cake from left to right. Whenever the left part has value $1/n$ of the entire cake for one of the agents, the agent takes that part of the cake and leaves; the procedure is then repeated among the remaining agents. In addition to ensuring proportionality, the Dubins-Spanier protocol has the important property that it always allocates to every agent a *connected* piece of the cake. Without this property, it may well be that an agent is presented with—in the famous words of Stromquist (1980)—a “union of crumbs”.

The proportionality guarantee of the Dubins-Spanier protocol holds for a variety of resources that one may wish to divide, since a resource of any shape or form can be “projected” onto a line, which we can then run the protocol on. However, the connectivity property does not necessarily translate from the line back to the original shape. A simple illustrating example is when the resource has the shape of a ring (e.g., a donut or a ring road): projecting the ring onto a line and applying Dubins-Spanier may result in an agent receiving disconnected pieces of the ring. For this example, the difficulty can be circumvented by cutting the ring at an arbitrary point, stretching it into a line, and running the protocol to achieve proportionality. Nevertheless, one may already begin to suspect that this type of fix no longer works when the shapes get more complex.

In this paper, we consider a natural setting where the cake is represented by the set of edges of an arbitrary undirected graph. The graph corresponds to a resource in the form of a network, such as a road network to be divided among construction companies. Each company has its own preference on different parts of the network, and it is desirable for every company to receive a single connected component, or not too many such components. The canonical cake-cutting setting—where the cake is assumed to be an interval—is a special case of our setting, with the graph consisting of a single edge. We show that proportionality cannot always be attained in our generalized setting if connectivity is required. Therefore, our goal in this work is to provide the best possible approximation of proportionality that can be achieved for arbitrary graphs and agents’ preferences. Furthermore, we study a number of variants and extensions, including when each agent can get more than one connected piece—we show that better guarantees can indeed be obtained as the number of permitted pieces increases—or when the item to be divided is undesirable (often referred to as a *chore*).

1.1 Our Results

We assume throughout the paper that the agents are endowed with valuation functions that are additive and normalized with each agent having value 1 for the entire cake; these assumptions are standard in the cake-cutting literature (Procaccia 2016). We also assume that the pieces of cake that different agents receive may intersect in a finite number of points. Our formal model is described in Section 2.

In Section 3, we provide general guarantees for any number of agents. For n agents with arbitrary valuations and any graph, we establish the existence of a connected allocation that gives every agent a utility of at least $\frac{1}{2n-1}$. We also show that this bound is tight even when the agents have identical valuations and the graph is a star with $2n - 1$ edges. In addition, for every specific star graph, we determine the optimal utility that can be guaranteed for each number of agents.

In Section 4, we delve deeper into the case of two agents. While we know from Section 3 that both agents can be guaranteed a utility of $1/3$ in general, for certain graphs it is possible to do better. Perhaps surprisingly, we show that the optimal guarantee for each graph is always either $1/3$ or $1/2$; the latter case corresponds to a proportional allocation. The classification depends on a graph property that we call *almost bridgeless*—a graph satisfies this property if we can add an edge so that the resulting graph contains no “bridges”, where a bridge refers to an edge that is not contained in any cycle. We show that a guarantee of $1/2$ can be obtained if the graph is almost bridgeless, while $1/3$ is the best possible guarantee otherwise. Moreover, we consider allocating more than one connected piece to each agent in arbitrary graphs: we show that for any positive integer k , both agents can be guaranteed a utility of $\frac{1}{2} - \frac{1}{2 \cdot 3^k}$ if we allow the agents to receive a total of $k + 1$ connected pieces, and this is tight. We also study approximate *equitability* and prove the existence of a connected allocation for which the agents’ utilities differ by no more than $1/3$; we again establish the tightness of the bound.

Finally, in Section 5, we turn our attention to chore division. In the case of two agents there is a simple reduction between the settings of cake and chores, so all of our results in Section 4 carry over to chore division. By contrast, when there are more than two agents, the relationship between the two settings is much less clear. We show that there exists a connected allocation that incurs cost at most $\frac{2}{n+1}$ for $n \leq 5$, and that no better bound can be obtained for any n .

We remark that all of our positive results are constructive: for each result, we exhibit a moving-knife protocol that achieves the desired guarantee. Moreover, one can make these protocols discrete, so that they only use the *cut* and *evaluation* queries allowed by the Robertson-Webb query model in order to access the valuation functions of the agents (Robertson and Webb 1998).

