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Approximate Probabilistic Inference via Word-Level Counting * †

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Abstract

Hashing-based model counting has emerged as a promising approach for large-scale probabilistic inference on graphical models. A key component of these techniques is the use of xor-based 2-universal hash functions that operate over Boolean domains. Many counting problems arising in probabilistic inference are, however, naturally encoded over finite discrete domains. Techniques based on bit-level (or Boolean) hash functions require these problems to be propositionalized, making it impossible to leverage the remarkable progress made in SMT (Satisfiability Modulo Theory) solvers that can reason directly over words (or bit-vectors). In this work, we present the first approximate model counter that uses word-level hashing functions, and can directly leverage the power of sophisticated SMT solvers. Empirical evaluation over an extensive suite of benchmarks demonstrates the promise of the approach.

1 Introduction

Probabilistic inference on large and uncertain data sets is increasingly being used in a wide range of applications. It is well-known that probabilistic inference is polynomially inter-reducible to model counting (Roth 1996). In a recent line of work, it has been shown (Chakraborty, Meel, and Vardi 2013; Chakraborty et al. 2014; Ermon et al. 2013; Ivrii et al. 2015) that one can strike a fine balance between performance and approximation guarantees for propositional model counting, using 2-universal hash functions (Carter and Wegman 1977) on Boolean domains. This has propelled the model-counting formulation to emerge as a promising "assembly language" (Belle, Passerini, and Van den Broeck 2015) for inferencing in probabilistic graphical models.

In a large class of probabilistic inference problems, an important case being lifted inference on first order representations (Kersting 2012), the values of variables come from finite but large (exponential in the size of the representation) domains. Data values coming from such domains are naturally encoded as fixed-width words, where the width is logarithmic in the size of the domain. Conditions on observed values are, in turn, encoded as word-level constraints, and the corresponding model-counting problem asks one to count the number of solutions of a word-level constraint. It is therefore natural to ask if the success of approximate propositional model counters can be replicated at the word-level.

The balance between efficiency and strong guarantees of hashing-based algorithms for approximate propositional model counting crucially depends on two factors: (i) use of XOR-based 2-universal bit-level hash functions, and (ii) use of state-of-the-art propositional satisfiability solvers, viz. CryptoMiniSAT (Soos, Nohl, and Castelluccia 2009), that can efficiently reason about formulas that combine disjunctive clauses with XOR clauses.

In recent years, the performance of SMT (Satisfiability Modulo Theories) solvers has witnessed spectacular improvements (Barrett et al. 2012). Indeed, several highly optimized SMTsolvers for fixed-width words are now available in the public domain (Brummayer and Biere 2009; Jha, Limaye, and Seshia 2009; Hadarean et al. 2014; De Moura and Bjørner 2008). Nevertheless, 2-universal hash functions for fixed-width words that are also amenable to efficient reasoning by SMT solvers have hitherto not been studied. The reasoning power of SMTsolvers for fixed-width words has therefore remained untapped for word-level model counting. Thus, it is not surprising that all existing work on probabilistic inference using model counting (viz. (Chistikov, Dimitrova, and Majumdar 2015; Belle, Passerini, and Van den Broeck 2015; Ermon et al. 2013)) effectively reduce the problem to propositional model counting. Such approaches are similar to "bit blasting" in SMT solvers (Kroening and Strichman 2008).

The primary contribution of this paper is an efficient word-level approximate model counting algorithm SMTApproxMC that can be employed to answer inference queries over high-dimensional discrete domains. Our algorithm uses a new class of word-level hash functions that are 2-universal and can be solved by word-level SMTsolvers capable of reasoning about linear equalities on words. Therefore, unlike previous works, SMTApproxMC is able to leverage the power of sophisticated SMT solvers.

To illustrate the practical utility of SMTApproxMC, we implemented a prototype and evaluated it on a suite of benchmarks. Our experiments demonstrate that

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SMTApproxMC can significantly outperform the prevalent approach of bit-blasting a word-level constraint and using an approximate propositional model counter that employs XOR-based hash functions. Our proposed word-level hash functions embed the domain of all variables in a large enough finite domain. Thus, one would not expect our approach to work well for constraints that exhibit a hugely heterogeneous mix of word widths, or for problems that are difficult for word-level SMT solvers. Indeed, our experiments suggest that the use of word-level hash functions provides significant benefits when the original word-level constraint is such that (i) the words appearing in it have long and similar widths, and (ii) the SMTsolver can reason about the constraint at the word-level, without extensive bit-blasting.

