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A Semantical Analysis of Second-Order Propositional Modal Logic

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Abstract

This paper is aimed as a contribution to the use of formal modal languages in Artificial Intelligence. We introduce a multi-modal version of Second-order Propositional Modal Logic (SOPML), an extension of modal logic with propositional quantification, and illustrate its usefulness as a specification language for knowledge representation as well as temporal and spatial reasoning. Then, we define novel notions of (bi)simulation and prove that these preserve the interpretation of SOPML formulas. Finally, we apply these results to assess the expressive power of SOPML.

1 Introduction

Modal logic (Blackburn, de Rijke, and Venema 2001; Blackburn, van Benthem, and Wolter 2007) has become one of the most popular formal frameworks in Artificial Intelligence and knowledge representation (van Harmelen, Lifschitz, and Porter 2007). There is a number of reasons for this. At the core of the semantics of modal logic lies the notion of *world*, or *state*. Indeed, the notion of state is very natural when studying computational concepts, i.e., systems evolving over time, or notions of agency (states that are preferred, desired, or epistemically possible) and of interaction (e.g., states can be winning, losing, terminal, initial, ...). Indeed, distributed computing (Halpern and Moses 1990), temporal systems (Manna and Pnueli 1992), multi-agent systems (van der Hoek and Wooldridge 2008) and game theory (van der Hoek and Pauly 2006) have all been studied within modal logic, and this list is by no means exhaustive. Importantly, the states in the models for modal logic are connected by means of indexed relations R_a that model (program) transitions, epistemic or desired alternatives, or the effect of possible moves (here, a can represent a specific program, a dimension of time (say, future or past), an agent, a move, etc.). Each accessibility relation R_a in the semantics is then paired with a necessity operator $[a]$ in the modal language, where $[a]\varphi$ may read: after every execution of a, in each future along dimension a, in every state considered possible or desired by agent a, or in every state that is the result of performing move a, φ holds.

So, the language of modal logic provides a crisp, variable-free way of expressing a variety of properties of interest. It is important to realise that there is not just one modal logic: although there is an axiomatisation $\mathbf{K}$ that characterises the class $\mathcal{K}$ of validities on all models, this does not mean that all logics for, say, agency are the same. It only means that they are extensions of $\mathbf{K}$. As a simple example, the property $[a]\varphi \to \varphi$ (i) seems reasonable when $[a]$ denotes 'knowing that', but is perhaps less desirable when it means 'believing that'. One of the reasons for the success of modal logic is that often, a syntactic scheme corresponds to an additional constraint on the accessibility relation R_a: in the case of (i), reflexivity of R_a is, in a precise sense, sufficient and necessary for its validity.

To appreciate this, we use a little bit more detail (precise definitions are given in Section 2.) Central in the semantics of modal logic is the notion of Kripke frame $\mathcal{F}$, which includes a set W of states and some accessibility relations R_a, for indexes $a \in I$. We can then define a notion of validity $\models$ on frames and formulate the result mentioned above:

$$R_a \text{ is reflexive} \quad \text{iff} \quad \mathcal{F} \models [a]\varphi \to \varphi, \text{ for all } \varphi \qquad (1)$$

Characterisations like (1) are referred to as *correspondence results* (van Benthem 1976), because they establish a correspondence between a first-order property on frames and a modal validity. Another example of correspondence is that between $\forall x \forall y (R_a(x, y) \to R_b(x, y))$ and $[b]\varphi \to [a]\varphi$ (saying, e.g., that whatever is achieved by program b, is also achieved by a, or that a knows at least as much as b).

Mathematically elegant and powerful correspondence theory may be, it also has shortcomings. First note that in (1), truth is *global*, i.e., holds throughout the frame. This means that for instance (using a doxastic/epistemic reading of (i)), we cannot model situations in which a's beliefs are true, but b does not know that: the validity of (i) entails b's knowledge of (i). That is, we cannot express that for all φ, we have $[a]\varphi \to \varphi$, while for some ψ, also $\neg[b]([a]\psi \to \psi)$ holds. Notice that in such cases, as well as in (1), quantification appears at the meta-, and therefore the outermost, level. It is therefore impossible to distinguish the following two situations: in the first, b knows that a has perfect information and is a perfect reasoner, therefore, b knows that whatever a believes must be correct. The second situation is one in which b systematically has some way to verify for every property φ, that whenever a believes it, then φ is true.

As observed in (Belardinelli and van der Hoek 2015), by allowing for quantification over propositions — thus obtaining the language of Epistemic Quantified Boolean Logic (EQBL), both of these problems can be addressed. To wit, in the first example, the EQBL formula $\forall p([a]p \rightarrow p) \wedge \exists p \neg [b]([a]p \rightarrow p)$ expresses that all beliefs of agent a are correct, but b does not know this, or, that although program a leaves the truth of propositions intact, if we first execute program b this is no longer the case. The two different readings in the second example can simply be represented by $[b]\forall p([a]p \rightarrow p)$ and $\forall p[b]([a]p \rightarrow p)$, respectively. The reader may recognise the distinction between *de dicto* and *de re* quantification here (Fitting and Mendelsohn 1998).

Finally, at the level of models, it is also impossible to distinguish the following two situations. The beliefs of agent a may be correct because the model is reflexive (for R_a), but it might as well be correct because of some particular choices for the valuation in that model: take two states u and v that are a-connected (but the relation is not reflexive). In this case, in u agent a's beliefs are correct, but this does not hold for arbitrary valuations in the model. Since this is what we effectively quantify over, in u the formula $\forall p[a]p \rightarrow p)$ will typically not hold.