1.2 Further Related Work

While our graphical cake model is new to the best of our knowledge, graphs have been studied in the context of allocating *indivisible* items. In particular, the items correspond to vertices of an undirected graph, and each agent must be

allocated a connected subgraph of the graph. A line of work has explored existence and complexity questions for several fairness notions, both in the case of goods (Bouveret et al. 2017; Lonc and Truszczynski 2018; Bilò et al. 2019; Igarashi and Peters 2019; Suksompong 2019; Bei et al. 2021) and chores (Bouveret, Cechlárová, and Lesca 2019).

Connectivity constraints are commonly considered in the cake-cutting literature, where each agent is allocated a single subinterval of the interval cake (Stromquist 1980, 2008; Su 1999; Bei et al. 2012; Cechlárová and Pillárová 2012; Aumann, Dombb, and Hassidim 2013; Cechlárová, Doboš, and Pillárová 2013; Aumann and Dombb 2015). Segal-Halevi et al. (2017) studied the fair division of land and introduced geometric constraints to the setting by requiring that allocated pieces be of certain shape—such requirements are important since a long but narrow piece of land is likely to be of little practical use. Similarly to our setting, a proportional allocation does not always exist in the presence of these constraints, and the authors examined the approximations of proportionality that can be obtained.

A related line of work also considers a graphical model of resource allocation, but uses graphs to model the relationship between agents instead of the resource (Abebe, Kleinberg, and Parkes 2017; Bei, Qiao, and Zhang 2017; Chevalleyre, Endriss, and Maudet 2017; Aziz et al. 2018). Specifically, a graph represents the acquaintance relationship between the agents, and agents only evaluate their shares relative to those of other agents with whom they are acquainted.

2 Preliminaries

Let $N = \{1, 2, \dots, n\}$ be the set of agents, and $G = (V, E)$ be a finite and connected undirected graph representing the cake, with no loops but possibly with multiple edges joining the same pair of vertices.¹ Denote by m the number of edges. Each edge in E can be viewed as an interval of the cake. For any points x, y on an edge, we write $[x, y]$ or $[y, x]$ to denote the interval of the cake between x and y ; we sometimes identify a vertex $v \in V$ with the corresponding endpoint of the edges adjacent to v .

A *piece of cake* is a finite union of disjoint intervals, where each interval is a subinterval of an edge and the intervals in the piece may belong to different edges. As is commonly done in cake cutting, we assume that all intervals are closed intervals. Two intervals are said to be *disjoint* if they intersect in at most one point, and two pieces of cake are said to be *disjoint* if they intersect in a finite number of points.² A piece of cake is *connected* if for any two points in the piece, it is possible to get from one point to the other along the graph G by only traversing this piece of cake. Each agent

¹A loop can be represented in our model by adding a new vertex inside the loop, thereby breaking the loop into two edges joining the same pair of vertices.

²If we adopt the stricter convention that each point can only be allocated to one agent (so intervals can be open, half-open, or closed) and two intervals are “disjoint” only if their intersection is empty, there are strong negative results. For example, on a star graph, at most one agent would be able to receive intervals from more than one branch in a connected allocation.

i has a nonnegative *valuation function* (or *utility function*) f_i , which specifies the agent’s value for each piece of cake. An *instance* consists of the graph G , the agents, and their valuation functions. We assume that the valuation functions are

- *normalized*: the value of an agent for the entire cake is 1;
- *divisible*: for each interval $[x, y]$ and $0 \leq \lambda \leq 1$, there is a point $z \in [x, y]$ such that $f_i([x, z]) = \lambda \cdot f_i([x, y])$;
- *additive*: the value of an agent for a piece of cake is the sum of her values for the intervals in the piece.

These assumptions are standard in the cake-cutting literature (Procaccia 2016).

An *allocation* of the cake is denoted by a vector $A = (A_1, \dots, A_n)$, where each A_i is a piece of cake, and A_i and A_j are disjoint for all $i \neq j$. An allocation is said to be *complete* if the entire cake is allocated, and *connected* if each A_i is a connected piece of cake. The *egalitarian welfare* of an allocation is defined as $\min_{i \in N} f_i(A_i)$. An allocation is *proportional* if its egalitarian welfare is at least $1/n$. The *inequity* of an allocation is defined as $\max_{i, j \in N} |f_i(A_i) - f_j(A_j)|$; an allocation with inequity 0 is said to be *equitable*.

We make analogous assumptions for chore division (Section 5). Each agent has a nonnegative *cost function* f_i for the chore, which is normalized, divisible, and additive. The *egalitarian cost* of an allocation is defined as $\max_{i \in N} f_i(A_i)$. Naturally, we require the entire chore to be allocated, so we restrict our attention to complete allocations of the chore.

