From Exact to Anytime Solutions for Marginal MAP

Junkyu Lee
University of California, Irvine
Irvine, CA 92697, USA
junkyul@ics.uci.edu

Radu Marinescu
IBM Research – Ireland
radu.marinescu@ie.ibm.com

Rina Dechter
University of California, Irvine
Irvine, CA 92697, USA
dechter@ics.uci.edu

Alexander Ihler
University Irvine, CA 92697, USA
ihler@ics.uci.edu

Abstract
This paper explores the anytime performance of search-based algorithms for solving the Marginal MAP task over graphical models. The current state-of-the-art for solving this challenging task is based on best-first search exploring the AND/OR graph with the guidance of heuristics based on mini-bucket and variational cost-shifting principles. Yet, those schemes are uncompromising in that they solve the problem exactly, or not at all, and often suffer from memory problems. In this work, we explore the well known principle of weighted search for converting best-first search solvers into anytime schemes. The weighted best-first search schemes report a solution early in the process by using inadmissible heuristics, and subsequently improve the solution. While it was demonstrated recently that weighted schemes can yield effective anytime behavior for pure MAP tasks, Marginal MAP is far more challenging (e.g., a conditional sum must be evaluated for every solution). Yet, in an extensive empirical analysis we show that weighted schemes are indeed highly effective anytime solvers for Marginal MAP yielding the most competitive schemes to date for this task.

Introduction
Graphical models provide a powerful framework for reasoning about conditional dependency structures over many variables. The Maximum a Posteriori (MAP) task asks for the mode of the joint probability distribution, while the Marginal MAP (MMAP) task generalizes MAP by allowing a subset of the variables to be marginalized.

MMAP is known to be a very challenging task. The difficulty rises from not only the exponential size of the search space but also the hardness of evaluating a full instantiation of MAP variables, i.e., computing the marginal probability given evidence. Furthermore, variable elimination orders of MMAP are constrained and often more costly because the maximization and the summation do not commute, i.e., sum variables are eliminated before any MAP variable.

Early work for solving MMAP exactly was based on the depth-first branch and bound search guided by a heuristic based on evaluating (incrementally) a join-tree built along an unconstrained variable ordering (Park and Darwiche 2003; Yuan and Hansen 2009). The depth-first branch and bound search improves its performance when it traverses an AND/OR search graph, and when it is guided by heuristics based on weighted mini-bucket elimination with cost shifting schemes, as shown in (Marinescu, Dechter, and Ihler 2014). More recently, several best-first AND/OR search algorithms for MMAP were also proposed, and were shown to outperform considerably depth-first search if memory is available (Marinescu, Dechter, and Ihler 2015). They include the best-first AND/OR search algorithm which belongs to the AO* family, as well as a recursive best-first AND/OR search algorithm that operates within limited memory. In addition, variational algorithms for MMAP (Liu and Ihler 2013), and an anytime MMAP algorithm based on factor-set-elimination (Maua and De Campos 2012) have also been developed.

Best-first search, which typically expands fewer nodes than depth-first search, is a particularly favorable choice for MMAP because it can save on the number of evaluations of the marginal probability conditioned on the MAP variables. However, best-first search faces two drawbacks: (1) it may require enormous amounts of memory, (2) it only returns the optimal solution at the end of processing. Recursive best-first search (Korf 1993) remedies the memory issue, but still lacks anytime behavior.

Contributions: Here, we focus on providing good anytime solutions, and develop the anytime version of the exact AND/OR search algorithms. We use the principle of weighted search that was shown to provide a useful tool for converting best first algorithms into anytime scheme (Pohl 1970). Unlike local search based methods (Park and Darwiche 2004), our proposed algorithms come with solution quality guarantees; unlike variational schemes (Liu and Ihler 2013) our algorithms will prove optimality if given enough time.

We discuss the effectiveness of the newly introduced anytime algorithms against state-of-the-art exact search algorithms over varying strength of heuristics and time bounds. Through extensive empirical evaluations, we show that the proposed anytime algorithms dominate exact search algorithms and provide anytime solutions to the problems that were beyond the reach of exact algorithms.

