

Optimal Pricing for Distance-Based Transit Fares

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Abstract

Numerous urban planners advocate for differentiated transit pricing to improve both ridership and service equity. Several metropolitan cities are considering switching to a more “fair fare system”, where passengers pay according to the distance travelled, rather than a flat fare or zone boundary scheme that discriminates against various marginalized groups.

In this paper, we present a two-part optimal pricing formula for switching to distance-based transit fares: the first formula maximizes forecasted revenue given a target ridership, and the second formula maximizes forecasted ridership given a target revenue. Both formulas hold for all price elasticities.

Our theory has been successfully tested on the SkyTrain mass transit network in Metro Vancouver, British Columbia, with over 400,000 daily passengers. This research has served Metro Vancouver’s transportation authority as they consider changing their fare structure for the first time in over 30 years.

Introduction

Nearly every metropolitan city in the world has a mass transit system, to reduce car congestion and improve environmental sustainability. In the United States, over ten billion trips are made each year on buses and trains, a 37% increase in public transit ridership over the past two decades, nearly double the population growth rate during this same period (American Public Transportation Association 2017).

There are several ways to charge passengers for public transit. The two simplest systems are a *flat fare* that charges a fixed amount regardless of distance travelled, and a *zone-based fare* scheme that is common to many cities, including the Canadian metropolitan region of Vancouver.

Metro Vancouver’s SkyTrain network is divided into three zones, where passengers pay either $f_1 = \$2.20$, $f_2 = \$3.25$, or $f_3 = \$4.30$, depending on the number of zones they pass through. While the zone-based system is simple to understand, it punishes passengers travelling short distances over an arbitrary zone boundary and benefits passengers who travel long distances within a single zone. This is clear from the given SkyTrain map, where various one-station trips cost f_2 while one 17-station trip costs only f_1 .

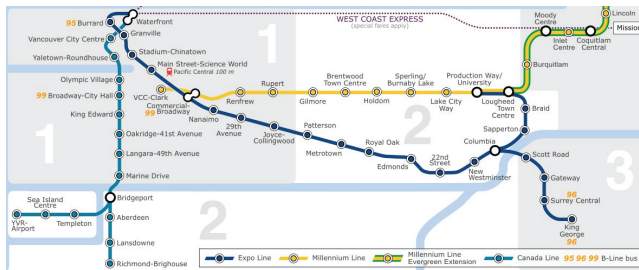
TransLink, the transit authority of Metro Vancouver, implemented the three-zone fare structure back in 1984, when the region’s public transit was based entirely on buses. In 2016, TransLink decided to launch its first-ever transit fare review, to explore other possibilities for pricing transit given the major changes in travel patterns due to the growth of the SkyTrain rapid transit network over the past three decades.

In Phase 1 of the review, TransLink surveyed over 28,000 people, finding that 64% disagreed that “the current zone-based fare structure works well”, compared to only 19% who agreed with the statement. Among all priorities identified by the respondents, the most important was to “make fares lower for shorter-distance trips” (TransLink 2016).

In Phase 2, TransLink surveyed nearly 13,000 residents, finding that the majority preferred *distance-based fares* for the SkyTrain (TransLink 2017). In a distance-based model, there are n different “tiers”, with a fare of X_i for travelling a distance of i units. Note that the units can be measured in kilometres, miles, or even the number of passed stations.

Distance-based fares have long been advocated by urban planners and transit scholars, as this paradigm charges passengers for the actual costs they impose on the transit system (Cervero 1981), and benefit low-income and elderly populations (Farber et al. 2014). Furthermore, distance-based fares eliminate the inherent unfairness of zone boundaries, and incentivize people to take short trips with a cheap public transit option, rather than by car or taxi.

As a result, public transit authorities in several large cities, including Edmonton and Toronto, are currently considering a switch to distance-based transit fares. The challenge is determining *how* to do this, to optimally convert m different zone fares (note that flat fares are just zone fares with $m = 1$) to n distance-based tiers, to maximize forecasted ridership and revenue. We answer this question by formulating it as an AI problem and finding an exact solution.