The aim of this paper is to re-emphasise the usefulness of EQBL and indeed, more generally, Second-order Propositional Modal Logic – SOPML, by building on the line of (Belardinelli and van der Hoek 2015). In particular, we aim at developing a tool for SOPML which resembles that of bisimulation in propositional modal logic: bisimulations bring to the fore when two models can be considered the same, and they can be used to test the limits of what can be expressed: when two models for a language L are bisimilar but disagree on some property $\Phi \in L'$, it shows that Φ is not expressible in L. We also introduce notions of game which approximate these (bi)simulations. Our aim is to provide formal tools so as to facilitate the use of SOPML as a language for knowledge representation, as well as temporal and spatial reasoning in AI.

Scheme of the paper. In Section 2 we introduce the syntax and semantics of SOPML. The main technical contributions appear in Section 3, where we develop the model-theoretic notion of (bi)simulation within SOPML, while in Section 4 we provide game-theoretic correspondents. Section 5 illustrates the theoretical contribution through applications to (in)expressibility results. Finally, in Section 6 we draw a comparison with the related literature and point to future directions of research. For reasons of space, proofs are omitted.

2 Syntax and Semantics of SOPML

Hereafter we assume a set AP of atomic propositions (or atoms), and a finite set I of indexes.

Definition 1 (SOPML) *The formulas in SOPML are defined in BNF as follows, for $p \in AP$ and $a \in I$:*

$$\psi \quad ::= \quad p \mid \neg\psi \mid \psi \rightarrow \psi \mid [a]\psi \mid \forall p \psi$$

The language of Second-order Propositional Modal Logic contains indexed modal formulas $[a]\phi$, for any $a \in I$, whose informal meaning is that "ϕ is necessary relatively to a". Also, a quantified formula $\forall p \phi$ intuitively says that "for all propositions, ϕ is true". The propositional connectives $\wedge$, $\vee$, modal operator $\langle a \rangle$, and quantifier $\exists$ are defined as standard. The name SOPML is related to second-order quantification, as it will become apparent later on. In particular, SOPML has been studied in relation to Monadic Second-order Logic – MSO (ten Cate 2006; Kaminski and Tiomkin 1996).

In this paper we consider also the *universal* fragment of SOPML (A-SOPML), defined by the following BNF:

$$\psi \quad ::= \quad p \mid \neg p \mid \psi \wedge \psi \mid \psi \vee \psi \mid [a]\psi \mid \forall p \psi$$

Notice that in A-SOPML negation applies only to atoms. Hence, it contains no formula of the form $\exists p \phi$ nor $\langle a \rangle \phi$.

To provide a meaning to SOPML formulas we introduce multi-modal Kripke frames and models.

Definition 2 (Kripke frame) *A Kripke frame is a tuple $\mathcal{F} = \langle W, D, R \rangle$ where*

- *W is a set of* possible worlds*;*
- *D is the* domain of propositions*, i.e., a subset of 2^W;*
- *$R : I \rightarrow 2^{W \times W}$ assigns a* binary relation *on W to each index in I.*

As standard in propositional modal logic – PML (Blackburn, de Rijke, and Venema 2001), for every index $a \in I$, R_a is an accessibility relation between worlds in W. In addition, Def. 2 includes a set $D \subseteq 2^W$ of "admissible" propositions to interpret atoms and quantifiers. Clearly, the Kripke frames in Def. 2 are related to general frames (Blackburn, de Rijke, and Venema 2001; Mares and Goldblatt 2006). However, there are some notable differences. Firstly, in general frames the domain D of propositions is a boolean algebra with operators. Secondly, the language interpreted on general frames is usually a plain modal logic, while here we address quantification as well. This makes our framework strictly more expressive than general frames. Finally, for each index $a \in I$ and $w \in W$, we define $R_a(w) = \{w' \mid R_a(w, w')\}$.

To assign a meaning to SOPML formulas we define *assignments* as functions $V : AP \rightarrow D$. Given a set $U \in D$, the assignment V^p_U assigns U to p and coincides with V on all other atoms. Notice that atoms can only be assigned propositions in $D \subseteq 2^W$. A *(Kripke) model* is then defined as a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$.

In the rest of the paper we analyse particular classes of Kripke frames and models.

Definition 3 *A Kripke frame $\mathcal{F}$ is*

boolean	*iff*	*D is a boolean algebra, i.e., it is closed under intersection, union and complementation*
full	*iff*	*$D = 2^W$*

A Kripke model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is boolean *(resp.* full*) whenever the underlying frame $\mathcal{F}$ is.*

Since the interest in literature and applications has focused on the classes above (Fine 1970; Mares and Goldblatt 2006), we will pre-eminently analyse them in the following. A model is *full* (resp. *boolean*) whenever the underlying frame is.

We finally define the notion of satisfaction for SOPML.

Definition 4 (Semantics of SOPML) *We define whether model $\mathcal{M} = \langle \mathcal{F}, V \rangle$* satisfies *formula φ at world w, or $(\mathcal{M}, w) \models \varphi$, as follows (clauses for propositional connectives are omitted as straightforward):*

$(\mathcal{M}, w) \models p$ *iff* $w \in V(p)$
$(\mathcal{M}, w) \models [a]\psi$ *iff for all* $w' \in R_a(w)$, $(\mathcal{M}, w') \models \psi$
$(\mathcal{M}, w) \models \forall p\psi$ *iff for all* $U \in D$, $(\langle \mathcal{F}, V^p_U \rangle, w) \models \psi$

We now introduce standard notions of truth and validity to be used hereafter. We write $(\mathcal{F}, V, w) \models \phi$ for $(\langle \mathcal{F}, V \rangle, w) \models \phi$. Then, a formula ϕ is *true* at w, or $(\mathcal{F}, w) \models \phi$, iff $(\mathcal{F}, V, w) \models \phi$ for every assignment V; ϕ is *valid* in a frame $\mathcal{F}$, or $\mathcal{F} \models \phi$, iff $(\mathcal{F}, w) \models \phi$ for every world w in $\mathcal{F}$; ϕ is *valid* in a class $\mathcal{K}$ of frames, or $\mathcal{K} \models \phi$, iff $\mathcal{F} \models \phi$ for every $\mathcal{F} \in \mathcal{K}$.