2 Preliminaries

A *word* (or *bit-vector*) is an array of bits. The size of the array is called the *width* of the word. We consider here *fixed-width* words, whose width is a constant. It is easy to see that a word of width k can be used to represent elements of a set of size 2^k. The first-order theory of fixed-width words has been extensively studied (see (Kroening and Strichman 2008; Bruttomesso 2008) for an overview). The vocabulary of this theory includes interpreted predicates and functions, whose semantics are defined over words interpreted as signed integers, unsigned integers, or vectors of propositional constants (depending on the function or predicate). When a word of width k is treated as a vector, we assume that the component bits are indexed from 0 through $k-1$, where index 0 corresponds to the rightmost bit. A *term* is either a word-level variable or constant, or is obtained by applying functions in the vocabulary to a term. Every term has an associated width that is uniquely defined by the widths of word-level variables and constants in the term, and by the semantics of functions used to build the term. For purposes of this paper, given terms t_1 and t_2, we use $t_1 + t_2$ (resp. $t_1 * t_2$) to denote the sum (resp. product) of t_1 and t_2, interpreted as unsigned integers. Given a positive integer p, we use $t_1 \mod p$ to denote the remainder after dividing t_1 by p. Furthermore, if t_1 has width k, and a and b are integers such that $0 \leq a \leq b < k$, we use $\mathsf{extract}(t_1, a, b)$ to denote the slice of t_1 (interpreted as a vector) between indices a and b, inclusively.

Let F be a formula in the theory of fixed-width words. The *support* of F, denoted $\mathsf{sup}(F)$, is the set of word-level variables that appear in F. A *model* or *solution* of F is an assignment of word-level constants to variables in $\mathsf{sup}(F)$ such that F evaluates to True. We use R_F to denote the set of *models* of F. The model-counting problem requires us to compute $|R_F|$. For simplicity of exposition, we assume henceforth that all words in $\mathsf{sup}(F)$ have the same width. Note that this is without loss of generality, since if k is the maximum width of all words in $\mathsf{sup}(F)$, we can construct a formula $\widehat{F}$ such that the following hold: (i) $|\mathsf{sup}(F)| = |\mathsf{sup}(\widehat{F})|$, (ii) all word-level variables in $\widehat{F}$ have width k, and (iii) $|R_F| = |R_{\widehat{F}}|$. The formula $\widehat{F}$ is obtained by replacing every occurrence of word-level variable x having width m $(< k)$ in F with $\mathsf{extract}(\widehat{x}, 0, m-1)$, where $\widehat{x}$ is a new variable of width k.

We write $\mathsf{Pr}\,[X : \mathcal{P}]$ for the probability of outcome X when sampling from a probability space $\mathcal{P}$. For brevity, we omit $\mathcal{P}$ when it is clear from the context.

Given a word-level formula F, an *exact model counter* returns $|R_F|$. An *approximate model counter* relaxes this requirement to some extent: given a *tolerance* $\varepsilon > 0$ and *confidence* $1 - \delta \in (0, 1]$, the value v returned by the counter satisfies $\mathsf{Pr}[\frac{|R_F|}{1+\varepsilon} \leq v \leq (1+\varepsilon)|R_F|] \geq 1 - \delta$. Our model-counting algorithm belongs to the class of approximate model counters.

Special classes of hash functions, called 2-*wise independent universal* hash functions play a crucial role in our work. Let $\mathsf{sup}(F) = \{x_0, \ldots x_{n-1}\}$, where each x_i is a word of width k. The space of all assignments of words in $\mathsf{sup}(F)$ is $\{0,1\}^{n.k}$. We use hash functions that map elements of $\{0,1\}^{n.k}$ to p bins labeled $0, 1, \ldots p-1$, where $1 \leq p < 2^{n.k}$. Let $\mathbb{Z}_p$ denote $\{0, 1, \ldots p-1\}$ and let $\mathcal{H}$ denote a family of hash functions mapping $\{0,1\}^{n.k}$ to $\mathbb{Z}_p$. We use $h \xleftarrow{R} \mathcal{H}$ to denote the probability space obtained by choosing a hash function h uniformly at random from $\mathcal{H}$. We say that $\mathcal{H}$ is a 2-wise independent universal hash family if for all $\alpha_1, \alpha_2 \in \mathbb{Z}_p$ and for all distinct $\mathbf{X}_1, \mathbf{X}_2 \in \{0,1\}^{n.k}$, $\mathsf{Pr}\left[h(\mathbf{X}_1) = \alpha_1 \wedge h(\mathbf{X}_2) = \alpha_2 : h \xleftarrow{R} \mathcal{H}\right] = 1/p^2$.

3 Related Work

The connection between probabilistic inference and model counting has been extensively studied by several authors (Cooper 1990; Roth 1996; Chavira and Darwiche 2008), and it is known that the two problems are inter-reducible. Propositional model counting was shown to be #P-complete by Valiant (Valiant 1979). It follows easily that the model counting problem for fixed-width words is also #P-complete. It is therefore unlikely that efficient exact algorithms exist for this problem. (Bellare, Goldreich, and Petrank 2000) showed that a closely related problem, that of almost uniform sampling from propositional constraints, can be solved in probabilistic polynomial time using an NP oracle. Subsequently, (Jerrum, Valiant, and Vazirani 1986) showed that approximate model counting is polynomially inter-reducible to almost uniform sampling. While this shows that approximate model counting is solvable in probabilstic polynomial time relative to an NP oracle, the algorithms resulting from this largely theoretical body of work are highly inefficient in practice (Meel 2014).