All omitted proofs can be found in the full version of this paper (Bei and Suksompong 2019).

3 Any Number of Agents

In this section, we present an egalitarian welfare guarantee that holds for any number of agents and arbitrary graphs, and derive improved guarantees when the graph is a star.

We begin by showing that it is always possible to give every agent a utility of at least $\frac{1}{2n-1}$, and this bound is tight. Similarly to the Dubins-Spanier protocol, our algorithm proceeds by identifying a piece that is valuable enough for one agent but at the same time not too valuable for the other agents, allocating such a piece to the former agent, and recursing on the latter agents.

Theorem 3.1. *For any graph G , there exists a connected allocation with egalitarian welfare at least $\frac{1}{2n-1}$. On the other hand, there exists a graph G and identical valuations of the agents such that any connected allocation yields egalitarian welfare at most $\frac{1}{2n-1}$.*

Proof. Let $\alpha := \frac{1}{2n-1}$. To show the second part of the theorem, let G be a star with $2n - 1$ edges such that every agent values each edge exactly α , and the value is distributed uniformly within the edge. Assume for contradiction that there is a connected allocation with egalitarian welfare strictly greater than α . Consider any agent i . The agent must receive intervals from at least two edges, and these intervals must be connected via the center vertex. Note that the unallocated parts of these edges cannot be allocated to other

agents, since any agent who receives an interval from such a part cannot receive intervals from other edges and would therefore obtain value less than α . Hence, at least two edges are only allocated to agent i . However, this means that there must be at least $2n$ edges in total, a contradiction.

We now prove the first part of the theorem. Let G be an arbitrary graph. We will show that there exists a moving-knife algorithm that produces a connected allocation with egalitarian welfare at least α . We proceed by induction on n ; the statement trivially holds for $n = 1$ since we can simply allocate the entire cake to the only agent. Assume that the statement holds for $n - 1$ agents, and consider an instance with n agents.

First, we claim that we can turn G into a tree. As long as G contains at least one cycle, pick an edge uv that belongs to a cycle, add a new vertex v' , and replace this edge by an edge uv' while keeping the remaining edges of the graph as before (see Figure 1). Since at least one cycle is removed by this operation and no new cycle is created, G eventually becomes a tree. Note that any connected allocation of the modified graph is a connected allocation in the original graph with the same value for every agent, so it suffices to prove the theorem for the modified graph.

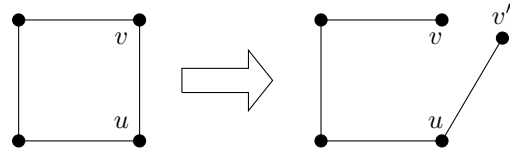


Figure 1: The cycle-removing operation

Choose an arbitrary vertex u of the tree G as its root. Let v be a vertex such that the subtree rooted at v yields value at least α to some agent, and the same does not hold for the subtree rooted at any child of v . Let w_1, \dots, w_k be the children of v . We consider two cases:

- **Case 1:** At least one of the k branches of v along with the corresponding subtree yields value at least α to some agent. Assume without loss of generality that the branch containing w_1 is one such branch. By our assumption, the subtree rooted at w_1 yields value less than α to all agents. Hence, by moving a knife from w_1 to v , we can find the point x closest to w_1 such that some agent i values the interval $[w_1, x]$ together with the subtree rooted at w_1 exactly α , and all other agents value this piece of cake at most α . We allocate this piece of cake to agent i , and make x a new vertex in the remaining graph, which has value at least $1 - \alpha$ for each of the remaining agents. By the inductive hypothesis, there exists a connected allocation of the remaining graph to the $n - 1$ agents such that every agent receives value at least $\frac{1}{2n-3} \cdot (1 - \alpha) = \frac{1}{2n-3} \cdot \frac{2n-2}{2n-1} > \alpha$, as desired.
- **Case 2:** Every branch of v along with the corresponding subtree yields value less than α to all agents. Let $t \in \{1, 2, \dots, k\}$ be the smallest number such that the first t branches and their subtrees together yield value at least α to some agent i . We allocate this piece of cake to

agent i . For every other agent, the first $t - 1$ branches are worth less than α and the t th branch is also worth less than α , so the piece of cake allocated to agent i is worth less than 2α . By the inductive hypothesis, there exists a connected allocation of the remaining graph to the $n - 1$ agents such that every agent receives value at least $\frac{1}{2n-3} \cdot (1 - 2\alpha) = \frac{1}{2n-1} = \alpha$, as desired.