Hereafter we consider classes $\mathcal{K}_{all}$ of all Kripke frames, $\mathcal{K}_{bool}$ of all boolean frames, and $\mathcal{K}_{full}$ of all full frames. If we define $\mathsf{Th}(\mathcal{K}) = \{\phi \in SOPML \mid \mathcal{K} \models \phi\}$, then we remark without proof that $\mathsf{Th}(\mathcal{K}_{all}) \subset \mathsf{Th}(\mathcal{K}_{bool}) \subset \mathsf{Th}(\mathcal{K}_{full})$. Hence, these are all distinct classes of validities.

Example 1 To illustrate the expressive power of SOPML in knowledge representation, we contrast it with Comparative Epistemic Logic – CEL (van Ditmarsch, van der Hoek, and Kooi 2012). CEL extends the language of PML with formulas $a \succcurlyeq b$, the intuitive interpretation of which is: *agent b knows at least as much as agent a*. Semantically, the clause for satisfaction of such formulas at world w in model $\mathcal{M}$ is given as follows:

$$(\mathcal{M}, w) \models a \succcurlyeq b \quad \text{iff} \quad R_b(w) \subseteq R_a(w) \tag{2}$$

In this sense $a \succcurlyeq b$ also expresses a local property of frame $\mathcal{F}$, namely the inclusion $R_b(w) \subseteq R_a(w)$. The comparison between a's and b's knowledge can be recast in SOPML as

$$\forall p(K_a p \to K_b p) \tag{3}$$

where K_a is the standard notation for modal operator $[a]$ in epistemic contexts. In particular, the RHS of (2) is tantamount to the satisfaction of (3) at w, whenever model $\mathcal{M}$ is full. More precisely, for generic models $\mathcal{M}$ we have that

$$(\mathcal{M}, w) \models a \succcurlyeq b \quad \Rightarrow \quad (\mathcal{M}, w) \models \forall p(K_a p \to K_b p)$$

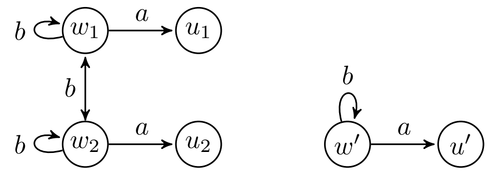
and the converse holds for full $\mathcal{M}$. As a result, formulas $a \succcurlyeq b$ and (3) have the same meaning in the class of full models, and therefore CEL can indeed be expressed in SOPML, as detailed in (Belardinelli and van der Hoek 2015).

Moreover, SOPML allows us to make distinctions that are not expressible in PML. Related to the example in the introduction, in SOPML we can state that *b knows that a's beliefs are not truthful* by using the SOPML formula

$$K_b \exists p(B_a p \wedge \neg p) \tag{4}$$

Notice that (4) expresses a *de dicto* reading of quantification w.r.t. agent b's knowledge, that is, b knows that there exists some fact believed by a, which is false, possibly without being able to explicitly point out the actual content of a's false belief. On the other hand, b could actually be aware of some fact, which is believed by a but false, as expressed in the following *de re* formula:

$$\exists p K_b(B_a p \wedge \neg p) \tag{5}$$



(a) frame $\mathcal{F}$ (b) frame $\mathcal{F}'$

Figure 1: Frames $\mathcal{F}$ and $\mathcal{F}'$ in Example 1.

We remark that (4) and (5) are not equivalent in general, being (5) strictly stronger than (4).

Specifically, to account for the difference between (4) and (5), consider frame $\mathcal{F}$ in Fig. 1(a), where the W- and R-components are as depicted, and $D = \{\{w\} \mid w \in W\}$. Clearly, $(\mathcal{F}, V, w_1) \models B_a p \wedge \neg p$ for $V(p) = \{u_1\}$, and similarly $(\mathcal{F}, V', w_2) \models B_a p \wedge \neg p$ for $V'(p) = \{u_2\}$. Hence, $(\mathcal{F}, w) \models (4)$ for $w \in \{w_1, w_2\}$. On the other hand, for no $U \in D$, $(\mathcal{F}, V^p_U, w) \models B_a p \wedge \neg p$. Therefore, $(\mathcal{F}, w) \not\models (5)$ for $w \in \{w_1, w_2\}$. As a result, in SOPML we can distinguish the two readings (4) and (5) of agent b's higher-level knowledge.

Finally, consider frame $\mathcal{F}'$ in Fig. 1(b) with $D' = \{\{w'\} \mid w' \in W'\}$. We can check that $(\mathcal{F}', w') \models (5)$ (and (4) as well). However, $\mathcal{F}$ and $\mathcal{F}'$, taken as frames for PML (i.e., supressing the domains D and D' of quantification, and using only the language of PML), are bisimilar (Blackburn, de Rijke, and Venema 2001), with bisimulation relation H s.t. $H(w', w_i)$ and $H(u', u_i)$ for $i \in \{1, 2\}$. Hence, $\mathcal{F}$ and $\mathcal{F}'$ cannot be distinguished by any PML formula, implying that the *de re* formula (5) cannot be expressed in PML. ■

3 A Model-Theoretic Analysis of SOPML

In this section we investigate the expressive power of SOPML by introducing truth-perserving (bi)simulations relations. In particular, since truth for SOPML formulas depends not only on assignments to atoms, but also on structural features of frames (namely, the set D of allowed propositions), (bi)simulations have to be defined on frames.

We start with some definitions. Given a relation $\sigma \subseteq W \times W'$ and $U \subseteq W$, $U' \subseteq W'$, let $\sigma(U) = \{w' \in W' \mid \sigma(w, w') \text{ for some } w \in W\}$ and $\sigma^{-1}(U') = \{w \in W \mid \sigma(w, w') \text{ for some } w' \in W'\}$.