Building on the work of Bellare, Goldreich and Petrank (2000), Chakraborty, Meel and Vardi (2013) proposed the first scalable approximate model counting algorithm for propositional formulas, called ApproxMC. Their technique is based on the use of a family of 2-universal bit-level hash functions that compute XOR of randomly chosen propositional variables. Similar bit-level hashing techniques were also used in (Ermon et al. 2013; Chakraborty et al. 2014) for weighted model counting. All of these works leverage the significant advances made in propositional satisfiability solving in the recent past (Biere et al. 2009).

Over the last two decades, there has been tremendous progress in the development of decision procedures, called Satisfiability Modulo Theories (or SMT) solvers, for combinations of first-order theories, including the theory of fixed-width words (Barrett, Fontaine, and Tinelli 2010; Barrett, Moura, and Stump 2005). An SMT solver uses a core propositional reasoning engine and decision procedures for individual theories, to determine the satisfiability of a formula in the combination of theories. It is now folklore that a well-engineered word-level SMT solver can significantly outperform the naive approach of *blasting* words into component bits and then using a propositional satisfiability solver (De Moura and Bjørner 2008; Jha, Limaye, and Seshia 2009; Bruttomesso et al. 2007). The power of word-level SMT solvers stems from their ability to reason about words directly (e.g. $a + (b - c) = (a - c) + b$ for every word a, b, c), instead of *blasting* words into component bits and using propositional reasoning.

The work of (Chistikov, Dimitrova, and Majumdar 2015) tried to extend ApproxMC (Chakraborty, Meel, and Vardi 2013) to non-propositional domains. A crucial step in their approach is to propositionalize the solution space (e.g. bounded integers are equated to tuples of propositions) and then use XOR-based bit-level hash functions. Unfortunately, such propositionalization can significantly reduce the effectiveness of theory-specific reasoning in an SMT solver. The work of (Belle, Passerini, and Van den Broeck 2015) used bit-level hash functions with the propositional abstraction of an SMT formula to solve the problem of *weighted model integration*. This approach also fails to harness the power of theory-specific reasoning in SMT solvers.

Recently, (de Salvo Braz et al. 2015) proposed SGDPLL(T), an algorithm that generalizes SMT solving to do lifted inferencing and model counting (among other things) modulo background theories (denoted T). A fixed-width word model counter, like the one proposed in this paper, can serve as a theory-specific solver in the SGDPLL(T) framework. In addition, it can also serve as an alernative to SGDPLL(T) when the overall problem is simply to count models in the theory T of fixed-width words, There have also been other attempts to exploit the power of SMT solvers in machine learning. For example, (Teso, Sebastiani, and Passerini 2014) used optimizing SMT solvers for structured relational learning using Support Vector Machines. This is unrelated to our approach of harnessing the power of SMT solvers for probabilistic inference via model counting.

4 Word-level Hash Function

The performance of hashing-based techniques for approximate model counting depends crucially on the underlying family of hash functions used to partition the solution space. A popular family of hash functions used in propositional model counting is $\mathcal{H}_{xor}$, defined as the family of functions obtained by XOR-ing a random subset of propositional variables, and equating the result to either 0 or 1, chosen randomly. The family $\mathcal{H}_{xor}$ enjoys important properties like 2-independence and easy implementability, which make it ideal for use in practical model counters for propositional formulas (Gomes, Sabharwal, and Selman 2007; Ermon et al. 2013; Chakraborty, Meel, and Vardi 2013). Unfortunately, word-level universal hash families that are 2-independent, easily implementable and amenable to word-level reasoning by SMT solvers, have not been studied thus far. In this section, we present $\mathcal{H}_{SMT}$, a family of word-level hash functions that fills this gap.