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Background

A graphical model is a tuple $M = (X, D, F)$, where $X = \{X_i : i \in V\}$ is a set of variables indexed by set $V$ and $D = \{D_i : i \in V\}$ is the set of their finite domains of values. $F = \{\psi_\alpha : \alpha \in F\}$ is a set of discrete positive real-valued local functions defined on subsets of variables, where we use $\alpha \subseteq V$ and $X_\alpha \subseteq X$ to indicate the scope of function $\psi_\alpha$, i.e., $X_\alpha = \text{var}(\psi_\alpha) = \{X_i : i \in \alpha\}$.

The function scopes yield a primal graph whose vertices are the variables and whose edges connect any two variables that appear in the same function. The graphical model $M$ defines a factorized probability distribution on $X$, $P(X) = \frac{1}{Z} \prod_{\alpha \in F} \psi_\alpha(X_\alpha)$. The partition function, $Z$, normalizes the probability.

Let $X_M = \{X_1, ..., X_m\}$ be a subset of $X$ called MAP variables and $X_S = X \setminus X_M$ be the complement of $X_M$, called sum variables. The Marginal MAP (MMAP) task seeks an assignment $x_M^*$ to variables $X_M$ having maximum probability. This requires access to the marginal distribution over $X_M$, which is obtained by summing out variables $X_S$:

$$x_M^* = \arg \max_{x_M} \sum_{X_S} \prod_{\alpha \in F} \psi_\alpha(X_\alpha) \quad (1)$$

**AND/OR Search Spaces**

Significant recent improvements in search for MMAP inference have been achieved by using AND/OR search spaces, which often capture problem structure far better than standard OR search methods (Marinescu, Dechter, and Ihler 2014; Dechter and Mateescu 2007). The AND/OR search space is defined relative to a pseudo tree of the primal graph, which captures problem decomposition.

**DEFINITION 1 (pseudo tree)** A pseudo tree of an undirected graph $G = (V, E)$ is a directed rooted tree $T = (V', E')$ such that every arc of $G$ not included in $E'$ is a backarc in $T$ connecting a node in $T$ to one of its ancestors. The arcs in $E'$ may not all be included in $E$.

We say that a pseudo tree $T$ of $G$ is valid for MAP variables $X_M$ if $T$ restricted to $X_M$, denoted by $T' = (V', E'')$ where $V' \subseteq V$, $E'' \subseteq E'$ and $V'$ correspond to $X_M$, forms a connected subgraph of $T$ that has the same root as $T$ and is called a start pseudo tree.

Given a graphical model $M = (X, D, F)$ with primal graph $G$ and valid pseudo tree $T$ of $G$, the AND/OR search tree $S_T$ based on $T$ has alternating levels of OR nodes corresponding to the variables, and AND nodes corresponding to the values of the OR parent’s variable, with edge weights extracted from the original functions $F$ (for details see Dechter and Mateescu (2007)). Identical sub-problems, identified by their context (the partial instantiation that separates the sub-problem from the rest of the problem graph), can be merged, yielding an AND/OR search graph. Merging all context-mergeable nodes yields the context minimal AND/OR search graph, denoted $C_T$. The size of $C_T$ is exponential in the induced width of $G$ along a depth-first traversal of $T$ (i.e., the constrained induced width).

**DEFINITION 2 (MMAP solution tree)** A solution subtree $x_M$ of $C_T$ relative to the MAP variables $X_M$ is a subtree of $C_T$ restricted to $X_M$ that: (1) contains the root of $C_T$; (2) if an internal OR node $n \in C_T$ is in $x_M$, then $n$ is labeled with a MAP variable and exactly one of its children is in $x_M$; (3) if an internal AND node $n \in C_T$ is in $x_M$ then all its OR children which denote MAP variables are in $x_M$.

Each node $n$ in $C_T$ can be associated with a value $v(n)$; for MAP variables $v(n)$ is the optimal marginal MAP value of the conditioned sub-problem rooted at $n$, while for a sum variable it is the likelihood of the partial assignment denoted by $n$. Clearly, $v(n)$ can be computed recursively based on the values of $n$’s successors: OR nodes by maximization or summation (for MAP or sum variables, respectively), and AND nodes by multiplication.