Definition 5 (Frame Simulation) *Let $\mathcal{F} = \langle W, D, R \rangle$ and $\mathcal{F}' = \langle W', D', R' \rangle$ be frames. A* simulation *is a relation $\sigma \subseteq W \times W'$ such that*

1. *$\sigma(w, w')$ implies*
   (a) *for every $v \in W$, $a \in I$, if $R_a(w, v)$ then for some $v' \in R'_a(w')$, $\sigma(v, v')$;*
   (b) *for every $U \in D$, $w \in U$ iff $w' \in \sigma(U)$.*
2. *for every $U \in D$, $\sigma(U) \in D'$;*

A state w' *simulates* w, or $w \preceq w'$, iff $\sigma(w, w')$ holds for some simulation $\sigma \subseteq W \times W'$. Generally, the relation $\preceq$ is not a simulation. To see this, consider frames, $\mathcal{F}_1 = \langle \{w_1, w_2\}, \{\{w_1\}, \{w_2\}\}, \{(w_1, w_2), (w_2, w_1)\} \rangle$, and $\mathcal{F}_2 = \langle \{x_1, x_2\}, \{\{x_1\}, \{x_2\}\}, \{(x_1, x_2), (x_2, x_1)\} \rangle$.

Obviously, the two frames are isomorphic. Now, for $U = \{w_1\}$, by definition $\preceq(U)$ is equal to $\{x_1, x_2\}$, as both x_1, x_2 can be chosen as the image of w_1. However, $\{x_1, x_2\} \notin D_2$. Nonetheless, $\preceq$ is a preorder, i.e., a reflexive and transitive relation. Finally, a frame $\mathcal{F}'$ *simulates* $\mathcal{F}$, or $\mathcal{F} \preceq \mathcal{F}'$, iff for every $w \in W$, $w \preceq w'$ for some $w' \in W'$.

We illustrate the newly introduced notion by an example.

Example 2 Consider frames $\mathcal{F} = \langle W, R, D\rangle$ and $\mathcal{F}' = \langle W', R', D'\rangle$ over $I = \{a, b, c\}$, depicted in Fig. 2, with

- $W = \{w_1, w_2, w_3\}$;
- $R_a = \{(w_1, w_3), (w_3, w_1)\}$, $R_b = \{(w_1, w_2), (w_2, w_1)\}$, $R_c = \{(w_2, w_3), (w_2, w_3)\}$;
- $D = 2^W$;
- $W' = \{u_s \mid s$ is a finite sequence on $\{1, 2, 3\}$ starting with 1, with no adjacent repetition$\}$;
- for every $i \in I$, $R'_i = \{(u_s, u_{s'}) \mid s' = s \cdot m$ and $R_i(w_{last(s)}, w_m)\}$;
- let $U'_n = \{u_s \mid last(s) = n\}$, then D' is the boolean algebra having U'_1, U'_2 and U'_3 as atoms.

Intuitively, frame $\mathcal{F}$ can be thought of as a scenario where robots a, b, and c move around locations w_1, w_2, w_3 (robot a moves between w_1 and w_3, etc.). We check that the relation $\sigma \subseteq W \times W'$ such that $\sigma(w_n, u_s)$ holds iff $last(s) = n$, is a simulation. Firstly, if $\sigma(w_n, u_s)$ and $R_i(w_n, w_m)$ then $s' = s{\cdot}m$ is such that $R'_i(u_s, u_{s'})$ and $\sigma(w_m, u_{s'})$. Secondly, if $\sigma(w_n, u_s)$ then $last(s) = n$. Hence, for every $U \in D$, $u_s \in \sigma(U)$ implies that for some $w_k \in U$, $\sigma(w_k, u_s)$, that is, $last(s) = k = n$ and $w_n \in U$. Thirdly, for every $U \in D$, the set $\sigma(U) = \{u_s \mid last(s) = n \text{ and } w_n \in U\} = \bigcup_{w_n \in U} U'_n$ belongs to D'.

Finally, we observe that for every $w_n \in W$, $\sigma(w_n, u_s)$ for $last(s) = n$. Thus, frame $\mathcal{F}'$ simulates $\mathcal{F}$. ■

Consider now the following remark on the relation between simulations and properties of frames.

Remark 1 If a frame $\mathcal{F}'$ simulates a boolean (resp. full) frame $\mathcal{F}$, then $\mathcal{F}'$ need not be boolean (resp. full). Nor $\mathcal{F}'$ being boolean (resp. full) implies that $\mathcal{F}$ is also boolean (resp. full). ■

Hence, similar frames need not to belong to the same class. Below we will compare these results with those available for bisimulations.

We now state that simulations preserve the satisfaction of the universal fragment of SOPML.

Theorem 1 *If $w \preceq w'$, then for all φ in A-SOPML,*

$$(\mathcal{F}', w') \models \varphi \quad \textit{implies} \quad (\mathcal{F}, w) \models \varphi$$

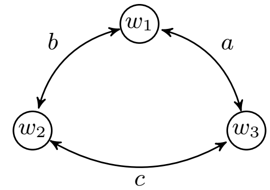
As a direct consequence of Theorem 1 we obtain the following corollary.

Corollary 2 *If $\mathcal{F} \preceq \mathcal{F}'$, then for all φ in A-SOPML,*

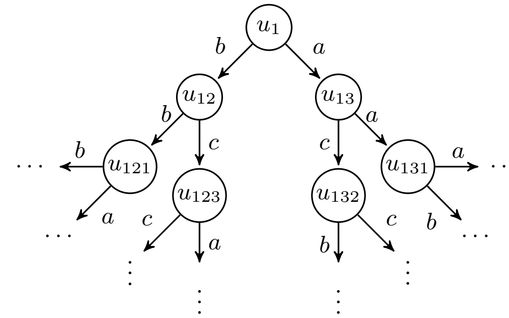
$$\mathcal{F}' \models \varphi \quad \textit{implies} \quad \mathcal{F} \models \varphi$$

As a result, the notion of simulation introduced in Def. 5 preserves the universal fragment of SOPML, similarly to the case for standard simulations and PML.