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Price elasticity measures the extent to which consumption patterns change in response to a change in price. In the context of public transit, the price elasticity is $-k = \frac{\Delta R}{\Delta F}$, where ΔR is the change in ridership and ΔF is the change in fares. For example, if a 20% fare increase causes a 4% decrease in ridership, then $k = 0.2$. Transit policy analysts have found that k is approximately 0.2 in North America (Tawfik 2014).

In the next section, we present a brief literature review, after which we define the specific problem that falls under the broad AI field of constraint optimization. We present our main theorem, an optimal pricing formula for converting zone fares to distance-based fares that holds for any price elasticity $-k$. Our two-part result is an application of the Cauchy-Schwarz Inequality, where we (a) maximize forecasted revenue given a target ridership, and (b) maximize forecasted ridership given a target revenue.

We then apply the theory to the TransLink system with over 400,000 daily passengers, and explain how we created a simple Java program that enables the transit authority to rapidly generate forecasted ridership and revenue numbers for various distance-based fare scenarios. We present the results of our analysis on a month's worth of TransLink passenger data, present our proposed recommendation for five tiers with the optimal fare prices, and conclude the paper with some avenues for future research.

Background and Related Work

Transit policy researchers use modelling and simulations to forecast ridership and revenue under various scenarios; for example, a recent thesis (Tawfik 2014) calculated the average per-trip SkyTrain fare across various distance intervals to conclude that switching to distance-based fares has the potential to increase TransLink's ridership and revenue.

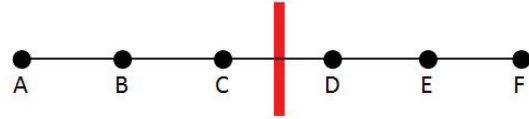
Several mathematicians have explored how to calculate optimal distance-based transit fares: one used a hybrid artificial bee colony algorithm (Huang et al. 2016), and another used a discrete choice model (Borndorfer, Karbsteina, and Pfetsch 2012) that was then applied to the transit system in Potsdam, Germany. But the most promising approach was a quadratic programming model (Daskin, Schofer, and Haghani 1988) as it took elasticity into account, maximizing the total revenue (a quadratic function) subject to ridership constraints. This paper solved the quadratic program (QP) using the Frank-Wolfe algorithm, taking the weighted average of numerous linear programming (LP) solutions.

Quadratic programming has numerous applications in Artificial Intelligence, as illustrated by recently-published AAAI papers on binary optimization problems (Yuan and Ghanem 2017), multi-label prediction (Han et al. 2010), and the calibration of scores in scientific peer review (Roos, Rothe, and Scheuermann 2011). AI researchers have also developed powerful techniques to compress convex QPs to make them more compact, which then makes them more efficient to solve (Mladenov, Kleinhans, and Kersting 2017).

In this paper, we solve this real-life transit policy problem using quadratic programming. Our optimal pricing formula is extremely versatile in that it is both fast and exact: unlike other QPs that are solved using multi-step algorithms, ours is a direct solution, using the Cauchy-Schwarz inequality.

An Example Problem

Consider the following simplified problem, with six stations equally spaced on a straight line, with labels from A to F. Suppose there is a "boundary" in the middle so that passengers are charged a two-zone fare for travelling from C to D, but only a one-zone fare for the longer trip from D to F.



Suppose f_1 and f_2 are the prices of a one-zone and two-zone fare, respectively. In this example problem, we will assume $f_1 = 4$ and $f_2 = 5$, with the units in dollars.

Let $(X_1, X_2, X_3, X_4, X_5)$ be the ordered 5-tuple corresponding to a distance-based fare model, where X_i is the cost of travelling i stations in either direction.