As discussed earlier, let $\mathsf{sup}(F) = \{x_0, \ldots x_{n-1}\}$, where each x_i is a word of width k. We use $\mathbf{X}$ to denote the n-dimensional vector $(x_0, \ldots x_{n-1})$. The space of all assignments to words in $\mathbf{X}$ is $\{0,1\}^{n.k}$. Let p be a prime number such that $2^k \leq p < 2^{n.k}$. Consider a family $\mathcal{H}$ of hash functions mapping $\{0,1\}^{n.k}$ to $\mathbb{Z}_p$, where each hash function is of the form $h(\mathbf{X}) = (\sum_{j=0}^{n-1} a_j * x_j + b) \mod p$, and the a_j's and b are elements of $\mathbb{Z}_p$, represented as words of width $\lceil \log_2 p \rceil$. Observe that every $h \in \mathcal{H}$ partitions $\{0,1\}^{n.k}$ into p bins (or cells). Moreover, for every $\xi \in \{0,1\}^{n.k}$ and $\alpha \in \mathbb{Z}_p$, $\mathsf{Pr}\left[h(\xi) = \alpha : h \xleftarrow{R} \mathcal{H}\right] = p^{-1}$. For a hash function chosen uniformly at random from $\mathcal{H}$, the expected number of elements per cell is $2^{n.k}/p$. Since $p < 2^{n.k}$, every cell has at least 1 element in expectation. Since $2^k \leq p$, for every word x_i of width k, we also have $x_i \mod p = x_i$. Thus, distinct words are not aliased (or made to behave similarly) because of modular arithmetic in the hash function.

Suppose now we wish to partition $\{0.1\}^{n.k}$ into p^c cells, where $c > 1$ and $p^c < 2^{n.k}$. To achieve this, we need to define hash functions that map elements in $\{0,1\}^{n.k}$ to a tuple in $(\mathbb{Z}_p)^c$. A simple way to achieve this is to take a c-tuple of hash functions, each of which maps $\{0,1\}^{n.k}$ to $\mathbb{Z}_p$. Therefore, the desired family of hash functions is simply the iterated Cartesian product $\mathcal{H} \times \cdots \times \mathcal{H}$, where the product is taken c times. Note that every hash function in this family is a c-tuple of hash functions. For a hash function chosen uniformly at random from this family, the expected number of elements per cell is $2^{n.k}/p^c$.

An important consideration in hashing-based techniques for approximate model counting is the choice of a hash function that yields cells that are neither too large nor too small in their expected sizes. Since increasing c by 1 reduces the expected size of each cell by a factor of p, it may be difficult to satisfy the above requirement if the value of p is large. At the same time, it is desirable to have $p > 2^k$ to prevent aliasing of two distinct words of width k. This motivates us to consider more general classes of word-level hash functions, in which each word x_i can be split into thinner slices, effectively reducing the width k of words, and allowing us to use smaller values of p. We describe this in more detail below.

Assume for the sake of simplicity that k is a power of 2, and let q be $\log_2 k$. For every $j \in \{0, \ldots q-1\}$ and for every $x_i \in \mathbf{X}$, define $\mathbf{x_i}^{(j)}$ to be the 2^j-dimensional vector of slices of the word x_i, where each slice is of width $k/2^j$. For example, the two slices in $\mathbf{x_1}^{(1)}$ are $\mathsf{extract}(x_1, 0, k/2 - 1)$ and $\mathsf{extract}(x_1, k/2, k-1)$. Let $\mathbf{X}^{(j)}$ denote the $n.2^j$-dimensional vector $(\mathbf{x_0}^{(j)}, \mathbf{x_1}^{(j)}, \ldots \mathbf{x_{n-1}}^{(j)})$. It is easy to see that the m^{th} component of $\mathbf{X}^{(j)}$, denoted $\mathbf{X}_m^{(j)}$, is $\mathsf{extract}(x_i, s, t)$, where $i = \lfloor m/2^j \rfloor$, $s = (m \mod 2^j) \cdot (k/2^j)$ and $t = s + (k/2^j) - 1$. Let p_j denote the smallest prime larger than or equal to $2^{(k/2^j)}$. Note that this implies $p_{j+1} \leq p_j$ for all $j \geq 0$. In order to obtain a family of hash functions that maps $\{0,1\}^{n.k}$ to $\mathbb{Z}_{p_j}$, we split each word x_i into slices of width $k/2^j$, treat these slices as words of reduced width, and use a technique similar to the one used above to map $\{0,1\}^{n.k}$ to $\mathbb{Z}_p$. Specifically, the family $\mathcal{H}^{(j)} = \left\{h^{(j)} : h^{(j)}(\mathbf{X}) = \left(\sum_{m=0}^{n.2^j-1} a_m^{(j)} * \mathbf{X}_m^{(j)} + b^{(j)}\right) \mod p_j\right\}$ maps $\{0,1\}^{n.k}$ to $\mathbb{Z}_{p_j}$, where the values of $a_m^{(j)}$ and $b^{(j)}$ are chosen from $\mathbb{Z}_{p_j}$, and represented as $\lceil \log_2 p_j \rceil$-bit words.