**Example 1** Figure 1 shows an example of simple graphical model. Figure 1(a) is the primal graph of 8 bi-valued variables and 10 binary functions, where the MAP and sum variables are $X_M = \{A, B, C, D\}$ and $X_S = \{E, F, G, H\}$. Figure 1(b) is a valid pseudo tree whose MAP variables form a start pseudo tree. Figure 1(c) displays the context minimal AND/OR search graph based on the valid pseudo tree. (the contexts are shown next to the pseudo tree nodes). A MMAP solution subtree corresponding to the MAP assignment $(A = 0, B = 1, C = 1, D = 0)$ is shown in red.

**Earlier Work: Exact AND/OR Search for MMAP**

Current state-of-the-art approaches for exact MMAP solving are based on either depth-first or best-first AND/OR search guided by mini-bucket heuristics enhanced with variational cost-shifting ideas (Marinescu, Dechter, and Ihler 2014; 2015).

**AND/OR Branch and Bound** (A0BB-MMAP) (Marinescu, Dechter, and Ihler 2014) explores in a depth-first
manner the context minimal AND/OR search graph and therefore takes advantage of problem decomposition. During search, AOBF-MMAP keeps track of the value of the best solution found so far (a lower bound on the optimal MMAP cost) and uses this value and the heuristic function to prune away portions of the search space that are guaranteed not to contain the optimal solution in a typical branch and bound manner.

Best-First AND/OR Search (AOBF-MMAP) (Marinescu, Dechter, and Ihler 2015) is a variant of AO* (Nilsson 1980) applicable to graphical models that explores the context minimal AND/OR search graph in a best-first rather than depth-first manner. This enables AOBF-MMAP to visit a significantly smaller search space than AOBF-MMAP which sometimes translates into important time savings, as well as considerably fewer conditional likelihood evaluations. Extensive empirical evaluations (Marinescu, Dechter, and Ihler 2015) showed that when given enough memory AOBF-MMAP is often superior to AOBF-MMAP. For completeness, we provide the pseudo code of AOBF-MMAP in Algorithm 1.

Recursive Best-First AND/OR Search (RBFAOO-MMAP) was recently introduced to address the memory issues of AOBF-MMAP. RBFAOO-MMAP (described by Algorithm 2) extends Recursive Best-First Search (RBFS) (Korf 1993) to MMAP queries over graphical models and thus it uses a threshold controlling mechanism to drive the search in a depth-first like manner. Specifically, let \( q(n) \), called the q-value, be an upper bound of the solution cost at node \( n \), and \( \theta(n) \) be the threshold at \( n \) indicating the availability of a second best solution besides \( q(n) \). RBFAOO-MMAP keeps examining the subtree rooted at \( n \) until either \( q(n) > \theta(n) \) or the subtree is solved optimally. Therefore, it gradually grows the search space by updating the q-values of the internal nodes and re-expanding them, unlike AOBF-MMAP. The algorithm may operate in linear space (no caching); however, for efficiency, it may use a fixed size cache table to store some of the nodes (based on contexts). In practice, RBFAOO-MMAP outperformed dramatically both AOBF-MMAP and AOBF-MMAP on a variety of benchmarks (Marinescu, Dechter, and Ihler 2015).

Anytime AND/OR Search for MMAP

A serious drawback of these highly competitive best-first search schemes is that they lack anytime behavior, namely they either solve the problem exactly, or not at all. In many situations, it is desirable to obtain a good solution relatively quickly and spend the remaining time improving it. Therefore, in this paper we explore the potential of the well known principle of weighted search and convert the best-first search algorithms into anytime search schemes for Marginal MAP.

Weighted best-first AND/OR search was introduced re-
cently as an effective alternative to anytime depth-first search schemes for pure MAP tasks (Flerova, Marinescu, and Dechter 2014). These algorithms were evaluated extensively on a wide range of benchmark problems and were shown to be very powerful and highly competitive with Breadth Rotating AOBB search (Otten and Dechter 2011), one of the best performing anytime MAP solvers. The anytime best-first AND/OR search algorithms for MMAP presented in this paper are direct extensions of the weighted best-first search algorithms for pure MAP (Flerova, Marinescu, and Dechter 2014).