For each $1 \leq i \leq 5$ and $1 \leq j \leq 2$, let $r_{i,j}$ be the number of passengers who currently pay f_j for travelling in j zones, and would pay X_i in a distance-based system for travelling i stations. From the above diagram, we see that $r_{i,1} = 0$ for $3 \leq i \leq 5$. We will assume that the $r_{i,j}$ values are:

i	1	1	2	2	3	4	5
j	1	2	1	2	2	2	2
$r_{i,j}$	300	100	200	100	400	300	200
f_j	4	5	4	5	5	5	5

Based on the above assumptions, the total ridership and revenue are $\sum_{i=1}^5 \sum_{j=1}^2 r_{i,j} = 1600$ and $\sum_{i=1}^5 \sum_{j=1}^2 r_{i,j} f_j = 7500$.

Let $-k$ be the price elasticity, with $k > 0$. Thus, if we decrease the fare of the C to D trip by $p\%$, then we would forecast a $pk\%$ increase of passengers travelling from C to D. In this problem, we will assume that $k = 0.2$.

Suppose we change from zone fares to the distance-based 5-tuple $(X_1, X_2, X_3, X_4, X_5)$. For each $1 \leq i \leq 5$, we can predict the number of passengers who would travel distance i , as a function of the elasticity $-k$, and use this to forecast the ridership and revenue under this new pricing scheme.

If $(X_1, X_2, X_3, X_4, X_5) = (3.5, 4, 4.5, 5, 5.5)$, we can show that the total ridership increases to 1622, but the total revenue falls to 7077. In other words, for a 1% increase in ridership, we have created a 6% decrease in revenue.

The challenge is to properly set each X_i , knowing that a fare increase will lead to a ridership decrease, and vice-versa. A natural approach is to fix one variable and maximize the other. Specifically, we can either (a) maximize the forecasted revenue given a fixed target ridership, or (b) maximize the forecasted ridership given a fixed target revenue.

The main theorem of this paper, presented in the following section, solves both parts using the Cauchy-Schwarz Inequality, and finds an explicit formula for each X_i . We then illustrate both parts of our optimal pricing formula on our example problem, and apply this theory to propose an optimal distance-based fare system for the SkyTrain network.

Main Theorem

Let f_1, f_2, \dots, f_m be the current fares of the m different zones, and let X_1, X_2, \dots, X_n be the proposed distance-based fares, for each of the n possible distance options. (In this theorem, the lower case variables represent the known constants, while the upper case variables represent the unknowns that we are trying to determine.)

Let $-k$ be the price elasticity, and let $r_{i,j}$ be the number of riders who currently pay f_j for travelling across j zones and would pay X_i by travelling a distance of i units.

Define the following constants:

$$z_i = \sum_{j=1}^m r_{i,j}, \quad c_i = \sum_{j=1}^m \frac{r_{i,j}}{f_j}, \quad \tilde{z} = \sum_{i=1}^n z_i, \quad \tilde{c} = \sum_{i=1}^n c_i.$$

- (a) Let rid be the target ridership. Then for each $1 \leq i \leq n$, the optimal fare price X_i that maximizes total revenue is

$$X_i = \left(\frac{1+k}{2k} \right) \frac{z_i}{c_i} - \left(\frac{rid}{\tilde{c}k} - \frac{1+k}{2\tilde{c}k} \cdot \tilde{z} \right)$$

- (b) Let rev be the target revenue. Then for each $1 \leq i \leq n$, the optimal fare price X_i that maximizes total ridership is

$$X_i = \left(\frac{1+k}{2k} \right) \frac{z_i}{c_i} - \sqrt{-\frac{rev}{\tilde{c}k} + \frac{(1+k)^2}{4\tilde{c}k^2} \sum_{i=1}^n \frac{z_i^2}{c_i}}$$

Proof of the Main Theorem

For each $1 \leq i \leq n$, let Y_i be the forecasted ridership for a commute of distance i . By the definition of price elasticity,

$$Y_i = \sum_{j=1}^m r_{i,j} \left(1 - k \cdot \frac{X_i - f_j}{f_j} \right) = (1+k)z_i - kX_i c_i.$$

Thus, the total forecasted ridership and revenue, over all n distances, are $\sum_{i=1}^n Y_i$ and $\sum_{i=1}^n X_i Y_i$, respectively. Note that ridership is a linear function of the X_i 's, while the revenue function is a quadratic function of the X_i 's.