In general, we may wish to define a family of hash functions that maps $\{0,1\}^{n.k}$ to $\mathcal{D}$, where $\mathcal{D}$ is given by $(\mathbb{Z}_{p_0})^{c_0} \times (\mathbb{Z}_{p_1})^{c_1} \times \cdots (\mathbb{Z}_{p_{q-1}})^{c_{q-1}}$ and $\prod_{j=0}^{q-1} p_j^{c_j} < 2^{n.k}$. To achieve this, we first consider the iterated Cartesian product of $\mathcal{H}^{(j)}$ with itself c_j times, and denote it by $\left(\mathcal{H}^{(j)}\right)^{c_j}$, for every $j \in \{0, \dots q-1\}$. Finally, the desired family of hash functions is obtained as $\prod_{j=0}^{q-1} \left(\mathcal{H}^{(j)}\right)^{c_j}$. Observe that every hash function h in this family is a $\left(\sum_{l=0}^{q-1} c_l\right)$-tuple of hash functions. Specifically, the r^{th} component of h, for $r \leq \left(\sum_{l=0}^{q-1} c_l\right)$, is given by $\left(\sum_{m=0}^{n.2^j-1} a_m^{(j)} * \mathbf{X}_m^{(j)} + b^{(j)}\right) \mod p_j$, where $\left(\sum_{i=0}^{j-1} c_i\right) < r \leq \left(\sum_{i=0}^{j} c_i\right)$, and the $a_m^{(j)}$s and $b^{(j)}$ are elements of $\mathbb{Z}_{p_j}$.

The case when k is not a power of 2 is handled by splitting the words x_i into slices of size $\lceil k/2 \rceil$, $\lceil k/2^2 \rceil$ and so on. Note that the family of hash functions defined above depends only on n, k and the vector $C = (c_0, c_1, \dots c_{q-1})$, where $q = \lceil \log_2 k \rceil$. Hence, we call this family $\mathcal{H}_{SMT}(n,k,C)$. Note also that by setting c_i to 0 for all $i \neq \lfloor \log_2(k/2) \rfloor$, and c_i to r for $i = \lfloor \log_2(k/2) \rfloor$ reduces $\mathcal{H}_{SMT}$ to the family $\mathcal{H}_{xor}$ of XOR-based bit-wise hash functions mapping $\{0,1\}^{n.k}$ to $\{0,1\}^r$. Therefore, H_{SMT} strictly generalizes $\mathcal{H}_{xor}$.

We summarize below important properties of the $\mathcal{H}_{SMT}(n,k,C)$ class. All proofs are available in (Chakraborty et al. 2015).

Lemma 1. *For every $\mathbf{X} \in \{0,1\}^{n.k}$ and every $\alpha \in \mathcal{D}$,* $\mathsf{Pr}[h(\mathbf{X}) = \alpha \mid h \xleftarrow{R} \mathcal{H}_{SMT}(n,k,C)] = \prod_{j=0}^{|C|-1} {p_j}^{-c_j}$

Theorem 1. *For every $\alpha_1, \alpha_2 \in \mathcal{D}$ and every distinct $\mathbf{X}_1, \mathbf{X}_2 \in \{0,1\}^{n.k}$, $\mathsf{Pr}[(h(\mathbf{X}_1) = \alpha_1 \wedge h(\mathbf{X}_2) = \alpha_2) \mid h \xleftarrow{R} \mathcal{H}_{SMT}(n,k,C)] = \prod_{j=0}^{|C|-1} (p_j)^{-2.c_j}$. Therefore, $\mathcal{H}_{SMT}(n,k,C)$ is pairwise independent.*

Gaussian Elimination The practical success of XOR-based bit-level hashing techniques for propositional model counting owes a lot to solvers like CryptoMiniSAT (Soos, Nohl, and Castelluccia 2009) that use Gaussian Elimination to efficiently reason about XOR constraints. It is significant that the constraints arising from $\mathcal{H}_{SMT}$ are linear modular equalities that also lend themselves to efficient Gaussian Elimination. We believe that integration of Gaussian Elimination engines in SMT solvers will significantly improve the performance of hashing-based word-level model counters.

5 Algorithm

We now present SMTApproxMC, a word-level hashing-based approximate model counting algorithm. SMTApproxMC takes as inputs a formula F in the theory of fixed-width words, a tolerance ε (> 0), and a confidence $1-\delta \in (0,1]$. It returns an estimate of $|R_F|$ within the tolerance ε, with confidence $1-\delta$. The formula F is assumed to have n variables, each of width k, in its support. The central idea of SMTApproxMC is to randomly partition the solution space of F into "small" cells of roughly the same size, using word-level hash functions from $\mathcal{H}_{SMT}(n,k,C)$, where C is incrementally computed. The check for "small"-ness of cells is done using a word-level SMT solver. The use of word-level hash functions and a word-level SMT solver allows us to directly harness the power of SMT solving in model counting.