**Weighted AOBF-MMAP and RBFAOO-MMAP** The fixed-weighted version of the AOBF-MMAP and RBFAOO-MMAP algorithms can be obtained easily by multiplying\(^1\) the heuristic function \(h(n)\) of a node \(n\) in the AND/OR search graph by a weight \(w > 1\) (i.e., substituting \(h(n)\) by \(w \cdot h(n)\)). Notice that only the portion of the search space that corresponds to the MAP variables is considered. If \(h(n)\) is admissible, which is the case for minbucket heuristics, then the cost of the solution discovered by weighted AOBF-MMAP (or weighted RBFAOO-MMAP) is \(w\)-optimal, namely it is guaranteed to be within a factor \(w\) from the optimal one. These schemes extend well known approaches such as WA* (Pohl 1970) and WAO* (P. Chakrabarti 1987) to MMAP queries.

**Algorithm 3: WAOBF-MMAP**

| Input: Graphical model \(\mathcal{M} = (\mathbf{X}, \mathbf{D}, \mathbf{F})\), pseudo tree \(T\), \(\mathbf{X}_M = \mathbf{X} \setminus \mathbf{X}_S\), heuristic \(h(\cdot)\), initial weight \(w_0\) |
| Output: \(w\)-optimal MMAP value \(V(T)\), and assignment \(T\) |
| 1. Initialize \(w = w_0\), \(C_T = \{s\}\), \(V(T) = -\infty\), \(T = \emptyset\) |
| 2. while \(w > 1\) do |
| 3. \(\langle V, T' \rangle \leftarrow \text{AOBF-MMAP}(\mathcal{M}, w \cdot h)\) |
| 4. Maintain and report current best solution as \((V(T), T)\) |
| 5. Decrease weight according to schedule policy |
| 6. return \((V(T), T)\) |

**Iterative Weighted AOBF-MMAP** Since weighted AOBF-MMAP yields \(w\)-optimal solutions, it can then be extended into an iterative anytime scheme called WAOBF-MMAP (Algorithm 3), by decreasing the weight from one iteration to the next (until it becomes 1).

Specifically, WAOBF-MMAP starts by running AOBF-MMAP with the initial weight \(w_0\) until it finds a suboptimal solution, and restarts the search with a decreased weight following a weight update schedule policy such as \(w_{i+1} = \sqrt{w_i}\). Clearly, different schedules are also possible, however, the latter proved quite effective in practice (see also (Flerova, Marinescu, and Dechter 2013) for additional details). Notice that the algorithm also output a \(w\)-optimality guaranty.

**Iterative Weighted RBFAOO-MMAP** The anytime weighted RBFAOO-MMAP, called here WRBFAOO-MMAP, is obtained by replacing the call to AOBF-MMAP in line 3 of Algorithm 3 with a call to RBFAOO-MMAP.

\(^1\)For numerical stability all algorithms presented solve Eq. 1 in log space, as an energy minimization problem.

**Algorithm 4: WRAOBF-MMAP**

| Input: Graphical model \(\mathcal{M} = (\mathbf{X}, \mathbf{D}, \mathbf{F})\), pseudo tree \(T\), \(\mathbf{X}_M = \mathbf{X} \setminus \mathbf{X}_S\), heuristic \(h(\cdot)\), initial weight \(w_0\) |
| Output: \(w\)-optimal MMAP value \(V(T)\), and assignment \(T\) |
| 1. Initialize \(w = w_0\), \(C_T = \{s\}\), \(V(T) = -\infty\), \(T = \emptyset\) |
| 2. while \(w > 1\) do |
| 3. \(\langle V, T', C_T' \rangle \leftarrow \text{AOBF-MMAP}(\mathcal{M}, w \cdot h, C_T, T)\) |
| 4. Maintain and report current best solution as \((V(T), T)\) |
| 5. Decrease weight according to schedule policy |
| 6. Revise \(C_T'\) according to \(w \cdot h\) from leaves to the root |
| 7. Update the best partial solution tree \(T\) from revised \(C_T'\) |
| 8. return \((V(T), T)\) |

**Anytime Repairing AOBF-MMAP** Running WAOBF-MMAP from scratch at each iteration seems redundant. Therefore, we introduce another anytime scheme, called WRAOBF-MMAP, that is based on the anytime repairing AOBF approach for pure MAP (Flerova, Marinescu, and Dechter 2014). Algorithm 4 describes WRAOBF-MMAP. As the name suggests, it does not discard the explicited AND/OR search graph \(C_T\) after finishing the \(i\)-th iteration but rather it revises the node values in \(C_T\) from leaves to the root node according to the newly inflated heuristic \(w_{i+1} \cdot h(n)\). The search continues with the revised best partial solution tree.