Note that $\tilde{z} = \sum_{i=1}^n z_i = \sum_{i=1}^n \sum_{j=1}^m r_{i,j}$ is simply the total ridership under the existing zone-based fare system.

Let $V_i = \frac{1+k}{2k} \cdot \frac{z_i}{c_i} - X_i$ for each $1 \leq i \leq n$.

Then, the forecasted ridership function equals

$$\sum_{i=1}^n Y_i = \sum_{i=1}^n ((1+k)z_i - kX_i c_i) = \frac{(1+k)\tilde{z}}{2} + k \sum_{i=1}^n c_i V_i$$

Similarly, the forecasted revenue function equals

$$\begin{aligned} \sum_{i=1}^n X_i Y_i &= \sum_{i=1}^n (X_i z_i (1+k) - k \cdot c_i X_i^2) \\ &= \sum_{i=1}^n \left(-k c_i \cdot \left(\frac{1+k}{2k} \cdot \frac{z_i}{c_i} - X_i \right)^2 + \frac{(1+k)^2}{4k} \cdot \frac{z_i^2}{c_i} \right) \\ &= -k \cdot \sum_{i=1}^n c_i V_i^2 + \frac{(1+k)^2}{4k} \left(\sum_{i=1}^n \frac{z_i^2}{c_i} \right) \end{aligned}$$

Because k is a positive constant, as are all the c_i 's and z_i 's, our forecasted ridership is maximized when $\sum_{i=1}^n c_i V_i$ is maximized. Conversely, our forecasted revenue is maximized when $\sum_{i=1}^n c_i V_i^2$ is minimized.

By the Cauchy-Schwarz Inequality,

$$\left(\sum_{i=1}^n c_i V_i^2 \right) \left(\sum_{i=1}^n c_i \right) \geq \left(\sum_{i=1}^n c_i V_i \right)^2,$$

with equality occurring if and only if $V_1 = \dots = V_n$.

In part (a), we have a target ridership of

$$rid = \sum_{i=1}^n Y_i = \frac{(1+k)\tilde{z}}{2} + k \sum_{i=1}^n c_i V_i.$$

Since $\sum_{i=1}^n c_i V_i$ is a constant, the Cauchy-Schwarz Inequality tells us that $\sum_{i=1}^n c_i V_i^2$ is minimized when each $V_i = V$ for some real number $V \geq 0$. Therefore, our forecasted revenue is maximized when V satisfies

$$\begin{aligned} rid &= \frac{(1+k)\tilde{z}}{2} + k \sum_{i=1}^n c_i V \\ \Rightarrow V &= \frac{rid}{\tilde{c}k} - \frac{1+k}{2\tilde{c}k} \cdot \tilde{z} \end{aligned}$$

For each $1 \leq i \leq n$, we defined $V_i = \frac{1+k}{2k} \cdot \frac{z_i}{c_i} - X_i$.

And so, if rid is the target ridership, we conclude that the revenue-maximizing distance-based fare X_i is

$$X_i = \left(\frac{1+k}{2k} \right) \frac{z_i}{c_i} - \left(\frac{rid}{\tilde{c}k} - \frac{1+k}{2\tilde{c}k} \cdot \tilde{z} \right)$$

In part (b), we have a target revenue of

$$rev = \sum_{i=1}^n X_i Y_i = -k \cdot \sum_{i=1}^n c_i V_i^2 + \frac{(1+k)^2}{4k} \left(\sum_{i=1}^n \frac{z_i^2}{c_i} \right)$$

Since $\sum_{i=1}^n c_i V_i^2$ is a constant, the Cauchy-Schwarz Inequality tells us that $\sum_{i=1}^n c_i V_i$ is maximized when each $V_i = V$ for some real number $V \geq 0$. Therefore, our forecasted ridership is maximized when V satisfies

$$\begin{aligned} rev &= -k \cdot \sum_{i=1}^n c_i V^2 + \frac{(1+k)^2}{4k} \left(\sum_{i=1}^n \frac{z_i^2}{c_i} \right) \\ \Rightarrow V &= \sqrt{-\frac{rev}{\tilde{c}k} + \frac{(1+k)^2}{4\tilde{c}k^2} \sum_{i=1}^n \frac{z_i^2}{c_i}} \end{aligned}$$