Simulations can naturally be extended to bisimulations. Also in this case, our focus is at the frame level.



(a) the Kripke frame $\mathcal{F}$.



(b) the Kripke frame $\mathcal{F}'$.

Figure 2: frames $\mathcal{F}$ and $\mathcal{F}'$ in Example 2. D components are omitted for clarity.

Definition 6 (Frame Bisimulation) *Let $\mathcal{F} = \langle W, D, R\rangle$ and $\mathcal{F}' = \langle W', D', R'\rangle$ be frames. A* bisimulation *is a relation $\beta \subseteq W \times W'$ such that both β and converse β^{-1} are simulations. That is,*

1. *$\beta(w, w')$ implies*
   (a) *for every $v \in W$, $a \in I$, if $R_a(w, v)$ then for some $v' \in R'_a(w')$, $\beta(v, v')$;*
   (b) *for every $U \in D$, $w \in U$ iff $w' \in \beta(U)$;*
   (c) *for every $v' \in W'$, $a \in I$, if $R'_a(w', v')$ then for some $v \in R_a(w)$, $\beta(v, v')$;*
   (d) *for every $U' \in D'$, $w' \in U'$ iff $w \in \beta^{-1}(U')$;*
2. *for every $U \in D$, $\beta(U) \in D'$;*
3. *for every $U' \in D'$, $\beta^{-1}(U') \in D$.*

States w and w' are *bisimilar*, or $w \approx w'$, iff $\beta(w, w')$ holds for some bisimulation $\beta \subseteq W \times W'$. Notice that relation $\approx$ is not necessarily a bisimulation on $W \times W'$, similarly to what shown above for simulations, but it is an equivalence relation. Finally, frames $\mathcal{F}$ and $\mathcal{F}'$ are *bisimilar*, or $\mathcal{F} \approx \mathcal{F}'$, iff *(i)* for every $w \in W$, $w \approx w'$ for some $w' \in W'$; and *(ii)* for every $w' \in W'$, $w \approx w'$ for some $w \in W$.

Example 3 Notice that frames $\mathcal{F}$ and $\mathcal{F}'$ in Example 2 are actually bisimilar. To prove this fact, we state that the converse relation $\sigma^{-1} \subseteq W' \times W$ such that $\sigma^{-1}(u_s, w_n)$ iff $last(s) = n$, is also a simulation. ■

We now state the following remark on the relationship between properties of frames and bisimulations.

Remark 2 Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are bisimilar. Then, $\mathcal{F}$ is boolean iff $\mathcal{F}'$ is. However, if $\mathcal{F}$ is full, then $\mathcal{F}'$ need not to be full. Nor $\mathcal{F}'$ being full implies that $\mathcal{F}$ is also full. ■

Compare the situation for bisimulations with the weaker results available in Remark 1 for simulations. Specifically, bisimulations preserve the class of boolean frames.

We now state the main preservation result of this section.

Theorem 3 *If $w \approx w'$, then for every formula φ in SOPML,*

$$(\mathcal{F}, w) \models \varphi \quad \textit{iff} \quad (\mathcal{F}', w') \models \varphi.$$

We can now infer that bisimulations in SOPML are 'stronger' then those for PML: whereas we noted that the frames of Fig. 1 are bisimilar in PML, as a result of Theorem 3, they are not bisimilar in the SOPML sense.

As a consequence of Theorem 3 we obtain the following.

Corollary 4 *If $\mathcal{F} \approx \mathcal{F}'$, then for all φ in SOPML,*

$$\mathcal{F} \models \varphi \quad \textit{iff} \quad \mathcal{F}' \models \varphi$$

Discussion. We now compare our definition of (bi)simulation for SOPML, with the notion of (bi)simulation for PML (Blackburn, de Rijke, and Venema 2001). Observe that if a frame $\mathcal{F}'$ simulates $\mathcal{F}$ in SOPML, with simulation relation σ, then for every model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ based on $\mathcal{F}$, model $\mathcal{M}' = \langle \mathcal{F}', \sigma(V) \rangle$ on $\mathcal{F}'$ PML-simulates $\mathcal{M}$. In particular, if $\sigma(w, w')$ then for every $v \in W$, $a \in I$, $R_a(w, v)$ implies that for some $v' \in R'_a(w')$, $\sigma(v, v')$ by condition $1(a)$ in Def. 5. Moreover, $w \in V(p) \in D$ iff $w' \in \sigma(V)(p) \in D'$ by conditions $1(b)$ and 2. Therefore, if $\mathcal{M}'$ satisfies any universal formula ϕ in PML, then ϕ also holds in $\mathcal{M}$. Hence, Def. 5 of simulation for frames in SOPML is really a generalisation of the model-theoretic notion in PML. Furthermore, if frames $\mathcal{F}'$ and $\mathcal{F}$ are bisimilar in SOPML, with bisimulation relation β, then models $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and $\mathcal{M}' = \langle \mathcal{F}', \beta(V) \rangle$ are also bisimilar in PML. Likewise, models $\mathcal{M}' = \langle \mathcal{F}', V' \rangle$ and $\mathcal{M} = \langle \mathcal{F}, \beta^{-1}(V) \rangle$ are PML-bisimilar as well. Also in this case, SOPML bisimulations on frames generalise PML bisimulations on models.

4 Simulation Games for SOPML

In this section we present simulation games for SOPML. Similarly to the case for PML, the existence of a winning strategy for Duplicator guarantees the preservation of (universal) SOPML formulas. We start by considering simulation games played by Spoiler and Duplicator.