**Anytime AND/OR Branch and Bound Search** Depth-first AND/OR Branch and Bound often lacks good anytime behavior because during search AOBB will solve to completion all but one independent subproblems rooted at an AND node. This behavior was first observed by (Otten and Dechter 2011) in the context of pure MAP inference. Therefore, Breadth Rotating AOBB (BRAOBB) was introduced as an anytime depth-first AND/OR search scheme that rotates through different subproblems in a round-robin manner. Extensive empirical evaluations showed that BRAOBB improves considerably AOBB’s anytime behavior (Otten and Dechter 2011). We describe next algorithm BRAOBB-MMAP which is a direct application of the technique to MMAP queries.

Algorithm 5 summarizes BRAOBB-MMAP, omitting the details about AOBB-MMAP (i.e., pruning, caching and propagation). For these details, see (Marinescu, Dechter, and Ihler 2014).

The algorithm maintains not only the \(\text{LOCAL}\) stack but also a \(\text{GLOBAL}\) queue to store and rotate the subproblems. Notice that rotation applies only to MAP variables. A separate stack called \(\text{LOCALS}\) is used to perform an exhaustive depth-first exploration of the summation subproblems (lines 3-4). BRAOBB-MMAP expands the chain portions of the pseudo tree corresponding to the MAP variables until the rotation limit \(R\) is reached or the current subproblem is fully solved (line 9). If the rotation limit \(R\) is reached before solving the current subproblem, the nodes in \(\text{LOCAL}\) are moved to the \(\text{GLOBAL}\) queue, and visited later. The partial solution tree \(T\) as well as the best solutions found so far need to be maintained according to the node expansion (lines 6-8). The algorithm returns the optimal solution upon
Algorithm 5: BRAOBB-BMMAP

Input: Graphical model $M = (X, D, F)$, pseudo tree $T$, $X_M = X \setminus S$, heuristic $h()$, rotation limit $R$
Output: the anytime MMAP value $V(T)$, and assignment $T$
1 Insert root to GLOBAL queue (FIFO)
2 while GLOBAL $\neq \emptyset$ do
3 \hspace{0.5cm} if LOCALS $\neq \emptyset$ then
4 \hspace{1cm} DFS traversal below $n$ using stack LOCALS
5 \hspace{1cm} LOCALM $\leftarrow$ front(GLOBAL)
6 \hspace{1cm} Maintain partial MMAP assignment $T'$
7 \hspace{1cm} if $T'$ is a complete MMAP assignment then
8 \hspace{1.5cm} Maintain and report best solution so far as $(V(T), T)$
9 \hspace{1cm} for $r \leftarrow 1$ to $R$ or until LOCAL = $\emptyset$ or until chldSubprob(LOCAL) $\neq \emptyset$ do
10 \hspace{1.5cm} Pop the top node $n$ in LOCAL
11 \hspace{1.5cm} if $n = (X, x)$ is OR node then
12 \hspace{1.75cm} foreach $x \in D$, do
13 \hspace{2cm} Create AND node $n' = (X, x)$
14 \hspace{1.75cm} Push $n'$ to top of LOCAL
15 \hspace{1.5cm} else if $n = (X, x)$ is AND node then
16 \hspace{1.75cm} $Y_1 \ldots Y_m \leftarrow$ children($X$)
17 \hspace{1.75cm} Create OR nodes $(Y_1) \ldots (Y_m)$
18 \hspace{1.5cm} if $m = 1$ then
19 \hspace{1.75cm} Push $(Y_1)$ to top of LOCAL
20 \hspace{1.5cm} else if $m > 1$ then
21 \hspace{1.75cm} for $j \leftarrow 1$ to $m$ do
22 \hspace{1.75cm} if $Y_j \in X_M$ then
23 \hspace{2cm} NEW $\leftarrow$ $(Y_j)\}$
24 \hspace{1.75cm} else if $Y_j \in X_S$ then
25 \hspace{2cm} Push NEW to back of GLOBAL
26 \hspace{1.75cm} \hspace{0.5cm} $\$ \break$
27 \hspace{1.75cm} \hspace{0.5cm} $\$ break$
28 \hspace{1.5cm} if $\text{succ}(n) = \emptyset$ then Propagate values upward
29 \hspace{1.5cm} if LOCAL $\neq \emptyset$ then Push LOCAL to back of GLOBAL
30 return $(V(T), T)$

completion or an anytime solution if interrupted.