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Illustration on our Example Problem

We apply our Main Theorem to our example problem. We have $(z_1, z_2, z_3, z_4, z_5) = (400, 300, 400, 300, 200)$ and $(c_1, c_2, c_3, c_4, c_5) = (95, 70, 80, 60, 40)$. Our zone fare ridership and revenue were 1600 and 7500, respectively.

Suppose $rid = 1600$, i.e., we don't want our ridership to decrease by switching to a distance-based fare system. Then, for any price elasticity $-k$, we can determine our set of revenue-maximizing distance-based fares using part (a) of our Main Theorem:

$$X_1 = \frac{40}{1311} \left(145 - \frac{7}{k} \right), X_2 = \frac{5}{483} \left(431 - \frac{17}{k} \right)$$

$$X_3 = X_4 = X_5 = \frac{5}{138} \left(133 + \frac{5}{k} \right)$$

If $k = 0.2$, then our optimal fares are

$$(X_1, X_2, X_3, X_4, X_5) = (3.36, 3.59, 5.72, 5.72, 5.72)$$

We can quickly calculate that $(Y_1, Y_2, Y_3, Y_4, Y_5) = (416.2, 309.9, 388.4, 291.3, 194.2)$, which implies that the ridership indeed stays constant at 1600, and the forecasted revenue becomes 7510, a slight increase from the zone-based system.

Of course, our Main Theorem applies for all values of rid , even numbers that might be impractical to attain.

For example, if we insist on a ten percent increase in ridership, to $rid = 1760$, then we get $(X_1, X_2, X_3, X_4, X_5) = (1.04, 1.26, 3.41, 3.41, 3.41)$, i.e., all fares going down significantly to increase passengers. Predictably, this leads to a massive drop in revenue, from 7500 down to 4170.

Similarly, if we allow for a ten percent decrease in ridership, to $rid = 1440$, then we get $(X_1, X_2, X_3, X_4, X_5) = (5.68, 5.90, 8.04, 8.04, 8.04)$, i.e., all fares going up significantly. Predictably, this leads to a massive increase in revenue, from 7500 up to 10107.

Suppose $rev = 7500$, i.e., we don't want our revenue to decrease by switching to a distance-based fare system. Then, for any price elasticity $-k$, we can determine our set of ridership-maximizing distance-based fares using part (b) of our Main Theorem. For example, we have:

$$X_1 = \frac{40(k+1)}{19k} - \sqrt{\frac{25}{9177} \left(\frac{1987}{k^2} - \frac{4006}{k} + 1987 \right)}$$

If $k = 0.2$, then our optimal fares are

$$(X_1, X_2, X_3, X_4, X_5) = (3.35, 3.57, 5.71, 5.71, 5.71)$$

We can quickly calculate that $(Y_1, Y_2, Y_3, Y_4, Y_5) = (416.4, 310.0, 388.5, 291.4, 194.3)$, which implies that the revenue indeed stays constant at 7500, and the forecasted ridership becomes 1600.5, a slight increase from the zone-based system.

This formula also explains when a target is impossible to attain. For example, if we set $rev = 15000$ or $rid = 3200$, then our X_i variables are no longer positive reals, and so we know that no distance-based model can double the revenue, or double the ridership, with an elasticity of -0.2 .

Since X_i is the optimal price of travelling i stops, we clearly need to have $X_1 \leq X_2 \leq \dots \leq X_n$, in order for the result to make sense in practice. Unfortunately, depending on the ridership numbers $r_{i,j}$, it is possible for this monotonicity condition to fail. Recall that each $X_i = \frac{1+k}{2k} \cdot \frac{z_i}{c_i} - V$, for some constant V . Thus, to check whether $X_i \leq X_j$, it suffices to verify that $\frac{z_i}{c_i} \leq \frac{z_j}{c_j}$.