The two cases together complete the induction. \square

For certain graphs, it is possible to improve upon the guarantee provided by Theorem 3.1—an obvious example is the graph consisting of a single edge, for which the Dubins-Spanier protocol yields an egalitarian welfare of at least $1/n$. We derive the optimal egalitarian welfare guarantee in the case where the graph is a star. For integers $n \geq 2$ and $k \geq 3$, define

$$f(n, k) = \begin{cases} \frac{1}{n + \lceil k/2 \rceil - 1} & \text{for } k < 2n - 1; \\ \frac{1}{2n - 1} & \text{for } k \geq 2n - 1. \end{cases}$$

Theorem 3.2. *Let $n \geq 2$ and $k \geq 3$, and let G be a star with k edges. There exists a connected allocation with egalitarian welfare at least $f(n, k)$. Moreover, the bound $f(n, k)$ is tight.*

4 Two Agents

In this section, we focus on the case of two agents. We establish the optimal egalitarian welfare that can be obtained for each graph. In addition, we explore the extent to which our guarantees can be improved if we allow more than one connected piece per agent, and also consider approximate equitability.

First, we make a general observation that the well-known “cut-and-choose” protocol allows us to find an allocation that gives utility $1/2$ to the first agent and $1/3$ to the second agent; this strengthens the case $n = 2$ of Theorem 3.1.

Theorem 4.1. *For $n = 2$ and any graph G , there exists a connected allocation such that the first agent receives value at least $1/2$ and the second agent receives value at least $1/3$.*

Proof. By Theorem 3.1, there exists a partition of the cake into two connected pieces such that the second agent values both pieces at least $1/3$. The first agent can then choose the piece that she prefers and obtain value at least $1/2$. \square

4.1 Graph-Specific Guarantees

Before we can state our results for specific graphs, we need some graph-theoretic terminology. Recall that a *bridge* of a graph is an edge that is not contained in any cycle. A graph is said to be *bridgeless* if it contains no bridges.

Definition 4.2. A graph is said to be *almost bridgeless* if we can add an edge so that the resulting graph is bridgeless.

Note that according to this definition, every connected bridgeless graph with at least two vertices is also almost bridgeless, since we can add a copy of an existing edge. A single edge is almost bridgeless, and so is a cycle of any length, whereas a star with at least three edges is not.

Next, we define an oriented labeling of a graph.

Definition 4.3. An *oriented labeling* of a graph with m edges is a labeling of the edges with numbers $1, 2, \dots, m$, using each number exactly once, together with a labeling of one endpoint of each edge i with i^- and the other endpoint with i^+ (so each vertex receives a number of labels equal to the number of edges adjacent to it). An oriented labeling is said to be *contiguous* if:

- For each $2 \leq i \leq m$, the edges labeled $1, 2, \dots, i - 1$ form a connected subgraph, and the vertex labeled i^- belongs to one of these edges.
- For each $1 \leq i \leq m - 1$, the edges labeled $i + 1, i + 2, \dots, m$ form a connected subgraph, and the vertex labeled i^+ belongs to one of these edges.

An example is illustrated in Figure 2.

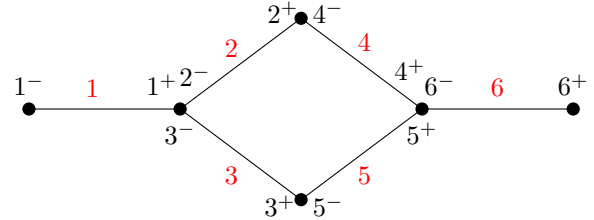


Figure 2: Example of an oriented labeling

As the following lemma shows, it turns out that a graph admitting a contiguous oriented labeling is equivalent to it being almost bridgeless.

Lemma 4.4. *A graph is almost bridgeless if and only if it admits a contiguous oriented labeling.*

Given a graph G with k vertices, a *bipolar numbering* of G is a labeling of the vertices with numbers $1, 2, \dots, k$, with each number used exactly once, such that every vertex with label greater than 1 has a neighbor with a smaller label and each vertex with label smaller than k has a neighbor with a larger label. Bilò et al. (2019) characterized the class of graphs that admit a bipolar numbering as the graphs with the property that if the vertices of the graph represent indivisible items of possibly different values to the two agents, then there always exists a connected “envy-free up to one item” allocation. We show that the class of graphs that admit a bipolar numbering forms a strict subclass of the almost bridgeless graphs (which, by Lemma 4.4, is equivalent to the class of graphs that admit a contiguous oriented labeling).

Proposition 4.5. *Any graph that admits a bipolar numbering is almost bridgeless, but the converse does not hold.*

We are now ready to show our classification result: the optimal egalitarian welfare that can always be obtained for a graph is $1/2$ if the graph is almost bridgeless, and $1/3$ otherwise. The former is shown in Theorem 4.6, while the latter follows from Theorems 3.1 and 4.8.