**Empirical Evaluation**

We evaluate empirically the newly introduced anytime AND/OR search algorithms and compare them with the exact algorithms on problem instances generated from the PASCAL2 Inference Challenge benchmarks (Elidan, Globerson, and Heinemann 2012). All competing algorithms use the static weighted mini-bucket heuristic with moment matching, WMB-MM($i$), whose strength can be controlled by a parameter $i$-bound (Dechter and Rish 2003; Liu and Ihler 2011). The experiment supports different levels of strength for the guiding heuristics, from weak to strong, and time bounds up to 2 hours with 24 GB memory. The starting weight was 64 for WAOBF, WRAOBF, and RBFAOO, and the overestimation parameter was 1 for RBFAOO and WRBFAOO. For the 5 best-first algorithms, the size of the cache table was 4 GB, whereas for the 2 depth-first branch and bound algorithms the size of the cache table was only limited by the total memory limit of 24 GB for a fair comparison.

Our benchmark is a subset of PASCAL2 benchmark on 3 domains: grid with 15 networks; pedigree and

![Graphical model](image)

Figure 2: Percentage of instances solved ($i=18$). The height of the bottom bar represents the percent of instances solved optimally, and the height of the stacked bar represents the percent of instances with any solution.

promedas with 10 networks. For each network, we generated 5 MMAP problem instances where half of the variables are MAP variables: 1 easy instance such that the MAP variables were selected from a breadth-first traversal of a pseudo tree obtained from an unconstrained min-fill ordering; and 4 hard instances where the MAP variables were selected uniformly at random.

To compare anytime performance, we consider responsiveness and the solution quality: how quickly the algorithm can generate a solution (responsiveness), and how good is the solution provided by a given time bound (quality). The responsiveness was measured by the coverage of each algorithm in terms of optimal and anytime solutions at varying time bounds. Specifically, anytime coverage refers to the percentage of problem instances with a solution within the time bound, whereas optimal coverage only accounts for the optimally solved instances. The quality of anytime solution for each algorithm was evaluated by counting the number of problem instances with better anytime solutions. The metric for evaluating the score of each algorithm will be given later in this section.

**Performance Regimes of MMAP Algorithms**

In Table 1, we summarize the anytime performance regimes of 7 MMAP algorithms with respect to the responsiveness at 1 hour time bound with WMB-MM($i = 18$) heuristics, and the quality of anytime solutions at 1 hour time bound with WMB-MM($i = 12$) heuristics on 3 benchmark domains.

RBFAOO and AOBF are the two worst performing algorithms in terms of both anytime responsiveness and quality of solution. Algorithms WRBFAOO and WAOBF are the best performing algorithms in terms of solution quality, and the anytime responsiveness of both algorithms was close to that of depth-first search based algorithms, showing the benefit of weighted best-first search.

**Responsiveness of MMAP Algorithms**

In Figure 2, we present both optimal and anytime coverage results for 7 MMAP algorithms over varying time bounds from 1 second to 2 hours. Note that, all algorithms shown in Figure 2 used the WMB-MM($i = 18$) heuristic. Considering the optimality of the solution at 2 hour time bound,
Table 1: Anytime performance regimes for each of 7 evaluated algorithms. 3 exact MMAP algorithms (AOBB-MMAP, AOBF-MMAP, RBFAOO-MMAP) and 4 new anytime algorithms (BRAOBB-MMAP, WAOBF-MMAP, WRBFAOO-MMAP, WRAOBF-MMAP) are evaluated in grid, pedigree, and promedas domain. Responsiveness shows the percentage of instances solved at 1 hour time bound with WMB-MM(\(i = 18\)) heuristic. Quality score is the sum of individual quality scores against other 6 algorithms that compare the quality of anytime solutions at 1 hour time bound with WMB-MM(\(i = 12\)) heuristic. The metric for the score is defined in the following section.