In our six-station example problem, it is straightforward to prove that $X_1 \leq X_2$ if and only if $r_{1,2} \cdot r_{2,1} \leq r_{1,1} \cdot r_{2,2}$. Thus, when this inequality is violated, our X_i values will not be monotonic. For example, if we change $r_{1,2}$ from 100 to 200, then we'll see that $(X_1, X_2, X_3, X_4, X_5) = (3.73, 3.54, 5.68, 5.68, 5.68)$, i.e., $X_1 > X_2$.

When monotonicity is violated, a simple solution is to bundle fares into tiers. This is a practical solution that yields a result that is close to the optimal fares that maximize ridership or revenue. In the above scenario, we could set two fare "tiers", where one tier is travelling 2 or fewer stops, and the second tier is travelling 3 or more stops. The first fare tier is set at $\frac{X_1+X_2}{2}$ and the second fare tier is set at $\frac{X_3+X_4+X_5}{3}$.

It is not a coincidence that $X_3 = X_4 = X_5$ in all of our scenarios. This is because all trips requiring 3 or more steps requires us to cross the "boundary" from C to D, and is therefore charged a two-zone fare. Thus, $r_{3,1} = r_{4,1} = r_{5,1} = 0$, which implies that $\frac{z_i}{c_i} = r_{i,2} \cdot \frac{f_2}{r_{i,2}} = f_2$, for all $3 \leq i \leq 5$. By our Main Theorem, this implies that our optimal price must satisfy $X_3 = X_4 = X_5 = \frac{1+k}{2k} \cdot f_2 - V$. We will see a similar result in our TransLink data, where n different distance options collapse to a smaller number.

In our example problem, our optimal prices achieved a result that were better than the zone fare prices: by fixing ridership as $rid = 1600$, we found X_i values that increased revenue, and by fixing revenue at $rev = 7500$, we found X_i values that increased ridership. This is usually not the case.

To illustrate, consider the scenario where each of our seven $r_{i,j}$ values is set at 100. Then our zone fare ridership and revenue are 700 and 3300, respectively. By setting $rid = 700$, we get a distance-based forecasted revenue of $3287 < 3300$, and by setting $rev = 3300$, we get a distance-based forecasted ridership of $699 < 700$.

There is a simple explanation for this. In the first scenario, only $\frac{100}{1600} = \frac{1}{16}$ of passengers are overpaying, paying two zones to travel one station. (This is the commute from C to D in our six-station network.) But in the second scenario, this fraction rises to $\frac{100}{700} = \frac{1}{7}$. These passengers are the beneficiaries of switching to a distance-based system.

By the definition of price elasticity, if these r passengers have their fare f drop by $p\%$, then the revenue generated from the commute from C to D decreases from rf to $rf \left(1 + \frac{pk}{100} \right) \left(1 - \frac{p}{100} \right)$. If $p = 10$ and $k = 0.2$, then ridership increases 2% while revenue falls by 8.2%.

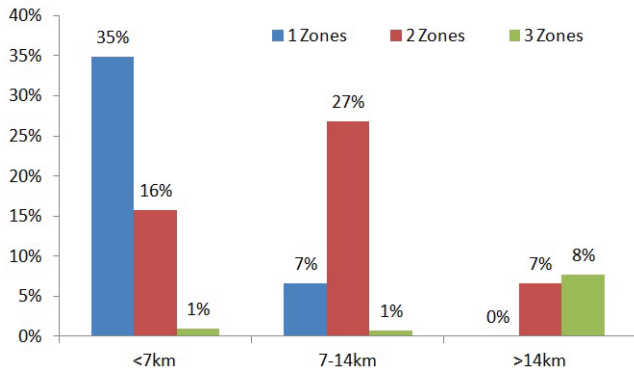
If many passengers are currently overpaying in a zone-based system, the percentage of lost revenue will exceed the percentage of gained ridership; thus, despite its increased fairness, switching to a distance-based system may yield results that are just slightly worse than the zone fare prices. As we see in the next section, this is the case for TransLink.