Theorem 4.6. *For $n = 2$ and any almost bridgeless graph G , there exists a connected proportional allocation.*

Proof. Suppose that G is almost bridgeless. By Lemma 4.4, it admits a contiguous oriented labeling. We move a knife

over the edges of G in increasing order of the label. For each edge with label i , the knife goes from vertex i^- to vertex i^+ . If the knife is currently on edge i , we stop when the piece of cake containing edges $1, 2, \dots, i - 1$, together with the interval of edge i between i^- and the current position of the knife, yields value exactly $1/2$ to one of the agents. We allocate this piece of cake to the agent who receives value $1/2$, and the remainder of the cake to the other agent. Both agents receive value at least $1/2$ and, by definition of the labeling, obtain a connected allocation of the cake. \square

Lemma 4.7. *Let F be a nonempty set of bridges in a graph. If no path contains all bridges in F , there exist three bridges in F such that no path contains all three bridges.*

Theorem 4.8. *For $n = 2$ and any graph G that is not almost bridgeless, there exist identical valuation functions of the two agents such that any connected allocation yields egalitarian welfare at most $1/3$.*

Proof. Suppose that G is not almost bridgeless. Then no path can contain all bridges of G : if there exists such a path, we can eliminate all bridges by adding an edge that connects the endpoints of this path. By Lemma 4.7, there exist three bridges of G such that no path contains all three bridges. For each of the three bridges, the other two bridges must lie on the same side of it, since otherwise we can construct a path that contains all three bridges. See Figure 3.

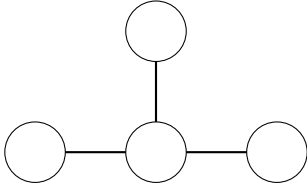


Figure 3: Illustration for the proof of Theorem 4.8. The circles denote components and the edges correspond to the three bridges.

Assume that both agents value each of the three bridges exactly $1/3$ (and every other edge 0), and the value is distributed uniformly within each bridge. Suppose for contradiction that there exists a connected allocation with egalitarian welfare strictly greater than $1/3$. This means that each agent must receive intervals from at least two bridges. However, when an agent receives intervals from two bridges, each interval must contain the endpoint of the bridge that is on the same side as the other two bridges. This is impossible since there are only three bridges, yielding the desired contradiction. \square

4.2 Beyond One Connected Piece

As we mentioned in the introduction, an important motivation for considering connected allocations is to avoid situations where an agent receives a “union of crumbs”. In light of this motivation, it is interesting to explore whether we can obtain improved guarantees if we allow the agents to receive a small number of connected pieces. We demonstrate

in this subsection that such improvements are indeed possible by presenting a tight bound of $\frac{1}{2} - \frac{1}{2 \cdot 3^k}$ on the egalitarian welfare that can be guaranteed when a total of $k + 1$ connected pieces are permitted. We first establish the lower bound. Note that by following the algorithm in the proof of Theorem 3.1, we obtain the following lemma, which we will use in the proof of the lower bound.

Lemma 4.9. *Let H be a connected piece of cake in a graph G , and suppose that all agents have value exactly x for H . For any $\alpha \leq x$, there exists a partition of H into two connected pieces such that one of the agents has value at least α for the first piece, while all of the remaining agents have value at most 2α for this piece.*

Theorem 4.10. *Let k be a positive integer. For $n = 2$ and any graph G , there exists an allocation in which the two agents receive a total of at most $k + 1$ connected pieces and the egalitarian welfare is at least $\frac{1}{2} - \frac{1}{2 \cdot 3^k}$.*

Proof. Let G be an arbitrary graph. It suffices to show that there exists a partition of G into two parts with at most $k + 1$ connected pieces in total such that both parts yield value at least $\frac{1}{2} - \frac{1}{2 \cdot 3^k}$ to the first agent. Indeed, given such a partition, we can let the second agent choose the part that she prefers and obtain value at least $1/2$. We therefore consider only the first agent from now on.

We proceed by induction on k ; the base case $k = 1$ follows from Theorem 4.1. Suppose that the statement holds for $k - 1$, i.e., there exists a partition of G into two parts with at most k connected pieces in total such that both parts yield value at least $\frac{1}{2} - \frac{1}{2 \cdot 3^{k-1}}$ to the agent. Assume without loss of generality that the second part has value at least $1/2$, so the first part has value $\frac{1}{2} - x$ for some $0 \leq x \leq \frac{1}{2 \cdot 3^{k-1}}$. Since the second part consists of at most $k - 1$ connected pieces, it contains a connected piece of value at least $\frac{1}{2k-2}$. Denote this piece by H .