The pseudocode for SMTApproxMC is presented in Algorithm 1. Lines 1– 3 initialize the different parameters. Specifically, pivot determines the maximum size of a "small" cell as a function of ε, and t determines the number of times SMTApproxMCCore must be invoked, as a function of δ. The value of t is determined by technical arguments in the proofs of our theoretical guarantees, and is not based on experimental observations Algorithm SMTApproxMCCore lies at the heart of SMTApproxMC. Each invocation of SMTApproxMCCore either returns an approximate model count of F, or $\perp$ (indicating a failure). In the former case, we collect the returned value, m, in a list M in line 8. Finally, we compute the median of the approximate counts in M, and return this as $\mathrm{FinalCount}$.

Algorithm 1 SMTApproxMC($F, \varepsilon, \delta, k$)

```
1:  counter ← 0; M ← emptyList;
2:  pivot ← 2 × ⌈e^{3/2} (1 + 1/ε)^2⌉;
3:  t ← ⌈35 log_2(3/δ)⌉;
4:  repeat
5:      m ← SMTApproxMCCore(F, pivot, k);
6:      counter ← counter + 1;
7:      if m ≠ ⊥ then
8:          AddToList(M, m);
9:  until (counter < t)
10: FinalCount ← FindMedian(M);
11: return FinalCount;
```

The pseudocode for SMTApproxMCCore is shown in Algorithm 2. This algorithm takes as inputs a word-level SMT formula F, a threshold pivot, and the width k of words in $\mathsf{sup}(F)$. We assume access to a subroutine BoundedSMT that accepts a word-level SMT formula φ and a threshold pivot as inputs, and returns pivot + 1 solutions of φ if $|R_\varphi| >$ pivot; otherwise it returns R_φ. In lines 1– 2 of Algorithm 2, we return the exact count if $|R_F| \leq$ pivot. Otherwise, we initialize C by setting $C[0]$ to 0 and $C[1]$ to 1, where $C[i]$ in the pseudocode refers to c_i in the previous section's discussion. This choice of initialization is motivated by our experimental observations. We also count the number of cells generated by an arbitrary hash function from

Algorithm 2 SMTApproxMCCore(F, pivot, k)

1: $Y \leftarrow$ BoundedSMT(F, pivot);
2: **if** $|Y| \leq$ pivot) **then return** $|Y|$;
3: **else**
4: $\quad C \leftarrow$ emptyVector; $C[0] \leftarrow 0$; $C[1] \leftarrow 1$;
5: $\quad i \leftarrow 1$; numCells $\leftarrow p_1$;
6: $\quad$ **repeat**
7: $\quad\quad$ Choose h at random from $\mathcal{H}_{SMT}(n, k, C)$;
8: $\quad\quad$ Choose α at random from $\prod_{j=0}^{i} \left(\mathbb{Z}_{p_j}\right)^{C[j]}$;
9: $\quad\quad Y \leftarrow$ BoundedSMT($F \wedge (h(\mathbf{X}) = \alpha)$, pivot);
10: $\quad\quad$ **if** ($|Y| >$ pivot) **then**
11: $\quad\quad\quad C[i] \leftarrow C[i] + 1$;
12: $\quad\quad\quad$ numCells $\leftarrow$ numCells $\times\, p_i$;
13: $\quad\quad$ **if** ($|Y| = 0$) **then**
14: $\quad\quad\quad$ **if** $p_i > 2$ **then**
15: $\quad\quad\quad\quad C[i] \leftarrow C[i] - 1$;
16: $\quad\quad\quad\quad i \leftarrow i + 1$; $C[i] \leftarrow 1$;
17: $\quad\quad\quad\quad$ numCells $\leftarrow$ numCells $\times\, (p_{i+1}/p_i)$;
18: $\quad\quad\quad$ **else**
19: $\quad\quad\quad\quad$ break;
20: $\quad$ **until** ($(0 < |Y| \leq$ pivot) or (numCells $> 2^{n.k}$))
21: $\quad$ **if** (($|Y| >$ pivot) **or** ($|Y| = 0$)) **then return** $\perp$;
22: $\quad$ **else return** $|Y| \times$ numCells;

$\mathcal{H}_{SMT}(n, k, C)$ in numCells. The loop in lines 6–20 iteratively partitions R_F into cells using randomly chosen hash functions from $\mathcal{H}_{SMT}(n, k, C)$. The value of i in each iteration indicates the extent to which words in the support of F are sliced when defining hash functions in $\mathcal{H}_{SMT}(n, k, C)$ – specifically, slices that are $\lceil k/2^i \rceil$-bits or more wide are used. The iterative partitioning of R_F continues until a randomly chosen cell is found to be "small" (i.e. has ≥ 1 and $\leq$ pivot solutions), or the number of cells exceeds $2^{n.k}$, rendering further partitioning meaningless. The random choice of h and α in lines 7 and 8 ensures that we pick a random cell. The call to BoundedSMT returns at most pivot$+1$ solutions of F within the chosen cell in the set Y. If $|Y| >$ pivot, the cell is deemed to be large, and the algorithm partitions each cell further into p_i parts. This is done by incrementing $C[i]$ in line 11, so that the hash function chosen from $\mathcal{H}_{SMT}(n, k, C)$ in the next iteration of the loop generates p_i times more cells than in the current iteration. On the other hand, if Y is empty and $p_i > 2$, the cells are too small (and too many), and the algorithm reduces the number of cells by a factor of p_{i+1}/p_i (recall $p_{i+1} \leq p_i$) by setting the values of $C[i]$ and $C[i+1]$ accordingly (see lines15 –17). If Y is non-empty and has no more than pivot solutions, the cells are of the right size, and we return the estimate $|Y| \times$ numCells. In all other cases, SMTApproxMCCore fails and returns $\perp$.