Application to SkyTrain Passenger Data

The Metro Vancouver SkyTrain system consists of 53 stations and three fare zones, with each pair of stations separated by a geodesic distance of at most 22 kilometres. To get on the SkyTrain, passengers require a Compass Card, a contactless smart card that enables TransLink to track exactly where a passenger is entering and exiting the network.

This analysis was conducted on thirty days of Compass Card data from September 2016, with approximately 13 million total boardings. There were 46 stations considered in our analysis, excluding the three airport stations (which have an additional fare surcharge) and the stations on the brand-new Evergreen Line (which opened in December 2016).

The SkyTrain zone-based fares are $f_1 = 2.20$, $f_2 = 3.25$, and $f_3 = 4.30$. As seen below, the zone-based system benefits some passengers at the expense of others: while 7% pay f_1 for a trip longer than 7km, 16% + 1% = 17% of passengers pay either f_2 or f_3 for a trip shorter than 7 km.



TransLink provided us with an Excel sheet containing three 46×46 matrices. In these three matrices, each (X, Y) entry corresponded to (i) the number of total trips from X to Y , (ii) the geodesic distance from X to Y , and (iii) the number of fare zones in a commute from X to Y .

We created a Java program that enables the user to input the price elasticity $-k$, the number of distance tiers n , as well as the actual ranges for each tier. For example, if $n = 10$, we can set our ten distance tier prices to be X_1 for travelling between 0 to 2.2 kilometres, X_2 for 2.2 to 4.4 kilometres, all the way up to X_{10} for 19.8 to 22.0 kilometres.

From this input data, we read in our three Excel sheets to calculate $r_{i,j}$ for each $1 \leq i \leq n$ and $1 \leq j \leq 3$. By definition, the sum of the $r_{i,j}$ coefficients must be approximately 13,000,000. Finally, we ask the user to indicate a target value for either ridership or revenue. Once the variable's target is entered, the Java program uses the Main Theorem to determine the optimal fare prices (X_1, X_2, \dots, X_n) that maximizes the other variable.

The first step of our analysis is to determine n , the appropriate number of distance tiers. We first consider the case where the n tiers are equally spread out, each with a range of $\frac{22}{n}$ kilometres. Assuming an elasticity of $k = -0.2$ and a non-changing ridership of $rid \sim 13,000,000$, we apply the Main Theorem to determine the prices (X_1, X_2, \dots, X_n) that maximize revenue given the fixed ridership.

All scenarios lead to a net revenue loss, but the case $n = 5$ is optimal, leading to a net loss of 0.25%. As mentioned previously, this revenue loss can be explained by the much larger percentage of passengers who are overpaying in the zone-based system (17%), as compared to those who are underpaying (7%). For this case, the optimal fare prices are $(X_1, X_2, X_3, X_4, X_5) = (1.70, 2.10, 4.09, 4.48, 7.15)$.

The high $X_5 = 7.15$ fare arises because of the small number of passengers making trips that are between 17.6 and 22.0 kilometres, and thus this fare can be raised for these longer journeys without losing as many passengers. This pricing scheme leads to an overall average fare decrease of 2.35%, with over 70% seeing their SkyTrain fares go down by switching to five equally-spaced distance tiers.

Given that $X_5 = 7.15$ is significantly higher than the 3-zone fare of $f_3 = 4.20$, it is possible to set a cap on the maximum distance-based fare to ensure that the final policy is "politically pragmatic". For example, if we were to set $X_5 = 6.00$ and hold ridership constant, then the optimal fare is $(X_1, X_2, X_3, X_4, X_5) = (1.74, 2.14, 4.18, 4.58, 6.00)$, which leads to a 0.46% reduction in revenue.