<table>
<thead>
<tr>
<th>MMAP algorithms</th>
<th>responsiveness</th>
<th>quality score</th>
<th>responsiveness</th>
<th>quality score</th>
<th>responsiveness</th>
<th>quality score</th>
<th>responsiveness</th>
<th>quality score</th>
</tr>
</thead>
<tbody>
<tr>
<td>grid (75 instances)</td>
<td>pedigree (50 instances)</td>
<td>promedas (50 instances)</td>
<td>overall (175 instances)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>exact</td>
<td>AOBF</td>
<td>3260</td>
<td>69%</td>
<td>394%</td>
<td>87%</td>
<td>405%</td>
<td>89%</td>
<td>208%</td>
</tr>
<tr>
<td></td>
<td>RBFAOO</td>
<td>293%</td>
<td>84%</td>
<td>158%</td>
<td>42%</td>
<td>258%</td>
<td>50%</td>
<td>90%</td>
</tr>
<tr>
<td></td>
<td>WAOBF</td>
<td>379%</td>
<td>88%</td>
<td>364%</td>
<td>54%</td>
<td>266%</td>
<td>82%</td>
<td>339%</td>
</tr>
<tr>
<td></td>
<td>WRBFAOO</td>
<td>422%</td>
<td>90%</td>
<td>440%</td>
<td>60%</td>
<td>305%</td>
<td>86%</td>
<td>394%</td>
</tr>
<tr>
<td></td>
<td>WRAOBF</td>
<td>374%</td>
<td>88%</td>
<td>259%</td>
<td>58%</td>
<td>473%</td>
<td>86%</td>
<td>365%</td>
</tr>
<tr>
<td></td>
<td>BRAOBB</td>
<td>365%</td>
<td>58%</td>
<td>259%</td>
<td>94%</td>
<td>473%</td>
<td>86%</td>
<td>365%</td>
</tr>
</tbody>
</table>

RBFAOO found the optimal solution on the largest number of instances (59%), WRBFAOO followed it by 56%, while the remaining algorithms proved optimality for about 50%. Clearly, the newly introduced WRBFAOO, WAOBF, and WRAOBF provide a significant improvement in terms of responsiveness as they readily reported a solution on more than 80% of the instances at 1 second time bound, which is close to AOB and BRAOBB, respectively.

Solution Quality of MMAP Algorithms

Figure 3 reports the solution quality obtained by 5 MMAP algorithms (WRBFAOO, WAOBF, WRAOBF, BRAOBB, and AOB) at 1 hour time bound. The \(i\)-bound was \(i = 12\), which corresponds to guiding heuristics with moderate strength. The score is computed as follows. We record the solutions for each pair of algorithms A and B at a fixed time bound \(t\). If A won against B (i.e., A’s solution was better than B’s) on \(n\) instances and tied on \(m\) instances, the score of A against B is \(\frac{n + \frac{m}{2}}{N}\), where \(N\) is the total number of instances; the score of B against A is \(\frac{N - n - \frac{m}{2}}{N}\). In grid and pedigree, WRBFAOO is the winner, WAOBF is second, and WRAOBF is third. On the other hand, BRAOBB is the best algorithm in promedas, and AOB is second. The overall score across all domains shows that WRBFAOO is the winner, WAOBF is second, while AOB is the worst among the 5 competing algorithms.

The ranking between algorithms presented above remains the same on different settings with varying time bounds and \(i\)-bounds, which is not shown in this paper due to the space limit. BRAOBB is the best performer only in the extreme cases, typically characterized by very short time bounds \((t < 1 \text{ sec})\) or uninformative heuristics \((i < 6)\).

Example: grid instance 75-20-5-I4 in Table 2 is a case where weighted best-first search algorithms find a good solution in a short time bound. All three algorithms found the optimal solution in less than 1 minute, although they failed to prove optimality when the guiding heuristic was weak.

Comparing WRBFAOO and BRAOBB

We investigate further the anytime performance of WRBFAOO and BRAOBB based on two problem intrinsic parameters, the constrained induced width \(w_c\) and the conditioned induced width \(w_a\) of the summation problem only. Figure 4 visualizes the difficulty of individual problem instances and the winner at each point.
Table 2: Search statistics of anytime algorithms for grid, pedIGree, and promedas instances. \( m \) is the number of variables, \( n \) is the number of MAP variables, \( k \) is the maximum domain size, \( w_c \) is the constrained induced width, and \( w_s \) is the conditional induced width of the summation problem only. The current best solution in log scale, \( \text{sol} \), the number of AND map nodes, and \( \text{map} \), are displayed at two time bounds, 60 seconds and 10 minutes, and two \( i \)-bounds, 12 and 18. In addition, the number of AND nodes, \( o \), is shown for BRAOBB, while the weight, \( w \), is shown for the weighted best-first algorithms. The \( (t,7200) \) denotes time out at 7200 seconds, \( m(t) \) denotes out of memory at time \( t \), and \( o(t) \) indicates optimality found at time \( t \).