Similar to the analysis of ApproxMC (Chakraborty, Meel, and Vardi 2013), the current theoretical analysis of SMTApproxMC assumes that for some C during the execution of SMTApproxMCCore, $\log |R_F| - \log(\text{numCells}) - 1 = \log(pivot)$. We leave analysis of SMTApproxMC without above assumption to future work. The following theorems concern the correctness and performance of SMTApproxMC.

Theorem 2. *Suppose an invocation of* SMTApproxMC($F, \varepsilon, \delta, k$) *returns* FinalCount. *Then* $\Pr\left[(1+\varepsilon)^{-1}|R_F| \leq \text{FinalCount} \leq (1+\varepsilon)|R_F|\right] \geq 1 - \delta$

Theorem 3. SMTApproxMC($F, \varepsilon, \delta, k$) *runs in time polynomial in* $|F|$, $1/\varepsilon$ *and* $\log_2(1/\delta)$ *relative to an* NP*-oracle.*

The proofs of Theorem 2 and 3 can be found in (Chakraborty et al. 2015).

6 Experimental Methodology and Results

To evaluate the performance and effectiveness of SMTApproxMC, we built a prototype implementation and conducted extensive experiments. Our suite of benchmarks consisted of more than 150 problems arising from diverse domains such as reasoning about circuits, planning, program synthesis and the like. For lack of space, we present results for only for a subset of the benchmarks.

For purposes of comparison, we also implemented a state-of-the-art bit-level hashing-based approximate model counting algorithm for bounded integers, proposed by (Chistikov, Dimitrova, and Majumdar 2015). Henceforth, we refer to this algorithm as CDM, after the authors' initials. Both model counters used an overall timeout of 12 hours per benchmark, and a BoundedSMT timeout of 2400 seconds per call. Both used Boolector, a state-of-the-art SMT solver for fixed-width words (Brummayer and Biere 2009). Note that Boolector (and other popular SMT solvers for fixed-width words) does not yet implement Gaussian elimination for linear modular equalities; hence our experiments did not enjoy the benefits of Gaussian elimination. We employed the Mersenne Twister to generate pseudo-random numbers, and each thread was seeded independently using the Python `random` library. All experiments used $\varepsilon = 0.8$ and $\delta = 0.2$. Similar to ApproxMC, we determined value of t based on tighter analysis offered by proofs. For detailed discussion, we refer the reader to Section 6 in (Chakraborty, Meel, and Vardi 2013). Every experiment was conducted on a single core of high-performance computer cluster, where each node had a 20-core, 2.20 GHz Intel Xeon processor, with 3.2GB of main memory per core.

We sought answers to the following questions from our experimental evaluation:

1. How does the performance of SMTApproxMC compare with that of a bit-level hashing-based counter like CDM?
2. How do the approximate counts returned by SMTApproxMC compare with exact counts?

Our experiments show that SMTApproxMC significantly outperforms CDM for a large class of benchmarks. Furthermore, the counts returned by SMTApproxMC are highly accurate and the observed geometric tolerance(ε_{obs}) = 0.04.

Performance Comparison Table 1 presents the result of comparing the performance of SMTApproxMC vis-a-vis CDM on a subset of our benchmarks. In Table 1, column 1 gives the benchmark identifier, column 2 gives the sum of widths of all variables, column 3 lists the number of

Benchmark	Total Bits	Variable Types	# of Operations	SMTApproxMC time(s)	CDM time(s)
squaring27	59	{1: 11, 16: 3}	10	–	2998.97
squaring51	40	{1: 32, 4: 2}	7	3285.52	607.22
1160877	32	{8: 2, 16: 1}	8	2.57	44.01
1160530	32	{8: 2, 16: 1}	12	2.01	43.28
1159005	64	{8: 4, 32: 1}	213	28.88	105.6
1160300	64	{8: 4, 32: 1}	1183	44.02	71.16
1159391	64	{8: 4, 32: 1}	681	57.03	91.62
1159520	64	{8: 4, 32: 1}	1388	114.53	155.09
1159708	64	{8: 4, 32: 1}	12	14793.93	–
1159472	64	{8: 4, 32: 1}	8	16308.82	–
1159115	64	{8: 4, 32: 1}	12	23984.55	–
1159431	64	{8: 4, 32: 1}	12	36406.4	–
1160191	64	{8: 4, 32: 1}	12	40166.1	–