In the above scenarios, we assumed the five distance tiers were equal in range. However, our Java program is more versatile, and enables the user to specify any set of ranges for any number of tiers. For example, consider the $n = 5$ case where the five tiers have the following distance ranges:

Tier	Min (km)	Max (km)
Tier 1	0.0	1.5
Tier 2	1.5	4.5
Tier 3	4.5	9.0
Tier 4	9.0	15.0
Tier 5	15.0	22.0

This choice of distance ranges results in the optimal prices of $(X_1, X_2, X_3, X_4, X_5) = (1.37, 1.97, 2.06, 4.17, 5.81)$, which yields a 0.63% reduction in revenue. In this proposal, nearly 70% of passengers would pay X_1 or X_2 , which is less than the one-zone fare $f_1 = 2.20$.

Given that this is a more balanced distribution of passengers paying each of the five fares, this proposal might be more practical to implement, though it would lead to lower forecasted revenue as compared to the other fare prices.

The natural solution is to round up each X_i to the nearest quarter, which then creates a slight increase in forecasted revenue, while creating a slight decrease in forecasted ridership. In this case, we have $(X_1, X_2, X_3, X_4, X_5) = (1.50, 2.00, 2.25, 4.25, 6.00)$.

This final selection of distance-based tiers, with nearly-optimal prices, is our recommended proposal to TransLink. This is our recommendation because the number of distance tiers is set relatively low ($n = 5$), there is a clear spread between medium and long trips (with $X_4 - X_3 = 2$ and $X_5 - X_4 = 1.75$), and there is a significant discount in short-distance trips to incentivize ridership.

As mentioned earlier, many TransLink passengers are currently overpaying in the zone-based system; thus, by switching to distance-based fares, the percentage of lost revenue

will exceed the percentage of gained ridership. As it is impossible to simultaneously increase forecasted revenue and ridership, a trade-off is necessary. The above solution is an appropriate trade-off.

In addition to our Java program that generates optimal values for X_i , we created a separate Excel sheet so that TransLink can immediately measure the change in forecasted ridership and revenue when altering the distance fares X_i . This Excel sheet allows the user to input the elasticity $-k$, the number of distance tiers n , the distance ranges for each tier, along with their corresponding fare X_i . Using the formulas provided in the proof of the Main Theorem, the Excel sheet re-calculates the forecasted revenue and ridership whenever any X_i value is changed.

Conclusion

Our Main Theorem allows any transportation authority to determine the optimal way to switch from zone-based fares to distance-based fares, to maximize either ridership or revenue. Furthermore, our Java program and Excel sheet enables a transit organization to rapidly generate a solution, and then use these X_i values as a starting point for adjustment: perhaps to cap the maximum fare at a certain amount, or to round up each X_i to the nearest quarter to see how that change would affect ridership and revenue.

Future work includes adding more complexity to our model. For example, there may be two different price elasticities, $-k_1$ for short trips and $-k_2$ for long trips, in addition to differences in short-term and long-term price elasticities due to political changes that affect transit ridership, such as carbon taxes. Thus, the Main Theorem would need to be updated accordingly.

Furthermore, we can incorporate the more sophisticated method of calculating elasticity, known as *arc* elasticity (Litman 2017). For example, if $k = -0.2$ is the arc elasticity, then a 40% fare increase is equivalent to forty 0.2% reductions in ridership. Thus, the net loss in ridership is $1 - (1 - 0.002)^{40} = 7.7\%$, which is different from the simpler $40 \times (0.002) = 8\%$ reduction under our simpler *point-price* elasticity formula.

Since $1 - (1 - x)^n \sim 1 - (1 - nx) = nx$ for small x , our point-price elasticity formula is a good approximation. However, our Main Theorem would be sharpened by applying this more rigorous and precise definition of elasticity.

This research served as an initial exploratory analysis into distance-based pricing, helping TransLink explore an ideal number of tiers to confirm the feasibility of a distance-based fare system. This work has formed an important part of the two-year Transit Fare Review, which will assist TransLink as it moves towards its final recommendation, which will be tabled sometime in 2018.

The collaboration with TransLink was a tremendous success, which included a one-month internship at the organization for the student-author of this paper. One of the senior managers remarked that this third-year undergraduate student “forged new ground with this approach”, and that her “quadratic optimization model provided insight into aspects of pricing and ridership and made us think about its application and new possibilities.”

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