Definition 7 (Simulation Game) *A simulation game $\mathcal{G}$ starting from pointed frames $(\mathcal{F}, w)$ and $(\mathcal{F}', w')$ is defined as follows. Let $(\mathcal{F}, v, \vec{U})$, $(\mathcal{F}', v', \vec{U}')$ be the current state of the game, where $v \in W$ (resp. $v' \in W'$) and $\vec{U}$ (resp. $\vec{U}'$) is a possibly empty sequence of sets in D (resp. D').*

Then the game proceeds according to the following rules:

1. *Either Spoiler picks a set U belonging to D and Duplicator has to reply with a set U' belonging to D' such that $v \in U$ iff $v' \in U'$. The new state of the game is $(\mathcal{F}, v, \vec{U} \cdot U)$, $(\mathcal{F}', v', \vec{U}' \cdot U')$.*
2. *Or, for some $a \in I$, Spoiler picks a state $u \in R_a(v)$ and Duplicator has to reply with state $u' \in R'_a(v')$ such that for every i, $u \in U_i$ iff $u' \in U'_i$. The new state of the game is $(\mathcal{F}, u, \vec{U})$, $(\mathcal{F}', u', \vec{U}')$.*

If Duplicator cannot match a Spoiler's move, then Spoiler wins the game. Otherwise, Duplicator wins the game. A winning strategy is a strategy whereby Duplicator can reply to all of Spoiler's move, thus winning the game. It can be shown that if state $w' \in \mathcal{F}'$ simulates $w \in \mathcal{F}$, Duplicator has a winning strategy in game $(\mathcal{F}, w)$, $(\mathcal{F}', w')$.

Lemma 5 *If state w' is similar to w, then Duplicator has a winning strategy for the game starting in $(\mathcal{F}, w)$, $(\mathcal{F}', w')$.*

On the other hand, we state without proof that the existence of a winning strategy for Duplicator is not sufficient to guarantee the existence of a simulation.

Nonetheless, winning strategies enforce the following preservation result.

Theorem 6 *If Duplicator has a winning strategy for the game starting in state $(\mathcal{F}, w)$, $(\mathcal{F}', w')$, then for every universal formula φ in A-SOPML,*

$$(\mathcal{F}', w') \models \varphi \quad \textit{implies} \quad (\mathcal{F}, w) \models \varphi$$

Thus, even though simulation games are strictly weaker than simulations, they still preserve A-SOPML.

We also introduce a generalisation to bisimulation games.

Definition 8 (Bisimulation Games) *A bisimulation game $\mathcal{G}$ starting from pointed frames $(\mathcal{F}, w)$ and $(\mathcal{F}', w')$ is defined as follows. Let $(\mathcal{F}, v, \vec{U})$, $(\mathcal{F}', v', \vec{U}')$ be the state of the game, where $v \in W$ (resp. $v' \in W'$) and $\vec{U}$ (resp. $\vec{U}'$) is a possibly empty sequence of sets in D (resp. D').*

Then the game proceeds according to the following rules:

1. *Either Spoiler picks a set U belonging to D (resp. $U' \in D'$) and Duplicator has to reply with a set U' belonging to D' (resp. $U \in D$) such that $v \in U$ iff $v' \in U'$. The new state of the game is $(\mathcal{F}, v, \vec{U} \cdot U)$, $(\mathcal{F}', v', \vec{U}' \cdot U')$.*
2. *Or, for some $a \in I$, Spoiler picks a state $u \in R_a(v)$ (resp. $u' \in R'_a(v')$) and Duplicator has to reply with state $u' \in R'_a(v')$ (resp. $u \in R_a(v)$) such that for every i, $u \in U_i$ iff $u' \in U'_i$. The new state of the game is $(\mathcal{F}, u, \vec{U})$, $(\mathcal{F}', u', \vec{U}')$.*

As above, if Duplicator cannot match a Spoiler's move, then Spoiler wins the game. Otherwise, Duplicator wins the game. Informally, Duplicator's role is to maintain that the two pointed frames satisfy the same formulas: if Spoiler can reach a situation in the game with two 'different' states it means that either Duplicator did not play optimally, or that the two pointed frames disagree on a formula. A winning strategy is defined as usual.

Theorem 7 *If Duplicator has a winning strategy for the game starting in state $(\mathcal{F}, w)$, $(\mathcal{F}', w')$, then for every formula φ in SOPML,*

$$(\mathcal{F}', w') \models \varphi \quad \textit{iff} \quad (\mathcal{F}, w) \models \varphi$$

We conclude by discussing the two groups of preservation results. Both Theorems 1 and 3 and Theorems 6 and 7 provide results on the preservation of (universal) SOPML. However, (bi)simulations define global concepts, as these are defined on the whole state space $W \times W'$; while games are local, as at each point in the game the players have only a local view on the frames, centred on a couple of states. Hence, the nature of these two notions is profoundly different, with games provably weaker than (bi)simulations, and it is likely that they have different applications. For instance, (bi)simulations are used to prove inexpressibility results (as in the following section); while games can be used to show that two frames are not bisimilar, by providing moves for Spoiler to which Duplicator cannot reply.

5 Bisimulations and Expressivity

In this section we explore the expressivity of SOPML, also by using the (bi)simulations introduced in Section 3. Hereafter we focus on temporal and spatial properties normally used in AI. Hereafter we say that a property $\mathcal{P}$ is *expressible* in a language L and class $\mathcal{K}$ of frames iff for some formula $\phi \in L$, $\mathcal{K} \models \phi$ iff every frame in $\mathcal{K}$ has property $\mathcal{P}$. Sometimes we omit either L or $\mathcal{K}$, whenever these are clear by the context.