Table 1: Runtime performance of SMTApproxMC vis-a-vis CDM for a subset of benchmarks.

variables ($\mathrm{numVars}$) for each corresponding width (w) in the format $\{\mathrm{w} : \mathrm{numVars}\}$. To indicate the complexity of the input formula, we present the number of operations in the original SMT formula in column 4. The runtimes for SMTApproxMC and CDM are presented in columns 5 and column 6 respectively. We use "–" to denote timeout after 12 hours. Table 1 clearly shows that SMTApproxMC significantly outperforms CDM (often by 2-10 times) for a large class of benchmarks. In particular, we observe that SMTApproxMC is able to compute counts for several cases where CDM times out.

Benchmarks in our suite exhibit significant heterogeneity in the widths of words, and also in the kinds of word-level operations used. Propositionalizing all word-level variables eagerly, as is done in CDM, prevents the SMT solver from making full use of word-level reasoning. In contrast, our approach allows the power of word-level reasoning to be harnessed if the original formula F and the hash functions are such that the SMT solver can reason about them without bit-blasting. This can lead to significant performance improvements, as seen in Table 1. Some benchmarks, however, have heterogenous bit-widths and heavy usage of operators like $\mathsf{extract}(x, n_1, n_2)$ and/or word-level multiplication. It is known that word-level reasoning in modern SMT solvers is not very effective for such cases, and the solver has to resort to bit-blasting. Therefore, using word-level hash functions does not help in such cases. We believe this contributes to the degraded performance of SMTApproxMC vis-a-vis CDM in a subset of our benchmarks. This also points to an interesting direction of future research: to find the right hash function for a benchmark by utilizing SMT solver's architecture.

Quality of Approximation To measure the quality of the counts returned by SMTApproxMC, we selected a subset of benchmarks that were small enough to be bit-blasted and fed to sharpSAT (Thurley 2006) – a state-of-the-art exact model counter. Figure 1 compares the model counts computed by SMTApproxMC with the bounds obtained by scaling the exact counts (from sharpSAT) with the tolerance factor ($\varepsilon = 0.8$). The y-axis represents model counts on log-scale while the x-axis presents benchmarks ordered in

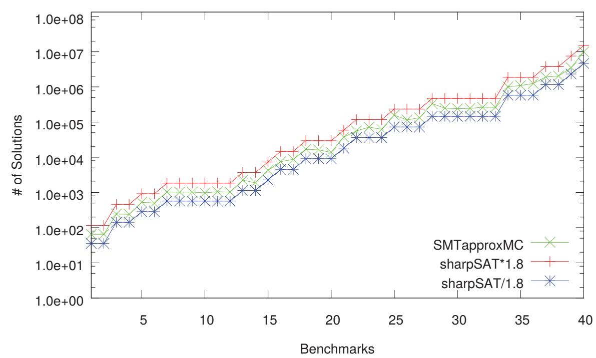

Figure 1: Quality of counts computed by SMTApproxMC vis-a-vis exact counts

ascending order of model counts. We observe that for *all* the benchmarks, SMTApproxMC computes counts within the tolerance. Furthermore, for each instance, we computed observed tolerance (ε_{obs}) as $\frac{\mathrm{count}}{|R_F|} - 1$, if $\mathrm{count} \geq |R_F|$, and $\frac{|R_F|}{\mathrm{count}} - 1$ otherwise, where $|R_F|$ is computed by sharpSAT and count is computed by SMTApproxMC. We observe that the geometric mean of ε_{obs} across all the benchmarks is only 0.04 – far less (i.e. closer to the exact count) than the theoretical guarantee of 0.8.

7 Conclusions and Future Work

Hashing-based model counting has emerged as a promising approach for probabilistic inference on graphical models. While real-world examples naturally have word-level constraints, state-of-the-art approximate model counters effectively reduce the problem to propositional model counting due to lack of non-bit-level hash functions. In this work, we presented, $\mathcal{H}_{SMT}$, a word-level hash function and used it to build SMTApproxMC, an approximate word-level model counter. Our experiments show that SMTApproxMC can significantly outperform techniques based on bit-level hashing.

Our study also presents interesting directions for future work. For example, adapting SMTApproxMC to be aware of SMT solving strategies, and augmenting SMT solving strategies to efficiently reason about hash functions used in counting, are exciting directions of future work.

Our work goes beyond serving as a replacement for other approximate counting techniques. SMTApproxMC can also be viewed as an efficient building block for more sophisticated inference algorithms (de Salvo Braz et al. 2015). The development of SMT solvers has so far been primarily driven by the verification and static analysis communities. Our work hints that probabilistic inference could well be another driver for SMT solver technology development.

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