Since $2k - 2 \leq 3^k$, we have $\frac{2x}{3} \leq \frac{1}{3^k} \leq \frac{1}{2k-2}$. By creating a duplicate of our agent and applying Lemma 4.9 with $\alpha = 2x/3$, we can partition H into two connected pieces in such a way that our agent has value in the range $[\frac{2x}{3}, \frac{4x}{3}]$ for the first piece. Move this piece from the second part of our partition of G to the first part. The resulting partition of G consists of at most $k + 1$ connected pieces in total, and the first part of this partition has value in the range $[\frac{1}{2} - \frac{x}{3}, \frac{1}{2} + \frac{x}{3}]$. This implies that both parts of the partition yield value at least $\frac{1}{2} - \frac{x}{3} \geq \frac{1}{2} - \frac{1}{2 \cdot 3^k}$, completing the induction. \square

Next, we show that the bound established in Theorem 4.10 is tight for every k . First we need the following technical lemma.

Lemma 4.11. *Let t be a positive integer, and let a_1, a_2, \dots, a_t be (not necessarily distinct) integers and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t \in \{\pm 1, \pm 2\}$. Then*

$$\left| \varepsilon_1 \cdot 3^{a_1} + \varepsilon_2 \cdot 3^{a_2} + \dots + \varepsilon_t \cdot 3^{a_t} - \frac{1}{2} \right| \geq \frac{1}{2 \cdot 3^t}.$$

Theorem 4.12. *Let k be a positive integer. There exists an instance with $n = 2$ and identical valuations such that any allocation in which the two agents receive a total of at most*

$k + 1$ connected pieces yields egalitarian welfare at most $\frac{1}{2} - \frac{1}{2 \cdot 3^k}$.

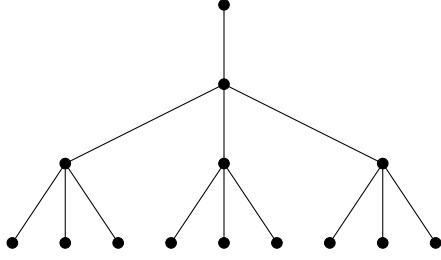


Figure 4: The graph in the proof of Theorem 4.12 for $k = 2$.

Proof. Let G be a rooted tree with $k + 2$ layers, where the first layer consists only of the root of the tree. The root has one child, and every vertex in subsequent layers up to the $(k + 1)$ st layer has three children. In particular, the $(k + 2)$ nd layer consists of 3^k leaves. The tree G for the case $k = 2$ is shown in Figure 4. Suppose that both agents value each edge adjacent to a leaf exactly $1/3^k$ with the value distributed uniformly within the edge, and do not value any other edge.

Consider an arbitrary allocation in which the two agents receive a total of at most $k + 1$ connected pieces. We will show that the egalitarian welfare is at most $\frac{1}{2} - \frac{1}{2 \cdot 3^k}$. If there are unallocated parts of the cake, we arbitrarily allocate these parts so that the number of connected pieces that each agent receives does not increase; this does not lower the egalitarian welfare of the allocation. Hence we may assume that the entire cake is allocated (i.e., the allocation is complete). Denote by X_1, \dots, X_p the connected pieces that agent 1 receives, and Y_1, \dots, Y_q the connected pieces that agent 2 receives, where $p + q \leq k + 1$. We assume that each edge has length 1, and refer to the distance along the (unique) path between two points in G simply as the distance between these two points. Note that every connected piece has a unique point closest to the root: if there are two such points, they must be connected via a point that is strictly closer to the root than both of them. For every connected piece Z , denote by w_z the unique point in Z closest to the root and $u(Z)$ the utility of the piece Z . We will define a connected piece Z^* as follows:

- If w_z is not a vertex of G , let Z^* be the set of points w such that the path from w to the root goes through w_z . In other words, Z^* is the part of the tree “below” w_z .
- If w_z is a vertex of G , let Z^* be the set of points w such that the intersection of Z and the path from w to the root has nonzero measure. Equivalently, Z^* consists of the edges adjacent to w_z that have a nontrivial overlap with Z , along with everything “below” these edges.

Assume without loss of generality that agent 1 receives a piece containing the root of the tree. We claim that $u(X_1) + \dots + u(X_p) = [u(X_1^*) + \dots + u(X_p^*)] - [u(Y_1^*) + \dots + u(Y_q^*)]$. First, note that for each Z^* where $Z \in \{X_1, \dots, X_p, Y_1, \dots, Y_q\}$, every connected piece X_i and Y_i is either contained in Z^* in its entirety or not at all.