First of all, we remind that the topological completeness of the real numbers is not expressible in PML: the proof of this fact makes use of a propositional bisimulation between the structure $(\mathbb{R}, <)$ of reals and the rationals $(\mathbb{Q}, <)$ (Baltag and Smets 2014). On the other hand, in SOPML we can express completeness by means of the following formula, where the modal operators $\Box$ and $\Diamond$ are interpreted on the strict linear order $<$, while $\blacksquare\phi$ (resp. $\blacklozenge\phi$) are shorthands for $\phi \wedge \Box\phi$ (resp. $\phi \vee \Diamond\phi$):

$$\forall p\,((\blacklozenge p \wedge \blacklozenge\blacksquare\neg p) \rightarrow \tag{6}$$
$$(\blacklozenge(p \wedge \Box\neg p)\vee \tag{7}$$
$$\exists q\,((q \leftrightarrow \blacksquare\neg p)\wedge \tag{8}$$
$$\exists s\,(\blacklozenge s \wedge \blacksquare(s \rightarrow q)\wedge \tag{9}$$
$$\blacksquare(\neg s \wedge q \rightarrow \blacksquare\neg s) \wedge \blacksquare(s \rightarrow \Box\neg s))))) \tag{10}$$

This formula states that (6) for every non-empty and upper bounded set p, either (7) p has a greatest element, or (8) there exists a set q of "strict" upper bounds, (9) which includes a non-empty subset s (10) that is a singleton and the least upper bound.

On the other hand, by using simulations we immediately obtain the following inexpressibility result.

Lemma 8 *Topological completeness is not expressible in the universal fragment A-SOPML.*

Clearly, the identity relation is a simulation from $(\mathbb{Q}, <)$ to $(\mathbb{R}, <)$, i.e., $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$, and if completeness were expressible as an A-SOPML formula ϕ, $(\mathbb{R}, <) \models \phi$ would imply $(\mathbb{Q}, <) \models \phi$, a contradiction.

Furthermore, consider the graph-theoretic property of 3-colorability, as formalised by the following SOPML formula, where operator $\Box$ is interpreted on the edges $E \subseteq W^2$ of a graph $G = \langle W, E\rangle$, while $\Box^+$ is interpreted on the transitive closure of E:

$$\exists p_1, p_2, p_3(\Box^+(p_1 \vee p_2 \vee p_3) \wedge \tag{11}$$
$$\bigwedge_{1,2,3} \Box^+(p_i \rightarrow \neg\Diamond p_i)) \tag{12}$$

The validity of this formula in a vertex $v \in G$ implies that (11) all vertices in the subgraph generated by v are either p_1, p_2, or p_3, and (12) no two adjacent vertices have the same colour. Thus, the subgraph generated by v is 3-colorable. Observe that frame $\mathcal{F}$ in Fig. 2(a) is trivially 3-colorable, and since states w_1 and u_1 are bisimilar, as an immediate consequence of Theorem 3, also frame $\mathcal{F}'$ is 3-colorable.

To illustrate further the (in)expressivity of SOPML through simulations, we consider one more graph-theoretic property: the existence of a Hamiltonian path, i.e., a path that visits all states exactly once. Again, frame $\mathcal{F}$ in Fig. 2(a) has a Hamiltonian path w_1, w_2, w_3. On the other hand, frame $\mathcal{F}'$ in Fig. 2(b) has no such path. Since $\mathcal{F}$ and $\mathcal{F}'$ are bisimilar, the following result immediately follows.

Lemma 9 *The property of having a Hamiltonian path is not expressible in SOPML.*

Indeed, it is known that such property is expressible in the language MSO_2, an extension of MSO, which is strictly more expressive than SOPML.

As a further example, we prove that neither finiteness nor infinity of the state space W are expressible in the class of boolean frames. To see this, consider frames $\mathcal{F}_1 = \langle \mathbb{N}, succ, \{\mathbb{N}, \emptyset\}\rangle$ and $\mathcal{F}_2 = \langle \{w'\}, \{(w', w')\}, \{\{w'\}, \emptyset\}\rangle$, which are clearly boolean. In particular, the relation β that maps every natural $n \in \mathbb{N}$ to w' is a SOPML bisimulation. Equivalently, it is easy to see that Duplicator has a winning strategy in game $(\mathcal{F}_1, n)$, $(\mathcal{F}_2, w')$, for every $n \in \mathbb{N}$: Duplicator has only to reply with w' to any $m \in \mathbb{N}$ chosen by Spoiler, and with $\{W'\}$ (resp. $\emptyset$) whenever Spoiler chooses $\mathbb{N}$ (resp. $\emptyset$). Thus, $\mathcal{F}_1$ and $\mathcal{F}_2$ validate the same SOPML formulas. However, $\mathcal{F}_1$ is infinite while $\mathcal{F}_2$ is finite.

Lemma 10 *Neither finiteness nor infinity are expressible in boolean frames.*

To conclude, we show that finiteness is not even expressible in full frames. Let $[n]$ be the set $\{0, \ldots, n\}$, $\mathcal{F}_n$ the frame $\langle [n], succ, 2^{[n]}\rangle$, and $\mathcal{F}_{\mathbb{N}} = \langle \mathbb{N}, succ, 2^{\mathbb{N}}\rangle$ the frame isomorphic to the natural numbers. Both $\mathcal{F}_n$ and $\mathcal{F}_{\mathbb{N}}$ are full. Let F be the class of all frames $\mathcal{F}_n$, for $n \in \mathbb{N}$, and consider the following result.

Lemma 11 $\mathsf{Th}(F)$ *is a subset of* $\mathsf{Th}(\mathcal{F}_{\mathbb{N}})$.

By Lemma 11, if ϕ expressed 'being finite', then it would be valid in F, and hence also in $\mathcal{F}_{\mathbb{N}}$, a contradiction. Thus, finiteness is not expressible even in the class of full frames.