![Diagram](image_url)

Note that, the time bound for determining the winner was 1 hour and the \( i \)-bound of the WMB-MM heuristic was 12. The area of each region changes over different combinations of the time bounds and the \( i \)-bounds, but the trend remains the same, which is not presented here due to space reasons. Example: In Table 2, WRBFAO showed the best anytime performance in pedigree39-I3. However, best-first algorithms couldn’t find the first solution for two promedas instances when the guiding heuristic was relatively weak.

### Compareding Weighted Best-First Algorithms

We next explore the anytime behavior of the weighted best-first search algorithms based on the average weight at different time bounds. All algorithms used the \( \sqrt{t} \) policy with the initial weight 64 (i.e., \( w_{i+1} = \sqrt{w_i} \)).

**WRBFAO:** Figure 5 shows that WRBFAO has the lowest average weight after 5 seconds on both strong and relatively weak heuristics, consistent with the ranking in Figure 3. This implies that the depth-first like node expansion strategy of WRBFAO can not only avoid the memory issues but also allows the algorithm to report a tighter \( w \)-optimal solution.

**WAOBF vs. WRAOBF:** For shorter time bounds \((t < 30)\), the weight of WRAOBF decreased faster than WAOBF and stronger heuristics foster the speed. This implies that repairing schemes save on the number of node expansions and find tighter \( w \)-optimal solutions when \( w \) is high (\( w \geq 8 \)).

On the other hand, the average weight of WAOBF and WRAOBF are indistinguishable at longer time bounds. This implies that repairing the entire explicatu search space incurs severe overhead, losing its advantage against expanding the search space afresh. Furthermore, the solution quality

**Tied Region** \((w_c \in [10, 58])\): WRBFAO and BRAOBB are tied in this region. WRBFAO failed to report a single solution on 2 instances with \( w_s > 20 \) due to the overhead of computing the exact solution to the summation problem.

**WRBFAO Region** \((w_c \in [60, 200])\): WRBFAO dominates BRAOBB, consistent with the ranking in Figure 3. WRBFAO failed to report any solution when \( w_s > 10 \).

**BRAOBB Region** \((w_c \in [202, 490])\): Only BRAOBB reported a solution in this region, except for 2 instances. Such extremely hard instances are mostly promedas, consistent with BRAOBB being a winner on promedas in Figure 3.
Figure 5: Average weights \((i = 12, i = 18)\). There are two groups of curves with respect to the \(i\) bound. Each curve shows the average weight over instances with a solution at 2 hours.

of WAOBF is superior to WRAOBF in Figure 3. This implies that restarting scheme effectively escapes from unfruitful search space by discarding the explicated search space.

**Example:** In Table 2, WRAOBF found the optimal solution for 90–30–5–I0 more than 2 times faster than other algorithms. We can see that the weight of WRAOBF at 10 minute was the smallest compared to the other two algorithms.

**Conclusion**

We introduced new anytime search algorithms for MMAP that explore the context-minimal AND/OR search space for graphical models, either depth-first or best-first, and are guided by static weighted mini-bucket heuristics with variational cost-shifting. Our extensive empirical evaluation showed that weighted best-first AND/OR search algorithms have a particularly effective anytime performance for MMAP in terms of both responsiveness and the quality of solutions. In addition, these schemes were superior to anytime depth-first search for relatively weak heuristics. Nevertheless, depth-first search is the only possibility when the heuristic is very weak, or the conditional likelihood evaluation is hard. For future work, we plan to explore approximations for the summation sub-task and hybrid search schemes that combine the various anytime algorithms we introduced.

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**References**


Flerova, N.; Marinescu, R.; and Dechter, R. 2014. Weighted best first search for MAP. In International Symposium on Artificial Intelligence and Mathematics.