Hence $u(Z^*)$ can be written as a sum of distinct $u(X_i)$'s and $u(Y_i)$'s. Moreover, one can verify from the definition that a connected piece Z is contained in W^* if and only if $Z = W$ or the path from w_z to the root has a nontrivial overlap with W . This path alternates between pieces X_i and Y_i and ends with a piece X_i . Therefore, each $u(X_i)$ is contained in $u(X_1^*) + \dots + u(X_p^*)$ exactly once more than in $u(Y_1^*) + \dots + u(Y_q^*)$, while each $u(Y_i)$ is contained in the two sums an equal number of times. This yields the claimed equality.

Next, observe that each $u(Z^*)$ can be written as $3^s/3^k$ for some nonnegative integer $s \leq k$, or $2 \cdot 3^s/3^k$ for some nonnegative integer $s \leq k - 1$, or $\delta/3^k$ for some $\delta \in [0, 1]$. Note also that since agent 1 receives a piece containing the root of the tree, for this piece X_i we have $u(X_i^*) = 1$. It follows that $[u(X_1^*) + \dots + u(X_p^*)] - [u(Y_1^*) + \dots + u(Y_q^*)]$ can be written as $1 - (S + \Delta)$, where S is a sum of a number of terms (say, r terms, where $r \leq k$) of the form $\varepsilon \cdot 3^a$ with $\varepsilon \in \{\pm 1, \pm 2\}$ and a an integer, and $|\Delta| \leq (k - r)/3^k$. Hence, letting $d := |u(X_1) + \dots + u(X_p) - \frac{1}{2}|$, we have

$$\begin{aligned} d &= \left| \sum_{i=1}^p u(X_i^*) - \sum_{i=1}^q u(Y_i^*) - \frac{1}{2} \right| \\ &= \left| (S + \Delta) - \frac{1}{2} \right| \geq \left| S - \frac{1}{2} \right| - |\Delta| \\ &\geq \frac{1}{2 \cdot 3^r} - \frac{k - r}{3^k} = \frac{3^{k-r} - 2(k - r)}{2 \cdot 3^k} \geq \frac{1}{2 \cdot 3^k}, \end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality follows from Lemma 4.11, and the last inequality holds since $3^b \geq 2b + 1$ for any integer $b \geq 0$. This implies that the egalitarian welfare is at most $\frac{1}{2} - \frac{1}{2 \cdot 3^k}$, as claimed. \square

Recall that the *height* of a rooted tree is the length of the longest path from the root to a leaf vertex. For example, a star rooted at the center vertex has height 1. Our next theorem shows that for graphs that can be represented as a tree of height at most 2, we can obtain full proportionality provided that we allow two connected pieces per agent.

Theorem 4.13. *For $n = 2$ and any graph G that can be represented as a rooted tree of height at most 2, there exists a proportional allocation such that each agent receives at most two connected pieces.*

4.3 Equitability

We end this section by briefly considering another well-established fairness notion: equitability. Interestingly, while for approximate proportionality it is useful to consider the *maximum* among the agents' values for the current piece (Theorem 3.1), for approximate equitability in the case of two agents, the appropriate quantity to consider is the *sum* of these values. Note that an empty allocation is always equitable but yields the lowest possible welfare of zero, so we are interested in complete allocations.

Theorem 4.14. *For $n = 2$ and any graph G , there exists a complete and connected allocation with inequity at most $1/3$. Moreover, the bound $1/3$ is tight.*

Proof. To show tightness, let G be a star with three edges such that every agent values each edge exactly $1/3$, and the value is distributed uniformly within the edge. In any complete and connected allocation, one of the agents must receive two full edges (and possibly part of the third). This implies that the inequity in such an allocation is at least $1/3$.

We now prove the first part of the theorem. Let G be an arbitrary graph. Our goal is to find a complete, connected allocation (A_1, A_2) such that $|f_1(A_1) - f_2(A_2)| \leq 1/3$. Since the allocation is complete, we may write $f_2(A_2) = 1 - f_2(A_1)$. The desired condition can be rewritten as $2/3 \leq f_1(A_1) + f_2(A_1) \leq 4/3$. We use a similar procedure as in Theorem 3.1, turning the graph into a tree and considering a minimal subtree that has value at least $2/3$ with respect to $f_1 + f_2$. We stop either when the knife cuts a subtree A_1 such that $f_1(A_1) + f_2(A_1) = 2/3$ (Case 1), or when a set of branches A_1 satisfies $f_1(A_1) + f_2(A_1) \geq 2/3$ for the first time (Case 2). An analogous argument shows that we must have $f_1(A_1) + f_2(A_1) \leq 2/3 + 2/3 = 4/3$, which yields the desired inequality. \square

5 Chore Division

In this section, we assume in contrast to the previous sections that the graph represents a *chore*, i.e., an item that yields negative value to the agents. This models, for example, a situation where we wish to divide the responsibilities of maintaining a road network among relevant organizations.