6 Conclusions

In this paper we initiated a semantical analysis of Second-order Propositional Modal Logic via (bi)simulations. We developed model-theoretic techniques for SOPML: we introduced notions of (bi)simulation and proved that they preserve the satisfaction of (universal) SOPML. Then, we defined game-theoretical counterparts to (bi)simulations and showed that, although weaker, these also preserve the truth of SOPML formulas and its universal fragment. We remarked that, while set-theoretical (bi)simulations might be more appropriate to prove inexpressibility results, their game-theoretic counterparts might be better computationally to actually show that two frames are not bisimilar. Finally, we used (bi)simulations to obtain some inexpressibility results. Specifically, we showed that being finite and having a Hamiltonian path are not expressible in SOPML; while other properties, viz. topological completeness and 3-colorability, *are* expressible. We conclude that SOPML can indeed be used as a modelling language for Artificial Intelligence, particularly for temporal and spatial reasoning, as well to describe higher-level knowledge of agents, that is, the knowledge agents have about other agents' knowledge and beliefs. In this respect, the development of model-theoretic techniques is key for applications.

Related Work. This contribution is inspired by a series of papers on LPML, an extension of propositional modal logic to express local properties (van Ditmarsch, van der Hoek, and Kooi 2009; 2011; 2012). But instead of introducing an *ad hoc* language (with an adjustment for each local property one has in mind) such as Comparative Epistemic Logic, here we make use of (multi-modal) Second-order Propositional Modal Logic. Mono-modal SOPML was first considered by Bull and Fine (Bull 1969; Fine 1970), mainly in relation with axiomatisability and (un)decidability questions. The high complexity (of satisfiability) of SOPML and some non-axiomatisability results might explain why SOPML has been studied far less than PML. More recently, the formal properties of SOPML have been investigated in the literature on modal logic. In (Kaminski and Tiomkin 1996) the authors proved that the expressive power of SOPML (for modalities weaker than 4.2) is the same as MSO; while (ten Cate 2006) provided SOPML with analogues of the van Benthem-Rosen and Goldblatt-Thomason theorems. More directly related to the present contribution, (Belardinelli and van der Hoek 2015) introduced Epistemic Quantified Boolean Logic (EQBL), an epistemic variant of SOPML, and provided axiomatisability and model-checking results. Differently from the reference, here we tackle general SOPML, including temporal and spatial reasoning, define original notions of (bi)simulation, and apply these model-theoretic techniques to analyse the expressivity of the language.

Future Work. From the point of view of logic-based AI, or, more specifically, knowledge representation and reasoning, it is important to address the question of what it means for a formula to follow from a knowledge base. We like to address the issue of entailment and the subsequent questions it raises in an AI context. Moreover, since SOPML allows for quantification in the object language, it would be interesting to extend SOPML and EQBL with a notion of announcement, and then compare it to the logic of arbitrary announcement logic (Balbiani et al. 2007), in terms of expressivity. More generally, quantification in Dynamic Epistemic Logic (van Ditmarsch, van der Hoek, and Kooi 2007) seems an interesting venue of research, where quantification can be over formulas ('after a reads the latter, he knows the same as b') or over new information ('no matter what a will be told, he won't believe that he knows more than b').

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References

Balbiani, P.; Baltag, A.; van Ditmarsch, H.; Herzig, A.; Hoshi, T.; and de Lima, T. 2007. What can we achieve by arbitrary announcements?: A dynamic take on fitch's knowability. In *TARK '07*, 42–51. ACM.

Baltag, A., and Smets, S., eds. 2014. *Johan van Benthem on Logic and Information Dynamics*. Springer.

Belardinelli, F., and van der Hoek, W. 2015. Epistemic quantified boolean logic: Expressiveness and completeness results. In Yang, Q., and Wooldridge, M., eds., *IJCAI*, 2748–2754. AAAI Press.

Blackburn, P.; de Rijke, M.; and Venema, Y. 2001. *Modal Logic*. Cambridge University Press.

Blackburn, P.; van Benthem, J.; and Wolter, F., eds. 2007. *Handbook of Modal Logic*. Amsterdam: Elsevier.

Bull, R. A. 1969. On modal logic with propositional quantifiers. *The Journal of Symbolic Logic* 34:257–263.

Fine, K. 1970. Propositional quantifiers in modal logic. *Theoria* 36(3):336–346.

Fitting, M., and Mendelsohn, R. 1998. *First-Order Modal Logic*. Kluwer Academic Publishers.

Halpern, J., and Moses, Y. 1990. Knowledge and common knowledge in a distributed environment. *Journal of the ACM* 37(3):549–587.

Kaminski, M., and Tiomkin, M. L. 1996. The expressive power of second-order propositional modal logic. *Notre Dame Journal of Formal Logic* 37(1):35–43.

Manna, Z., and Pnueli, A. 1992. *The Temporal Logic of Reactive and Concurrent Systems*, volume 1. Springer-Verlag.

Mares, E., and Goldblatt, R. 2006. An alternative semantics for quantified relevant logic. *JSL* 71(1):163–187.

ten Cate, B. 2006. Expressivity of second order propositional modal logic. *JPL* 35(2):209–223.

van Benthem, J. 1976. *Modal Correspondence Theory*. Ph.D. Dissertation, University of Amsterdam.

van der Hoek, W., and Pauly, M. 2006. Modal logic for games and information. In Blackburn et al. (2007). 1077–1148.

van der Hoek, W., and Wooldridge, M. 2008. Multi-agent systems. In *Handbook of Knowledge Representation* (2007). 887–928.

van Ditmarsch, H.; van der Hoek, W.; and Kooi, B. 2007. *Dynamic Epistemic Logic*. Springer.

van Ditmarsch, H. P.; van der Hoek, W.; and Kooi, B. P. 2009. Knowing more - from global to local correspondence. In Boutilier, C., ed., *Proc. 21st IJCAI*, 955–960.

van Ditmarsch, H. P.; van der Hoek, W.; and Kooi, B. P. 2011. Reasoning about local properties in modal logic. In Sonenberg L. et al., ed., *Proc. 10th AAMAS*, 711–718.

van Ditmarsch, H. P.; van der Hoek, W.; and Kooi, B. P. 2012. Local properties in modal logic. *Artificial Intelligence* 187:133–155.

van Harmelen, F.; Lifschitz, V.; and Porter, B. 2007. *Handbook of Knowledge Representation*. Elsevier.