For the case of two agents, all results in cake cutting (Section 4) can be translated to analogous results in chore division using a simple reduction. The idea is that given a chore instance, we can turn it into a cake instance by pretending that the cost functions are cake valuation functions, applying a result in the cake setting to obtain an initial allocation of the chore, and having the agents swap their assigned pieces to arrive at the final allocation. This reduction works for translating positive results to the chore setting. For negative results, we can use the reduction in the opposite direction, starting from a chore instance and reducing it to a cake instance. As an illustrating example, we show how to deduce an analogue of Theorem 4.1 in the chore setting.

Theorem 5.1. *In chore division, for $n = 2$ and any graph G , there exists a connected allocation such that the first agent incurs cost at most $1/2$ and the second agent incurs cost at most $2/3$.*

Proof. Consider an arbitrary chore division instance. If we treat the chore valuations as cake valuations, then by Theorem 4.1, there exists a (complete) connected allocation such that the first agent receives value at least $1/2$ and the second agent receives value at least $1/3$. Let the agents swap their assigned pieces in this allocation. In the resulting allocation, which is also connected, the first agent incurs cost at most $1 - 1/2 = 1/2$ and the second agent incurs cost at most $1 - 1/3 = 2/3$. \square

When there are more than two agents, the relationship between the cake and the chore settings becomes much less clear, and we do not know how to translate results from one

setting to the other. In the chore setting, we show that for each n , the egalitarian cost may need to be as high as $\frac{2}{n+1}$.

Proposition 5.2. *In chore division, there exists a graph G and identical valuations of the agents such that any connected allocation yields egalitarian cost at least $\frac{2}{n+1}$.*

The bound $\frac{2}{n+1}$ is tight for $n = 2$ due to Theorem 5.1. The following theorem shows that it remains tight as long as $n \leq 5$.

Theorem 5.3. *In chore division, for $n \leq 5$ and any graph G , there exists a connected allocation with egalitarian cost at most $\frac{2}{n+1}$.*

The proof of Theorem 5.3 involves some elaborate analysis and is left to the full version (Bei and Suksompong 2019). Here we give a simpler protocol for the case $n = 3$, yielding egalitarian cost at most $1/2$. First, pick two arbitrary agents. By Theorem 5.1, there exists a connected allocation to the two agents such that the first agent incurs cost at most $1/2$ and the second agent incurs cost at most $2/3$. Fix the piece assigned to the first agent, and divide the piece assigned to the second agent further between the second and third agents. By Theorem 5.1 again, there exists a connected allocation of the latter piece such that the third agent incurs cost at most $1/2$ and the second agent incurs cost at most $(2/3) \cdot (2/3) = 4/9 < 1/2$. Hence the egalitarian cost of the resulting allocation is at most $1/2$.

We conjecture that the bound $\frac{2}{n+1}$ is tight for all n , and leave it as an intriguing open question.

6 Conclusion and Future Work

In this paper, we introduce and study a generalized version of the classical cake-cutting problem, where the cake can be represented by an arbitrary graph instead of an interval. We establish bounds on the utilities that can be guaranteed to the agents for various classes of graphs, both for cake cutting and chore division, and demonstrate in several cases that our guarantees are tight. We also show that better guarantees are possible if we allow more connected pieces per agent, and exhibit an algorithm that computes an approximately equitable allocation.

Our work opens up a number of new directions for future research. Besides proportionality and equitability, another prominent fairness notion is *envy-freeness*, which stipulates that no agent prefers another agent's bundle to her own in the allocation. In the case of two agents, envy-freeness and proportionality are equivalent, and approximate proportionality bounds readily translate to corresponding approximate envy-freeness results. However, this equivalence ceases to hold when there are more than two agents. If the graph consists of a single edge, a connected envy-free allocation always exists for any number of agents (Stromquist 1980). It would be interesting to see whether one can obtain (approximate) envy-freeness guarantees for different classes of graphs. Furthermore, one could consider dividing a mixture of goods and chores (Bogomolnaia et al. 2017; Segal-Halevi 2018; Aziz et al. 2019) in our model as well.

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